

The Relationship Between the Maximum Principle and Dynamic Programming on FBSDE

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1 Problem formulation

We consider a stochastic optimization problem with objective function under non-linear expectation. Here the objective function is in general form as a solution of backward-SDE, and our system is described by the following coupled forward-backward systems of SDEs

Forward system

$$\begin{cases} dX(t) = b(t, X(t), u(t))dt + \sigma(t, X(t), u(t))dB(t) \\ X(0) = x(0) \end{cases} \quad (1)$$

Backward system

$$\begin{cases} dY(t) = -g(t, X(t), Y(t), Z(t), u(t))dt + Z(t)dB(t) \\ Y(T) = l(X(T)) \end{cases} \quad (2)$$

where $u(\cdot)$ is the decision(control) process.

And our utility function is given in the general form:

$$J(u(\cdot)) = E\left[\int_0^T f(t, X(t), Y(t), Z(t), u(t))dt + h(Y(0))\right] \quad (3)$$

where we are to maximize J .

Without loss of generality, we can assume that $h(\cdot)$ is identity function and $f(\cdot) = 0$, otherwise, we do a transform and let $Y^*(t) = h(Y(t)) + \int_t^T f(t, X(t), Y(t), Z(t), u(t))dt$, derive the dynamics and boundary condition of $Y^*(\cdot)$, and replace Y by Y^* .

The Hamiltonian is defined by

$$H(t, x, y, z, u) = g(t, x, y, z, u)\lambda(t) + b(t, x, u)p(t) + \sigma(t, x, u)q(t) \quad (4)$$

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Denote optimal control $\bar{u}(\cdot)$. And the dual forward-backward system is defined as:

Dual forward system

$$\begin{cases} d\lambda(t) = \frac{\partial}{\partial y}H(t, X(t, \bar{u}(\cdot)), Y(t, \bar{u}(\cdot)), Z(t, \bar{u}(\cdot)), \bar{u}(t))dt + \frac{\partial}{\partial z}H(t, X(t, \bar{u}(\cdot)), Y(t, \bar{u}(\cdot)), Z(t, \bar{u}(\cdot)), \bar{u}(t))dB(t) \\ \quad = g_y(t, X(t, \bar{u}(\cdot)), Y(t, \bar{u}(\cdot)), Z(t, \bar{u}(\cdot)), \bar{u}(t))\lambda(t)dt + g_z(t, X(t, \bar{u}(\cdot)), Y(t, \bar{u}(\cdot)), Z(t, \bar{u}(\cdot)), \bar{u}(t))\lambda(t)dB(t) \\ \lambda(0) = 1 \end{cases} \quad (5)$$

Dual backward system

$$\begin{cases} dp(t) = -\frac{\partial}{\partial x}H(t, X(t, \bar{u}(\cdot)), Y(t, \bar{u}(\cdot)), Z(t, \bar{u}(\cdot)), \bar{u}(t))dt + q(t)dB(t) \\ p(T) = \lambda(T)l_x(X(T)) \end{cases} \quad (6)$$

2 Maximum principle with convex control set and interior optimal control

Assume that control set U is convex, then the following two conditions are equivalent: (Oksendal [1], Wu [2])

1. $\frac{d}{dy}J(\bar{u} + y\beta)|_{y=0} = 0$ for all bounded $\beta \in U$
2. $\langle \frac{\partial}{\partial u}H(t, X(t, \bar{u}(\cdot)), Y(t, \bar{u}(\cdot)), Z(t, \bar{u}(\cdot)), u)|_{u=\bar{u}(t)}, v - \bar{u}(t) \rangle \geq 0, \forall v \in U$

Notice that condition 1 implicitly assumes that optimal control \bar{u} is an interior point in U .

3 The Relationship between the Maximum Principle and Dynamic Programming

Let value function $V(t, x) = \max_{u_s, s \geq t} E[\int_t^T g(s, X_s, Y_s, Z_s, u_s)ds + Y_T | \mathcal{F}_t, X_t = x]$. Then

$$\begin{cases} V(t, \bar{X}(t)) = Y(t, \bar{u}(\cdot)) \\ V_x(t, \bar{X}(t))\lambda(t) = p(t) \\ V_{xx}(t, \bar{X}(t)) \cdot \sigma(t, \bar{x}(t), \bar{u}(t)) \cdot \lambda(t) + V_x(t, \bar{X}(t)) \cdot \frac{\partial}{\partial z} \bar{H}(t, X(t, \bar{u}(\cdot)), Y(t, \bar{u}(\cdot)), Z(t, \bar{u}(\cdot)), \bar{u}(t)) = q(t) \end{cases} \quad (7)$$

Here dual backward process $p(t)$ can be interpreted as the shadow price of our objective function with respect to infinitesimal variation in underlining process $X(t)$ discounted back to time 0. And dual forward process $\lambda(t)$ can be interpreted as the 'stochastic discount factor' embedded in our recursive utility process $\bar{Y}(t)$ from time t back to time 0.

First we derive the HJB equation for our problem.

$$\begin{cases} 0 = V_t(t, x) + \max_u \{g[t, x, V(t, x), V_x(t, x)\sigma(t, x, u), u] + V_x(t, x)b(t, x, u) + \frac{1}{2}V_{xx}(t, x)\sigma^2(t, x, u)\} \\ V(T, x) = Y(T)|_{X(T)=x} = l(x) \end{cases} \quad (8)$$

By (8) we have

$$\begin{aligned}
& V_t(t, \bar{X}(t)) + g[t, x, V(t, \bar{X}(t)), V_x(t, \bar{X}(t))\sigma(t, \bar{X}(t), \bar{u}(t)), \bar{u}(t)] + V_x(t, \bar{X}(t))b(t, \bar{X}(t), \bar{u}(t)) \\
& + \frac{1}{2}V_{xx}(t, \bar{X}(t))\sigma^2(t, \bar{X}(t), \bar{u}(t)) = 0 \\
& \geq V_t(t, x) + g(t, x, V(t, x), V_x(t, x)\sigma(t, x, \bar{u}(t)), \bar{u}(t)) + V_x(t, x)b(t, x, \bar{u}(t)) + \frac{1}{2}V_{xx}(t, x)\sigma^2(t, x, \bar{u}(t)), \quad \forall x
\end{aligned}$$

Therefore we have

$$\frac{\partial}{\partial x} \{V_t(t, x) + g[t, x, V(t, x), V_x(t, x)\sigma(t, x, \bar{u}(t)), \bar{u}(t)] + V_x(t, x)b(t, x, \bar{u}(t)) + \frac{1}{2}V_{xx}(t, x)\sigma^2(t, x, \bar{u}(t))\}|_{x=\bar{X}(t)} = 0$$

This is equivalent to (we denote $\bar{F}(t) =: F(t, X(t, \bar{u}(\cdot)), Y(t, \bar{u}(\cdot)), V_x(t, \bar{X}(t))\sigma(t, \bar{u}(\cdot)), \bar{u}(t))$)

$$\begin{aligned}
0 &= V_{t,x}(t, \bar{X}(t)) + g_x[t, x, V(t, \bar{X}(t)), V_x(t, \bar{X}(t))\sigma(t, \bar{X}(t), \bar{u}(t)), \bar{u}(t)] \\
&+ g_y[t, x, V(t, \bar{X}(t)), V_x(t, \bar{X}(t))\sigma(t, \bar{X}(t), \bar{u}(t)), \bar{u}(t)] \cdot V_x(t, \bar{X}(t)) \\
&+ g_z[t, x, V(t, \bar{X}(t)), V_x(t, \bar{X}(t))\sigma(t, \bar{X}(t), \bar{u}(t)), \bar{u}(t)] \cdot V_{xx}(t, \bar{X}(t))\sigma(t, \bar{X}(t), \bar{u}(t)) \\
&+ g_z[t, x, V(t, \bar{X}(t)), V_x(t, \bar{X}(t))\sigma(t, \bar{X}(t), \bar{u}(t)), \bar{u}(t)] \cdot V_x(t, \bar{X}(t))\sigma_x(t, \bar{X}(t), \bar{u}(t)) \\
&+ V_{xx}(t, \bar{X}(t))b(t, \bar{X}(t), \bar{u}(t)) + V_x(t, \bar{X}(t))b_x(t, \bar{X}(t), \bar{u}(t)) \\
&+ \frac{1}{2}V_{xxx}(t, \bar{X}(t))\sigma^2(t, \bar{X}(t), \bar{u}(t)) + V_{xx}(t, \bar{X}(t))\sigma(t, \bar{X}(t), \bar{u}(t))\sigma_x(t, \bar{X}(t), \bar{u}(t))
\end{aligned}$$

Applying Ito's lemma on $[V_x(t, \bar{X}(t)) \cdot \lambda(t)]$ and the previous equation, we have

$$\left\{ \begin{aligned}
d[V_x(t, \bar{X}(t)) \cdot \lambda(t)] &= \{\bar{V}_{tx}(t) + \bar{V}_{xx}(t) \cdot \bar{b}(t) + \frac{1}{2}\bar{V}_{xxx}(t) \cdot \bar{\sigma}^2(t) + \bar{V}_x(t) \cdot \bar{g}_y(t) + \bar{V}_{xx}(t) \cdot \bar{g}_z(t) \cdot \bar{\sigma}(t)\} \lambda(t) dt \\
&+ \{\bar{V}_{xx}(t) \cdot \sigma(t) \cdot \lambda(t) + \bar{V}_x(t) \cdot \frac{\partial}{\partial z} \bar{H}(t)\} dB_t \\
&= -\{\bar{g}_x(t) \cdot \lambda(t) + \bar{b}_x \cdot [\bar{V}_x \cdot \lambda(t)] + \bar{\sigma}_x(t) \cdot [V_{xx}(t) \cdot \lambda(t) + \bar{V}_x(t) \cdot \frac{\partial}{\partial z} \bar{H}(t)]\} dt \\
&+ [V_{xx}(t) \cdot \lambda(t) + \bar{V}_x(t) \cdot \frac{\partial}{\partial z} \bar{H}(t)] dB(t) \\
V_x(T, \bar{X}(T)) \cdot \lambda(T) &= \lambda(T) \cdot l_x(X(T)) = p(T)
\end{aligned} \right.$$

By the uniqueness of the solutions to (6), we obtain the relationship (7).

4 An Example

Consider the asset allocation problem with control the proportion of wealth process $W(t)$ invested in risky asset $\pi(t)$ and consumption $c(t)$. Formulate the problem as follows

Dynamic of wealth process

$$\begin{cases} dW(t) = [r + \pi(t)(\mu - r) - c(t)]W(t)dt + \pi(t)W(t)\sigma dB(t) \\ W(0) = W_0 \end{cases} \quad (9)$$

Recursive utility process

$$\begin{cases} dY(t) = -[U(c(t)) - \beta Y(t)]dt + Z(t)dB(t) \\ Y(T) = U(W(T)) \end{cases} \quad (10)$$

Objective function

$$J(\pi(\cdot), c(\cdot)) = Y(0) \quad (11)$$

Here r , μ , σ and β are constants, and utility function $U(x) = \frac{x^{1-\gamma}}{1-\gamma}$.

Hamiltonian

$$H(t, w, y, z, \pi, c) = [U(c) - \beta y]\lambda(t) + \{[r + \pi(\mu - r)]w\}p(t) + \pi w \sigma q(t) \quad (12)$$

Discount factor

$$\begin{cases} d\lambda(t) = -\beta\lambda(t)dt \\ \lambda(0) = 1 \end{cases} \quad (13)$$

Time-t shadow price

$$\begin{cases} dp(t) = -\{[r + \bar{\pi}(t)(\mu - r)]p(t) + \bar{\pi}\sigma q(t)\}dt + q(t)dB(t) \\ p(T) = \lambda(T)U'(\bar{W}(T)) = e^{-\beta T}\bar{W}(T)^{-\gamma} \end{cases} \quad (14)$$

HJB equation

$$\begin{cases} 0 = V_t(t, w) - \beta V(t, w) + \max_{\pi, c}\{U(c) + [r + \pi(\mu - r) - c]wV_w(t, w) + \frac{1}{2}\pi^2 w^2 \sigma^2 V_{ww}(t, w)\} \\ V(T, w) = U(w) \end{cases} \quad (15)$$

Solution to (15)

$$\begin{cases} V(t, W(t)) = \frac{W(t)^{1-\gamma}}{1-\gamma} \\ \bar{\pi}(t) = \frac{\mu-r}{\gamma\sigma^2} \\ \bar{c}(t, W(t)) = \bar{W}(t) \end{cases} \quad (16)$$

Following relationship (7), we denote

$$\begin{cases} p^*(t) =: V_w(t, \bar{W}(t))\lambda(t) = \bar{W}(t)^{-\gamma}e^{-\beta t} \\ q^*(t) =: V_{ww}(t, \bar{W}(t)) \cdot \sigma(t, \bar{W}(t), \bar{\pi}(t)) \cdot \lambda(t) = -\gamma\bar{\pi}(t)\sigma p^*(t) \end{cases} \quad (17)$$

Plugging in (16), we can verify that $p^*(t)$ and $q^*(t)$ satisfy equation system (14). By the uniqueness of (13), we have $p^*(\cdot) = p(\cdot)$, $q^*(\cdot) = q(\cdot)$, which actually verifies the relationship (7).

References

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