The basics of "Dixit-Stiglitz lite"*

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9 June 2009

This document derives some of the most basic "Dixit-Stiglitz lite" equations step-by-step.[1] It is aimed at students encountering this demand system for the very first time, in hopes that it will ease their journey into homework assignments and journal articles that feature a "near-impenetrable soup of CES algebra" (Neary, 2001). Those seeking a less rudimentary introduction should consult the appendix of Baldwin, Forslid, Martin, Ottaviano, and Robert-Nicoud (2005).

1 Consumers

1.1 Preferences

The representative consumer's utility function is

$$U = \left(\int_0^n q(\omega)^\rho d\omega \right)^{\frac{1}{\rho}} \qquad 0 < \rho < 1 \tag{1}$$

where $q(\omega)$ is consumption of variety ω, n is the mass of varieties available to consumers, and ρ is a measure of substitutability. The consumer has "taste for variety" in that he or she prefers to consume a diversified bundle of goods. More details on this below.

*None of this material is original. I thank Peter Neary and Tony Venables for their lectures on this demand system and Enda Hargaden, Greg Thompson, and Assaf Zimring for feedback on previous versions of this document. Any errors are mine.

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[1] The popular "lite" specification imposes all three restrictions considered in Dixit and Stiglitz (1977). See Neary (2004), who coined the phrase. Note that this document omits the separable numeraire good x_0 and Cobb-Douglas upper tier ($u = x_0^{1-\mu} U^\mu$) for clarity of exposition. Many macro and trade models use preferences of the form of equation (1).

1.2 Demands

The consumer's constrained maximization problem may be solved by the Lagrangian $\mathcal{L} = U^\rho - \lambda(\int_0^n p(\omega)q(\omega)d\omega$ - I).[2] Take first derivatives.[3]

$$\frac{\partial \mathcal{L}}{\partial q(\omega)} = \rho q(\omega)^{\rho-1} - \lambda p(\omega) = 0 \tag{2}$$

Rearranging terms yields the Frisch demand function:

$$q(\omega) = \Big(\frac{\lambda p(\omega)}{\rho}\Big)^{\frac{1}{\rho-1}} \tag{3}$$

Taking the ratio of Frisch demands for two varieties ω_1 and ω_2 yields relative demand:

$$\frac{q(\omega_1)}{q(\omega_2)} = \Big(\frac{p(\omega_1)}{p(\omega_2)}\Big)^{\frac{1}{\rho-1}} \tag{4}$$

Relative demand will give us Marshallian demand functions, after a bit of manipulation. At this stage, it will be useful to introduce $\sigma \equiv \frac{1}{1-\rho}$ in order to keep notation concise. From (4), it is evident that the elasticity of substitution is the constant $\sigma = \frac{-d \ln q(\omega_1)/q(\omega_2)}{d \ln p(\omega_1)/p(\omega_2)}$, hence this is a CES demand function. Using σ and multiplying both sides by $q(\omega_2)$ yields:

$$q(\omega_1) = q(\omega_2)\Big(\frac{p(\omega_1)}{p(\omega_2)}\Big)^{-\sigma}$$

Now multiply both sides by $p(\omega_1)$ and take the integral with respect to ω_1.

$$\int_0^n p(\omega_1)q(\omega_1)d\omega_1 = \int_0^n q(\omega_2)p(\omega_1)^{1-\sigma}p(\omega_2)^\sigma d\omega_1$$

The left-hand side is the consumer's total expenditure on all varieties – the consumer's income.

$$I = q(\omega_2)p(\omega_2)^\sigma \int_0^n p(\omega_1)^{1-\sigma} d\omega_1$$

[2]It's easier to take derivatives of U^ρ, which is a strictly increasing transformation of the utility function and therefore yields the same optimization solutions.

[3]The consumer chooses $q(\omega)$ for each variety ω, so there is a continuum of first order conditions. Equation (2) describes each of them.

To obtain Marshallian demand for ω_2 in terms of prices and income, divide by $p(\omega_2)^\sigma \int_0^n p(\omega_1)^{1-\sigma} d\omega_1$:

$$q(\omega_2) = \frac{I p(\omega_2)^{-\sigma}}{\int_0^n p(\omega_1)^{1-\sigma} d\omega_1}$$

1.3 A price index

If we define an index of all varieties' prices to be $P \equiv \left(\int_0^n p(\omega)^{1-\sigma} d\omega \right)^{\frac{1}{1-\sigma}}$, then Marshallian demand is

$$q(\omega) = p(\omega)^{-\sigma} P^{\sigma-1} I = \left(\frac{p(\omega)}{P} \right)^{-\sigma} \frac{I}{P} \tag{5}$$

P is thus the true cost of living index, such that the expenditure function is $e(P, u) = Pu$, as can be seen by plugging our consumption solution from (5) into (1) while recalling that $\rho = \frac{\sigma-1}{\sigma}$.

$$\begin{aligned}
U &= \left(\int_0^n q(\omega)^\rho d\omega \right)^{\frac{1}{\rho}} \\
&= \left(\int_0^n p(\omega)^{1-\sigma} I^\rho P^{(\sigma-1)\rho} d\omega \right)^{\frac{1}{\rho}} \\
&= I P^{\sigma-1} \left(\int_0^n p(\omega)^{1-\sigma} d\omega \right)^{\frac{\sigma}{\sigma-1}} \\
&= I P^{\sigma-1} P^{-\sigma} \\
&= \frac{I}{P}
\end{aligned}$$

1.4 Taste for variety

Consumers prefer to diversify their consumption. Suppose that all varieties have the same price p and are therefore consumed in equal amounts q, so that $I = \int_0^n pq \, d\omega$. Then $I = npq$ and $q = \frac{I}{np}$. Therefore, recalling (1):

$$U = \left(\int_0^n q(\omega)^\rho d\omega \right)^{\frac{1}{\rho}} = \left(n \left(\frac{I}{np} \right)^\rho \right)^{\frac{1}{\rho}} = n^{\frac{1-\rho}{\rho}} I/p = n^{\frac{1}{\sigma-1}} I/p$$

where the last equality involves using $\sigma = \frac{1}{1-\rho}$ and $\rho = \frac{\sigma-1}{\sigma}$. You can see that utility is increasing in n and moreso the lower the value of σ.

2 Firms

2.1 Production

The Dixit-Stiglitz demand system is popular because it provides a tractable means of introducing monopolistic competition and increasing returns. The simplest means of introducing increasing returns is to assume that the production of a good involves a fixed cost in addition to a constant marginal cost, so that the average cost is decreasing in quantity. Rather than writing the production function, we write the labor demand function:

$$l(q) = f + cq \tag{6}$$

where l is labor demanded, f is the fixed cost of production, and c is the constant marginal cost.

It is assumed that there no economies of scope, so there is no reason for a firm to produce multiple varieties. Since consumers have an unbounded taste for variety, every firm will produce a distinct variety rather than producing another firm's type and losing profits to competition. The result is one variety per firm and one firm per variety. Thus, the distinction between the number of firms and the number of varieties collapses in our discussion of those topics.

2.2 Pricing

A firm's profits are:

$$\pi = pq - wcq - wf \tag{7}$$

A firm sets the price of its variety to maximise profits:

$$\begin{aligned} \frac{\partial \pi}{\partial p} &= q + (p - wc)\frac{\partial q}{\partial p} = 0 \\ p &= wc + \frac{-q}{\frac{\partial q}{\partial p}} \end{aligned} \tag{8}$$

Use Marshallian demand (5) to calculate $\frac{\partial q}{\partial p}$, noting that the firm's choice of p does not affect the price index P since there is a continuum of firms:

$$\frac{\partial q}{\partial p} = -\sigma p^{-\sigma-1} P^{\sigma-1} I \tag{9}$$

$$\frac{-q}{\frac{\partial q}{\partial p}} = \frac{-p^{-\sigma} P^{\sigma-1} I}{-\sigma p^{-\sigma-1} P^{\sigma-1} I} = \frac{p}{\sigma} \tag{10}$$

Plugging (10) into (8) and recalling $\rho = \frac{\sigma-1}{\sigma}$:

$$\begin{aligned} p &= wc + \frac{p}{\sigma} \\ p\Big(\frac{\sigma-1}{\sigma}\Big) &= wc \\ p &= \frac{wc}{\rho} \end{aligned} \tag{11}$$

The optimal pricing strategy is a proportional mark-up over cost that is independent of other firms' pricing strategies.

2.3 Free entry equilibrium

From the consumer's perspective the number of available varieties n is exogeneous, but we might expect more firms to enter if incumbents are earning positive profits. The free entry condition is zero profits, so that

$$\begin{aligned} \pi &= pq - wcq - wf \\ &= qwc\Big(\frac{1}{\rho} - 1\Big) - wf = 0 \\ q &= \frac{f}{c}(\sigma - 1) \end{aligned} \tag{12}$$

Firms enter the market until they are of sufficient number that no one earns a profit. Note that the scale of firms under free entry is determined solely by the cost structure (f and c) and the elasticity of substitution σ.

3 What's next?

This document has provided step-by-step algebra to get you through the first one or two homework questions you might face on this material. There's a whole world of Dixit-Stiglitz out there – and a whole book about it (Brakman and Heijdra, 2004)! Good luck.

References

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