The Cutoff Structure of Top Trading Cycles in School Choice*

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Abstract

The prominent Top Trading Cycles (TTC) mechanism has attractive properties for school choice, as it is strategy-proof, Pareto efficient, and allows school boards to guide the assignment by specifying priorities. However, the common combinatorial description of TTC does little to explain the relationship between student priorities and their eventual assignment. This creates difficulties in transparently communicating TTC to parents and in guiding policy choices of school boards.

We show that the TTC assignment can be described by \( n^2 \) admission thresholds, where \( n \) is the number of schools. These thresholds can be observed after mechanism is run, and can serve as non-personalized prices that allow parents to verify their assignment.

In a continuum model these thresholds can be computed directly from the distribution of preferences and priorities, providing a framework that can be used to evaluate policy choices. We provide closed form solutions for the assignment under a family of distributions, and derive comparative statics. As an application of the model we solve for the welfare maximizing investment in school quality, and find that a more egalitarian investment can be more efficient because it promotes more efficient sorting by students.

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1 Introduction

Since Abdulkadiroğlu and Sönmez (2003) formulated school choice as an assignment problem, many school districts have redesigned their school assignment mechanisms to give students more choice over their school assignment. Abdulkadiroğlu and Sönmez propose two mechanisms that are strategyproof for students and allow the school district to set a priority structure: the Deferred Acceptance (DA) mechanism (Gale and Shapley, 1962); and the Top Trading Cycles (TTC) mechanism (Shapley and Scarf (1974), attributed to David Gale). The mechanisms differ in that DA is stable but not necessarily efficient, and TTC is Pareto efficient for students but not necessarily stable. In many districts schools do not screen students and there is no need to sacrifice efficiency to guarantee stability\(^1\). Despite this, almost every district that redesigned their school choice mechanism chose to implement the DA mechanism (Pathak and Sönmez, 2013; IIPSC, 2017), instead of the more efficient TTC.\(^2\)

One of the main barriers to using TTC in practice is the difficulty designers had in communicating it to parents and school boards.\(^3\) The standard explanation of TTC is an algorithmic description in terms of sequentially clearing trade cycles, from which it is not directly apparent how priorities and preferences determine assignment. This makes it difficult for school boards to evaluate the effects of policy decisions on the TTC assignment and resulting welfare. It is also difficult for students to verify they were correctly assigned and be convinced that the mechanism is strategyproof.

\(^1\)There is no strategic concern that blocking pairs will match to each other outside the mechanism. This is because of two differences between school choice and two-sided settings like the medical match. First, in Boston and other districts schools cannot directly admit students without approval. Second, priorities are often determined by school zone, sibling status and lotteries, and are not controlled by the school, and so schools do not necessarily prefer students with higher priority. A notable exception is the NYC high school admissions system, see Abdulkadiroğlu, Pathak, and Roth (2009).

\(^2\)To the authors’ knowledge, the only instances of implementation of TTC in school choice systems are in the San Francisco school district (Abdulkadiroğlu et al. 2017) and previously in the New Orleans Recovery School District (Abdulkadiroğlu et al. 2017).

\(^3\)Pathak (2016) writes that:

“I believe that the difficulty of explaining TTC, together with the precedent set by New York and Boston’s choice of DA, are more likely explanations for why TTC is not used in more districts, rather than the fact that it allows for justified envy, while DA does not.”

In addition, Boston and NYC were early school redesigns that set a precedent in favor of DA. More details can be found in the discussion in Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005) and in Pathak (2016), Abdulkadiroğlu, Che, Pathak, Roth, and Tercieux (2017).
We explain the TTC assignment using \( n^2 \) cutoffs \( \{ p_b^c \} \). These cutoffs parallel prices in competitive equilibrium, where students’ priorities serve the role of endowments. Students can use priority at a school \( b \) to gain assignment for school \( c \) if their priority at school \( b \) is above the threshold \( p_b^c \). Publicly publishing the cutoffs \( \{ p_b^c \} \) lets each student verify that she is assigned to her most preferred school for which she has sufficient priority.\(^4\) To help convey to students that TTC is strategyproof, we present cutoffs \( \{ p_b^c \} \) that do not change with the preferences of any individual student.

To understand the mapping from the economy to the cutoffs and TTC assignment, we formulate TTC in a continuum model. We show that the TTC assignment in the continuum can be directly calculated by solving a system of equations, and give closed form solutions for parameterized economies. This framework allows us to derive comparative statics, and evaluate welfare under various policy choices. We validate our continuum TTC model by demonstrating that it produces cutoffs for discrete economies that describe the discrete TTC assignment.

The continuum TTC model allows us to calculate welfare. For example, the cutoff representation yields budget sets for each student, providing a tractable expression for welfare under random utility models. The model also provides comparative statics for assignment and welfare with respect to changes in school attributes and student preferences. For example, consider investment in a school’s quality that increases the utility for students who attend that school. Such an investment would change student preferences and therefore the TTC assignment. We decompose the marginal effect of such investment on welfare into the direct increase in utility of students assigned to the school, and the indirect effect that arises from changes in the assignment.

As an application of our model, we study optimal investment in school quality when all students prefer higher quality school but have unobserved taste shocks. We solve for the optimal investment under TTC for a parametric setting, and find that the optimal investment is equitable in the sense of making all schools equally over-demanded. A more equitable investment is more efficient because it allows students more choice, yielding better sorting on horizontal dimensions and therefore

\(^4\)This threshold representation allows us to give the following non-combinatorial description of TTC. For each school \( b \), each student receives \( b \)-tokens according to their priority at school \( b \), where students with higher \( b \)-priority receive more \( b \)-tokens. The TTC algorithm publishes prices \( \{ p_b^c \} \). Students can purchase a single school using a single kind of token, and the required number of \( b \)-tokens to purchase school \( c \) is \( p_b^c \). Note that \( \{ p_b^c \} \) can be obtained by running TTC and setting \( p_b^c \) to be the number of \( b \)-tokens of the lowest \( b \)-priority student that traded into \( c \) using priority at \( b \). We thank Chiara Margaria, Laura Doval and Larry Samuelson for suggesting this explanation.
higher welfare. This holds even if the schools are not symmetric, as the benefits from more efficient sorting can outweigh benefits from targeting schools with more efficient investment opportunities.

Second, we explore the design of priorities for TTC. The priority structure under TTC is “bossy” in the sense that a change in the relative priority among top priority students can change the cutoffs and the assignment of low priority students, without changing the assignment of any high priority student. Such changes to the relative priority among top students cannot be detected by supply-demand equations as in Azevedo and Leshno (2016), and therefore it is not possible to determine the TTC thresholds directly through a supply-demand equation. We characterize the range of possible assignments generated by TTC after changes to relative priority of high-priority students, and show that a small change to the priorities will only change the assignment of a few students.

A third application of our model is to provide comparisons between mechanisms in terms of assignments and welfare. We solve for welfare under TTC and DA in a particular setting and quantify how much welfare is sacrificed to guarantee stability. We compare TTC and DA across different school choice environments and corroborate a conjecture by Pathak (2016) that the difference between the mechanisms is smaller when students have a preference for and priority at their neighborhood school. We also compare TTC to the Clinch and Trade mechanism by Morrill (2015b) in large economies and find that it is possible for TTC to produce fewer blocking pairs than the Clinch and Trade mechanism.

A key idea that allows us to define TTC in the continuum is that the TTC algorithm can be characterized by its aggregate behavior over many cycles. Any collection of cycles must maintain trade balance, that is, the number of students assigned to each school is equal to the number of students who claimed or traded a seat at that school. In the continuum this necessary condition yields a system of equations that fully characterizes TTC. These equations also provide a recipe for calculating the TTC assignment.

A few technical aspects of the analysis may be of interest. First, we note that the trade balance equations circumvent many of the measure theoretic complications in defining TTC in the continuum. Second, a connection to Markov chain theory allows us to show that a solution to the marginal trade balance equations always exists, and to characterize the possible trades.
1.1 Related Literature

Abdulkadiroğlu and Sönmez (2003) first introduced school choice as a mechanism design problem and suggested the TTC mechanism as a solution with several desirable properties. Since then, TTC has been considered for use in a number of school choice systems. Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005) discuss how the city of Boston debated between using DA or TTC for their school choice systems and ultimately chose DA. Abdulkadiroğlu, Pathak, and Roth (2009) compare the outcomes of DA and TTC for the NYC public school system, and show that TTC gives higher student welfare. Kesten (2006) also studies the relationship between DA and TTC, and shows that they are equivalent mechanisms if and only if the priority structure is acyclic.

Our work contributes to the literature on threshold representations, which has been instrumental for empirical work on DA and variants of DA. Abdulkadiroğlu, Angrist, Narita, and Pathak (Forthcoming) use admission thresholds to construct propensity score estimates. Agarwal and Somaini (2014); Kapor, Neilson, and Zimmerman (2016) structurally estimate preferences from rank lists submitted to non-strategyproof variants of DA. Both build on the threshold representation of Azevedo and Leshno (2016). We hope that our threshold representation of TTC will be similarly useful for future empirical work on TTC.

Our finding that the TTC assignment can be represented in terms of cutoffs parallels the role of prices in competitive markets. Dur and Morrill (2016) show that the outcome of TTC can be expressed as the outcome of a competitive market where there is a price for each priority position at each school, and agents may buy and sell exactly one priority position. Their characterization provides a connection between TTC and competitive markets, but requires a price for each rank at each school and does not provide a method for directly calculating these prices without running TTC. He, Miralles, Pycia, Yan, et al. (2015) propose an alternative pseudo-market approach for discrete assignment problems that extends Hylland and Zeckhauser (1979) and also uses admission thresholds. Miralles and Pycia (2014) show a second welfare theorem for discrete goods, namely that any Pareto efficient assignment of discrete goods without transfers can be decentralized through prices and endowments, but requires an arbitrary endowment structure.

Our approach builds on existing axiomatic characterizations of TTC, and may extend to the class of Pareto efficient and strategyproofness mechanisms. Abdulka-
diroğlu, Che, Pathak, Roth, and Tercieux (2017) show that TTC minimizes the number of blocking pairs subject to strategyproofness and Pareto efficiency. Additional axiomatic characterizations of TTC were given by Dur (2012) and Morrill (2013, 2015a). These characterizations explore the properties of TTC, but do not provide another method for calculating the TTC outcome or evaluating welfare. Ma (1994), Pápai (2000) and Pycia and Ünver (2015) give characterizations of more general classes of Pareto efficient and strategy-proof mechanisms in terms of clearing trade cycles. While our analysis focuses on the TTC mechanism, we believe that our trade balance approach will be useful in analyzing these general classes of mechanisms.

Several variants of TTC have been suggested in the literature. Morrill (2015b) introduces the Clinch and Trade mechanism, which differs from TTC in that it identifies students who are guaranteed admission to their first choice and assigns them immediately without implementing a trade. Hakimov and Kesten (2014) introduce Equitable TTC, a variation on TTC that aims to reduce inequity. In Section 4.2 we use our model to analyze such variants of TTC and compare their assignments. Other variants of TTC can also arise from the choice of tie-breaking rules. Ehlers (2014) shows that any fixed tie-breaking rule satisfies weak efficiency, and Alcalde-Unzu and Molis (2011), Jaramillo and Manjunath (2012) and Saban and Sethuraman (2013) give specific variants of TTC that are strategy-proof and efficient. The continuum model allows us to characterize the possible outcomes from different tie-breaking rules.

Several papers also study TTC in large markets. Che and Tercieux (2015a,b) study the properties of TTC in a large market where the number of items grows as the market gets large. Hatfield, Kojima, and Narita (2016) study the incentives for schools to improve their quality under TTC and find that a school may be assigned some less preferred students when it improves its quality.

This paper contributes to a growing literature that uses continuum models in market design (Avery and Levin, 2010; Abdulkadirügli, Che, and Yasuda, 2015; Ashlagi and Shi, 2015; Che, Kim, and Kojima, 2013; Azevedo and Hatfield, 2015). Our description of the continuum economy uses the setup of Azevedo and Leshno (2016), who characterize stable matchings in terms of cutoffs that satisfy a supply and demand equation. Our results from Section 4.2 imply that the TTC cutoffs depend on the entire distribution and cannot be computed from simple supply and demand equations.
1.2 Organization of the Paper

Section 2 provides the standard description of the TTC mechanism in the discrete model and presents our cutoff characterization. Section 3 presents the continuum TTC model and provides our main results that allow for direct calculation of the TTC cutoffs. Section 4 explores several applications: quantifying the effects of improving school quality under school choice and solving for optimal investment, showing the "bossiness" of the TTC priorities, and comparing TTC with other mechanisms. Section A provides the intuition for the continuum TTC model. Omitted proofs can be found in the appendix.

2 TTC in School Choice

2.1 The Discrete TTC Model

In this section, we describe the standard model for the TTC mechanism in the school choice literature, and outline some of the properties of TTC in this setting.

Let $S$ be a finite set of students, and let $C$ be a finite set of schools. Each school $c \in C$ has a finite capacity $q_c > 0$. Each student $s \in S$ has a strict preference ordering $\succ^s$ over schools. Let $Ch^s(C) = \arg \max_{\succ^s} \{C\}$ denote $s$'s most preferred school out of the set $C$. Each school $c \in C$ has a strict priority ordering $\succ_c$ over students. To simplify notation, we assume that all students and schools are acceptable, and that there are more students than available seats at schools. It will be convenient to represent the priority of student $s$ at school $c$ by the student’s percentile rank $r_c^s = |\{s' \mid s \succ^s c s'\}|/|S|$ in the school’s priority ordering. Note that for any two students $s, s'$ and school $c$ we have that $s \succ^c s' \iff r_c^s > r_c^{s'}$ and that $0 \leq r_c^s < 1$.

A feasible assignment is $\mu : S \rightarrow C \cup \{\emptyset\}$ where $|\mu^{-1}(c)| \leq q_c$ for every $c \in C$. If $\mu(s) = c$ we say that $s$ is assigned to $c$, and we use $\mu(s) = \emptyset$ to denote that the student $s$ is unassigned. As there is no ambiguity, we let $\mu(c)$ denote the set $\mu^{-1}(c)$ for $c \in C \cup \{\emptyset\}$. A discrete economy is $E = (C, S, \succ^S, \succ_c, q)$, where $C$ is the set of schools, $S$ is the set of students, $q = \{q_c\}_{c \in C}$ is the capacity of each school, and $\succ^S = \{\succ^s\}_{s \in S}$, $\succ_c = \{\succ_c\}_{c \in C}$.

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5This is without loss of generality, as we can introduce auxiliary students and schools that represent being unmatched.
The discrete Top Trading Cycles algorithm (TTC) calculates an assignment $\mu_{\text{TTC}}$. The algorithm runs in discrete steps as follows.

**Algorithm 1** (Top Trading Cycles). Initialize unassigned students $S = S$, available schools $C = C$, capacities $\{q_c\}_{c \in C}$.

While there are still unassigned students and available schools:

- Each available school $c \in C$ tentatively offers a seat by pointing to its highest priority remaining student.
- Each student $s \in S$ that was tentatively offered a seat points to her most preferred remaining school.
- Select at least one trading cycle, that is, a list of students $s_1, \ldots, s_\ell$, $s_{\ell+1} = s_1$ such that $s_i$ points to the school pointing to $s_{i+1}$ for all $i$, or equivalently $s_{i+1}$ was offered a seat at $s_i$’s most preferred school. Assign all students in the cycles to the school they point to.\(^6\)
- Remove the assigned students from $S$, reduce the capacity of the schools they are assigned to by 1, and remove schools with no remaining capacity from $C$.

TTC satisfies a number of desirable properties. An assignment $\mu$ is Pareto efficient for students if no group of students can improve by swapping their allocations, and no individual student can improve by swapping her assignment for an unassigned object. A mechanism is Pareto efficient for students if it always produces an assignment that is Pareto efficient for students. A mechanism is strategy-proof for students if reporting preferences truthfully is a dominant strategy. It is well known that TTC, as used in the school choice setting, is both Pareto efficient and strategy-proof for students (Abdulkadiroğlu and Sönmez, 2003). Moreover, when type-specific quotas must be imposed, TTC can be easily modified to meet quotas while still maintaining constrained Pareto efficiency and strategy-proofness (Abdulkadiroğlu and Sönmez, 2003).

### 2.2 Cutoff Characterization

Our first main contribution is that the TTC assignment can be described in terms of $n^2$ cutoffs $\{p_{ij}^c\}$, one for each pair of schools.

\(^6\)Such a trading cycle must exist, since every vertex in the pointing graph with vertex set $S \cup C$ has out-degree 1.
Theorem 1. The TTC assignment is given by

$$\mu_{\text{TTC}}(s) = \max_{c} \{ c \mid r_b^s \geq p_b^c \text{ for some } b \} ,$$

where $p_b^c$ is the percentile in school $b$’s ranking of the worst ranked student at school $b$ that traded a seat at school $b$ for a seat at school $c$. If no such student exists, $p_b^c = 1$.

Cutoffs serve a parallel role to prices, with each student’s priority at each school serving the role of endowments. The cutoffs $p = \{p_b^c\}_{b,c}$ combine with each student $s$’s priorities $r^s$ to give $s$ a budget set $B(s,p) = \{ c \mid r_b^s \geq p_b^c \text{ for some } b \}$ of schools she can attend. TTC assigns each student to her favorite school in her budget set.

Theorem 1 provides an intuitive way for students to verify that they were correctly assigned by the TTC algorithm. The cutoff $p_b^c$ can be easily identified after the mechanism has been run. Instead of only communicating the assignment of each student, the mechanism can make the cutoffs publicly known. Students can calculate their budget set from their privately known priorities and the publicly given cutoffs, allowing them to verify that they were indeed assigned to their most preferred school in their budget set. In particular, if a student does not receive a seat at a desired school $c$, it is because she does not have sufficiently high priority at any school, and so $c$ is not in her budget set. We illustrate these ideas in Example 1.

Example 1. Consider a simple economy where there are two schools each with capacity $q = 120$, and a total of 300 students, $2/3$ of whom prefer school 1. Figure 1(a) illustrates the preferences and priorities of each of the students. A student’s priority determines a location in the square, with the horizontal axis indicating priority at school 1 and the vertical axis indicating priority at school 2.

The cutoffs $p$ and resulting budget sets $B(s,p)$ for each student are illustrated in Figure 1(b); the shaded areas show budget sets as a function of student priority. For example, a student has the budget set $\{1,2\}$ if she has sufficiently high priority at either school 1 or school 2. Note that students’ preferences are not indicated in Figure 1(b) as each student’s budget set is independent of her preferences. Figure 1(c) depicts the students’ assignments, which depend on their preferences. The plot has two squares: the left square gives the assignment of students who prefer school

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7Student priorities were selected such that there is little correlation between student priority at either school and between student priorities and preferences.
1; and the right square gives the assignment of students who prefer school 2. The assignment of each student is simply her favorite school in her budget set.

(a) An economy $E$ with 300 students.  
(b) Budget sets for the economy $E$.

Figure 1: The TTC budget sets and assignment for the economy $E$ in Example 1. All students find both schools acceptable, and students are labeled by their preferred school. Students’ priority ranks at school 1 are given by the horizontal axis and priority ranks at school 2 by the vertical axis. The budget sets along the axis for school $c$ list $B_c(s,p)$, the schools that enter a student’s budget set because of her rank at school $c$.

(c) TTC assignment for the economy $E$.

Figure 1 shows the role of priorities in determining the TTC assignment in Example 1. Students with higher priority have a larger budget set of school from which they can choose. A student can choose her desired school if her priority for some school is sufficiently high. Priority for each school is considered separately, and priority from multiple schools cannot be combined. For example, a student who has top priority for one school and bottom priority at the other school is assigned to her top
choice, but a student who has the median priority at both schools will not be assigned to school 1.

Remark. This example also shows that the TTC assignment cannot be expressed in terms of one cutoff for each school, as the assignment in Example 1 cannot be described by fewer than 3 cutoffs.

2.3 The Structure of TTC Budget Sets

The cutoff structure for TTC allows us to provide some insight into the structure of the assignment. For each student \( s \), let \( B_b(s,p) = \{ c \mid r_b^c \geq p_b^c \} \) denote the set of schools that enter student \( s \)'s budget set because of her priority at school \( b \). \( B_1(s,p) \) and \( B_2(s,p) \) are depicted along the \( x \) and \( y \) axes of Figure 1. Note that \( B_b(s,p) \) depends only on the \( n \) cutoffs \( p_b = \{ p_b^c \}_{c \in C} \). A student's budget set is the union \( B(s,p) = \cup_b B_b(s,p) \). The following proposition shows that budget sets \( B_b(s,p) \) can be given by cutoffs \( p_b \) that share the same ordering over schools for every \( b \). We let \( C^{(c)} = \{ c, c+1, \ldots, n \} \) denote the set of schools that have a higher index than \( c \).

**Proposition 1.** There exist cutoffs \( \{ p_b^c \} \) and a relabeling of school indices such that the cutoffs \( p \) describe the TTC assignment, 

\[
\mu_{\text{TTC}}(s) = \max_{\succ} \{ c \mid r_b^c \geq p_b^c \text{ for some } b \},
\]

and for any school \( b \) the cutoffs are ordered,\(^8\)

\[
p_b^1 \geq p_b^2 \geq \cdots \geq p_b^b = p_b^{b+1} = \cdots = p_b^n.
\]

Therefore, the set of schools \( B_b(s,p) \) student \( s \) can afford via her priority at school \( b \) is either the empty set \( \phi \) or

\[
B_b(s,p) = C^{(c)} = \{ c, c+1, \ldots, n \}
\]

for some \( c \leq b \). Moreover, each student's budget set \( B(s,p) = \cup_b B_b(s,p) \) is either \( B(s,p) = \phi \) or \( B(s,p) = C^{(c)} \) for some \( c \).

\(^8\)The cutoffs \( p \) defined in Theorem 1 do not necessarily satisfy this condition. However, the run of TTC produces the following relabeling of schools and cutoffs \( \tilde{p} \) that give the same assignment and satisfy the condition: the schools are relabeled in the order in which they reach capacity under TTC, and the cutoffs \( \tilde{p} \) are given by \( \tilde{p}_b^c = \min_{a \leq c} p_b^a \).
Proposition 1 allows us to give a simple illustration for the TTC assignment when there are \( n \geq 3 \) schools. For each school \( b \), we can illustrate the set of schools \( B_b(s, p) \) that enter a student’s budget set because of her priority at school \( b \) as in Figure 2 (under the assumption that schools are labeled in the correct order). This generalizes the illustration along each axis in Figure 1, and can be used for any number of schools.

It is possible that \( p_c^b = 1 \), meaning that students cannot use their priority at school \( b \) to trade into school \( c \).

2.4 Limitations

Although the cutoff structure is helpful in understanding the structure of the TTC assignment, there are several limitations to the cutoffs computed in Theorem 1 and Proposition 1. First, while the cutoffs can be determined by running the TTC algorithm, Theorem 1 does not provide a direct method for calculating the cutoffs from the economy primitives. In particular, it does not explain how the TTC assignment changes with changes in school priorities or student preferences. Second, the budget set \( B(s, p) \) given by the cutoffs derived in Theorem 1 does not correspond to the set of schools that student \( s \) can be assigned to by reporting some preferences.\(^9\),\(^{10}\)

\(^9\)More precisely, given economy \( E \) and student \( s \), let economy \( E' \) be generated by changing the preferences ordering of \( s \) from \( \succ^s \) to \( \succ' \). Let \( \mu_{\text{TTC}}(s \mid E) \) and \( \mu_{\text{TTC}}(s \mid E') \) be the assignment of \( s \) under the two economies, and let \( p \) be the cutoffs derived by Theorem 1 for economy \( E \). Theorem 1 shows that \( \mu_{\text{TTC}}(s \mid E) = \max_{\succ^s} B(s, p) \) but it may be \( \mu_{\text{TTC}}(s \mid E') \neq \max_{\succ'} B(s, p) \).

\(^{10}\)For example, let \( E \) be an economy with three schools \( C = \{1, 2, 3\} \), each with capacity 1. There are three students \( s_1, s_2, s_3 \) such that the top preference of \( s_1, s_2 \) is school 1, the top preference of
We therefore introduce the continuum model for TTC which allows us to directly calculate the cutoffs, allowing for comparative statics. These cutoffs will also correspond to refined budget sets which provide the sets of schools that students could be assigned to by unilaterally changing their preferences.

3 Continuum Model and Main Results

3.1 Model

We model the school choice problem with a continuum of students and finitely many schools, as in Azevedo and Leshno (2016). There is a finite set of schools denoted by $\mathcal{C} = \{1, \ldots, n\}$, and each school $c \in \mathcal{C}$ has the capacity to admit a mass $q_c > 0$ of students. A student $\theta \in \Theta$ is given by $\theta = (\succ^\theta, r^\theta)$. We let $\succ^\theta$ denote the student’s strict preferences over schools, and let $Ch^\theta(C) = \max_{\succ^\theta} (C)$ denote $\theta$’s most preferred school out of the set $C$. The priorities of schools over students are captured by the vector $r^\theta \in [0, 1]^C$. We say that $r^\theta_b$ is the rank of student $\theta$ at school $b$. Schools prefer students with higher ranks, that is $\theta \succ^\theta b \theta'$ if and only if $r^\theta_b > r^\theta_{b'}$.

**Definition 1.** A continuum economy is given by $\mathcal{E} = (\mathcal{C}, \Theta, \eta, q)$ where $q = \{q_c\}_{c \in \mathcal{C}}$ is the vector of capacities of each school, and $\eta$ is a measure over $\Theta$.

We make some assumptions for the sake of tractability. First, we assume that all students and schools are acceptable. Second, we assume there is an excess of students, that is, $\sum_{c \in \mathcal{C}} q_c < \eta(\Theta)$. Finally, we make the following technical assumption that ensures that the run of TTC in the continuum economy is sufficiently smooth and allows us to avoid some measurability issues.  

$s_3$ is school 3, and student $s_i$ has top priority at school $i$. Theorem 1 gives the budget set $\{1\}$ for student $s_1$, as $p^1 = \left(\frac{2}{3}, 1, 1\right)$, $p^2 = \left(1, \frac{2}{3}, 1\right)$ and $p^3 = \left(1, 1, \frac{2}{3}\right)$, since the only trades are of seats at $c$ for seats at the same school $c$. However, if $s_1$ reports the preference $2 \succ 1 \succ 3$ she will be assigned to school 2, so an appropriate definition of budget sets should include school 2 in the budget set for student $s_1$. Also note that no matter what preference student $s_1$ reports, she will not be assigned to school 3, so an appropriate definition of budget sets should not include school 3 in the budget set for student $s_1$.

We can incorporate an economy where two schools have perfectly aligned priories by considering them as a combined single school in the trade balance equations. The capacity constraints still consider the capacity of each school separately.
Assumption 1. The measure $\eta$ admits a density $\nu$. That is for any measurable subset of students $A \subseteq \Theta$
\[ \eta(A) = \int_A \nu(\theta) d\theta. \]
Furthermore, $\nu$ is piecewise Lipschitz continuous everywhere except on a finite grid,\(^{12}\) bounded from above, and bounded from below away from zero on its support.\(^{13}\)

Assumption 1 is general enough to allow embeddings of discrete economies, and
is satisfied by all the economies considered throughout the paper. However, it is not
without loss of generality, e.g. it is violated when all schools share the same priorities
over students.

An immediate consequence of Assumption 1 is that a school’s indifference curves
are of $\eta$-measure 0. That is, for any $c \in C$, $x \in [0,1]$ we have that $\eta(\{\theta \mid r^\theta_b = x\}) = 0$. This is analogous to schools having strict preferences in the standard discrete
model. Given this assumption, as $r^\theta_b$ carries only ordinal information, we may assume
each student’s rank is normalized to be equal to her percentile rank in the school’s
preferences, i.e. for any $b \in C$, $x \in [0,1]$ we have that $\eta(\{\theta \mid r^\theta_b \leq x\}) = x$.

It is convenient to describe the distribution $\eta$ by the corresponding marginal
distributions. Specifically, for each point $x \in [0,1]^n$ and subset of schools $C \subseteq C$, let
$H^c|_C(x)$ be the marginal density of students who are top ranked at school $b$ among
all students whose rank at every school $a$ is no better than $x_a$, and whose top choice
among the set of schools $C$ is $c$.\(^{14}\) We omit the dependence on $C$ when the relevant
set of schools is clear from context, and write $H^c_b(x)$. The marginal densities $H^c|_C(x)
$ uniquely determine the distribution $\eta$.

As in the discrete model, an assignment is a mapping $\mu : \Theta \to C \cup \{\emptyset\}$ specifying
the assignment of each student. With slight abuse of notation, we let $\mu(c) = 
\{\theta \mid \mu(\theta) = c\}$ denote the set of student assigned to school $c$. An assignment $\mu$ is
feasible if it respects capacities, i.e. for each school $c \in C$ we have $\eta(\mu(c)) \leq q_c$. Two
allocations $\mu$ and $\mu'$ are equivalent if they differ only on a set of students of zero
measure, i.e. $\eta(\{\theta \mid \mu(\theta) \neq \mu'(\theta)\}) = 0$.

\(^{12}\)A grid $G \subset \Theta$ is given by$G = \{\theta \mid \exists c \text{ s.t. } r^\theta_c \in D\}$, where $D = \{d_1,\ldots,d_L\} \subset [0,1]$ is a finite set of grid points. Equivalently, $\nu$ is Lipschitz continuous on the union of open hypercubes $\Theta \setminus G$.

\(^{13}\)That is, there exists $M > m > 0$ such that for every $\theta \in \Theta$ either $\nu(\theta) = 0$ or $m \leq \nu(\theta) \leq M$.

\(^{14}\)Formally $H^c|_C(x) \overset{def}{=} \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \eta(\{\theta \in \Theta \mid r^\theta \in [x - \varepsilon \cdot e^b, x) \text{ and } Ch^\theta(C) = c\})$, where $e^b$ is the unit vector in the direction of coordinate $b$. 

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Remark 1. In school choice, it is common for schools to have coarse priorities, and to refine these using a tie-breaking rule. Our economy $E$ captures the strict priority structure that results after applying the tie-breaking rule.

### 3.2 Main Results

Our main result establishes that in the continuum model the TTC assignment can be directly calculated from trade balance and capacity equations. This allows us to explain how the TTC assignment changes with changes in the underlying economy. It also allows us to derive cutoffs that are independent of a student’s reported preferences, giving another proof that TTC is strategyproof.

We remark that directly translating the TTC algorithm to the continuum setting by considering individual trading cycles is challenging, as a direct adaptation of the algorithm would require the clearing of cycles of zero measure. We circumvent the technical issues raised by such an approach by formally defining the continuum TTC assignment in terms of trade balance and capacity equations, which characterize the TTC algorithm in terms of its aggregate behavior over multiple steps. To verify the validity of our definition, we show in Subsection 3.3 that continuum TTC can be used to calculate the discrete TTC outcome. We provide further intuition in Appendix A.

We begin with some definitions. A function $\gamma(t) : [0, \infty) \to [0, 1]^C$ is a TTC path if $\gamma$ is continuous and piecewise smooth, $\gamma_c(t)$ is weakly decreasing for all $c$, and the initial condition $\gamma(0) = 1$ holds. A function $\bar{\gamma}(t) : [t_0, \infty) \to [0, 1]^C$ is a residual TTC path if it satisfies all the properties of a TTC path except the initial condition, and is defined only for $t \geq t_0 > 0$. For a set $\{t(c)\}_{c \in C} \subseteq \mathbb{R}_{>0}$ of times we let $t(c) \overset{def}{=} \min_c [t(c)]$ denote the minimal time. For a point $x \in [0, 1]^C$, let

$$D^c(x) \overset{def}{=} \eta \left( \{ \theta \mid r^\theta \not< x, Ch^\theta(C) = c \} \right)$$

denote the mass of students who want $c$ among all students with rank better than $x$. We will refer to $D^c(x)$ as the demand for $c$. Recall that $H^c_b(x)$ is the marginal density of students who want $c$ who are top ranked at school $b$ among all students with rank no better than $x$. Note that $D^c(x)$ and $H^c_b(x)$ depend implicitly on the set of available schools $C$.

**Definition 2.** Let $E = (C, \Theta, \eta, q)$ be an economy. We say that the (residual) TTC
path $\gamma(t)$ and positive stopping times $\{ t^{(c)} \}_{c \in C} \in \mathbb{R}^C_+$ satisfy trade balance and capacity for the economy $E$ if the following hold.

1. $\gamma(\cdot)$ satisfies the marginal trade balance equations given by

$$
\sum_{a \in C} \gamma'_a(t) H^c_a(\gamma(t)) = \sum_{a \in C} \gamma'_c(t) H^a_c(\gamma(t))
$$

for all $t \leq t^{(c^*)} = \min_c [ t^{(c)} ]$ for which the derivatives exist.

2. The minimal stopping time $t^{(c^*)}$ solves the capacity equations

$$
D^{c^*}(\gamma(t^{(c^*)})) = q_{c^*} \\
D^a(\gamma(t^{(c^*)})) \leq q_a \quad \forall a \in C
$$

and $\gamma_{c^*}(t)$ is constant for all $t \geq t^{(c^*)}$.

3. Define the residual economy $\tilde{E} = (\tilde{C}, \tilde{\Theta}, \tilde{\eta}, \tilde{q})$ by $\tilde{C} = C \setminus \{ c^* \}$, $\tilde{q}_c = q_c - D^c(\gamma(t^{(c^*)}))$ and $\tilde{\eta}(A) = \eta(A \cap \{ \theta : r^\theta \leq \gamma(t^{(c^*)}) \})$. Define the residual TTC path $\tilde{\gamma}(\cdot)$ by restricting $\gamma(\cdot)$ to $t \geq t^{(c^*)}$ and coordinates within $\tilde{C}$. Then $\tilde{\gamma}$ and the stopping times $\{ t^{(c)} \}_{c \in \tilde{C}}$ satisfy trade balance and capacity for $\tilde{E}$.

**Theorem 2.** Let $E = (C, \Theta, \eta, q)$ be an economy. There exist a TTC path $\gamma(\cdot)$ and stopping times $\{ t^{(c)} \}_{c \in C}$ that satisfy trade balance and capacity for $E$, and the TTC assignment is given by

$$
\mu_{\text{TTC}}(\theta) = \max \{ c : r^\theta_c \geq p^c_b \text{ for some } b \} ,
$$

where the $n^2$ TTC cutoffs $\{ p^c_b \}$ are given by

$$
p^c_b = \gamma_b(t^{(c)}) \quad \forall b, c.
$$

Moreover, any $\gamma(\cdot), \{ t^{(c)} \}_{c \in C}$ that satisfy trade balance and capacity yield the same assignment $\mu_{\text{TTC}}$.

In other words, Theorem 2 provides the following a recipe for calculating the TTC assignment. First, find $\hat{\gamma}(\cdot)$ that solves equation 1 for all $t$. Second, calculate $t^{(c^*)}$ from the capacity equations (2) for $\hat{\gamma}(\cdot)$. Set $\gamma(t) = \hat{\gamma}(t)$ for $t \leq t^{(c^*)}$. To determine
the remainder of \( \gamma (\cdot) \), apply the same steps to the residual economy \( \tilde{E} \) which has one less school.\(^{15}\) This recipe is illustrated in Example 2.

Theorem 2 shows that the cutoffs can be directly calculated from the primitives of the economy. In contrast to the cutoff characterization in the standard model (Theorem 1), this allows us to understand how the TTC assignment changes with changes in capacities, preferences or priorities. We remark that the existence of a smooth curve \( \gamma \) follows from our assumption that \( \eta \) has a density that is piecewise Lipschitz and bounded, and the existence of \( t^{(c)} \) satisfying the capacity equations (2) follows from our assumptions that there are more students than seats and all students find all schools acceptable.

The following immediate corollary of Theorem 2 shows that in contrast with the cutoffs given by the discrete model, the cutoffs given by Theorem 2 always satisfy the cutoff ordering.

**Corollary 1.** Let the schools be labeled such that \( t^{(1)} \leq t^{(2)} \leq \cdots \leq t^{(n)} \). Then the cutoff ordering in Proposition 1 holds, namely

\[
p^{1}_b \geq p^{2}_b \geq \cdots \geq p^{b}_b = p^{b+1}_b = \cdots = p^{[C]}_b \quad \text{for all } b.
\]

To illustrate how Theorem 2 can be used to calculate the TTC assignment and understand how it depends on the parameters of the economy, we consider the following simple economy. This parameterized economy yields a tractable closed form solution for the TTC assignment. For other economies the equations may not necessarily yield tractable expressions, but the same calculations can be be used to numerically solve for cutoffs for any economy satisfying our smoothness requirements.

**Example 2.** We demonstrate how to use Theorem 2 to calculate the TTC assignment for a simple parameterized continuum economy. The economy \( \mathcal{E} \) has two schools 1, 2 with capacities \( q_1 = q_2 = q \) with \( q < 1/2 \). A fraction \( p > 1/2 \) of students prefer school 1, and student priorities are uniformly distributed on \([0, 1]\) independently for each school and independently of preferences. This economy is described by

\[
H(x_1, x_2) = \begin{bmatrix}
px_2 \quad (1-p) x_2 \\
px_1 \quad (1-p) x_1
\end{bmatrix},
\]

\(^{15}\)Continuity of the TTC path provides an initial condition for \( \tilde{\gamma} \), namely that \( \tilde{\gamma}_c \left( t^{(c')} \right) = \gamma_c \left( t^{(c')} \right) \) for all \( c \).
where \( H_c^b (\mathbf{x}) \) is given by the \( b \)-row and \( c \)-column of the matrix. A particular instance of this economy with \( q = 4/10 \) and \( p = 2/3 \) can be viewed as a smoothed continuum version of the economy in Example 1.

We start by solving for \( \gamma \) from the trade balance equations (1), which simplify to the differential equation

\[
\frac{\gamma_2'(t)}{\gamma_1'(t)} = \frac{1 - p}{p} \frac{\gamma_2(t)}{\gamma_1(t)}.
\]

Since \( \gamma(0) = 1 \), this is equivalent to \( \gamma_2(t) = (\gamma_1(t))^{\frac{1}{p} - 1} \). Hence for \( 0 \leq t \leq \min \{t^{(1)}, t^{(2)}\} \) we set

\[
\gamma(t) = \left(1 - t, (1 - t)^{\frac{1}{p} - 1}\right).
\]

We next compute \( t^{(e^*)} = \min \{t^{(1)}, t^{(2)}\} \). Observe that because \( p > 1/2 \) it must be that \( t^{(1)} < t^{(2)} \). Otherwise, we have that \( t^{(2)} = \min \{t^{(1)}, t^{(2)}\} \) and \( D_1(\gamma(t^{(2)})) \leq q \), implying that \( D_2(\gamma(t^{(2)})) = \frac{1-p}{p} D_1(\gamma(t^{(2)})) < q \). Therefore, we solve \( D_1(\gamma(t^{(1)})) = q \) to get that \( t^{(1)} = 1 - \left(\frac{p-2}{p}\right)^p \) and that

\[
p_1^1 = \gamma_1(t^{(1)}) = \left(1 - \frac{q}{p}\right)^p, \quad p_2^1 = \gamma_2(t^{(1)}) = \left(1 - \frac{q}{p}\right)^{1-p}.
\]

For the remaining cutoffs, we eliminate school 1 and reiterate the same steps for the residual economy where \( C' = \{2\} \) and \( q_2' = q_2 - D_2(\gamma(t^{(1)})) = q(2 - 1/p) \).

For the residual economy equations (1) are trivial, and we set

\[
\gamma(t) = \left(1 - t^{(1)}, p_2^2 - (t - t^{(1)})\right)
\]

for \( t^{(1)} \leq t \leq t^{(2)} \). Solving the capacity equation (2) for \( t^{(2)} \) yields that

\[
p_1^2 = \gamma_1(t^{(2)}) = \left(1 - \frac{q}{p}\right)^p, \quad p_2^2 = \gamma_2(t^{(2)}) = (1 + 2q) \left(1 - \frac{q}{p}\right)^{-p}.
\]

The original trade balance equations are

\[
\begin{align*}
\gamma_1'(t) p_2 \gamma_2(t) + \gamma_2'(t) p \gamma_1(t) &= \gamma_1'(t) p_2 \gamma_2(t) + \gamma_1'(t) (1-p) \gamma_2(t), \\
\gamma_1'(t) (1-p) \gamma_2(t) + \gamma_2'(t) (1-p) \gamma_1(t) &= \gamma_2'(t) p_1 \gamma_1(t) + \gamma_2'(t) (1-p) \gamma_1(t).
\end{align*}
\]
For instance, if we plug in $q = 2/5$ and $p = 2/3$ to match the economy in Example 1, the calculation yields the cutoffs $p^1_1 = p^2_1 \approx .54$, $p^1_2 \approx .73$ and $p^2_2 \approx .37$, which are approximately the same cutoffs as those for the discrete economy in Example 1.

Example 2 illustrates how the TTC cutoffs can be directly calculated from the trade balance equations and capacity equations, without running the TTC algorithm. Example 2 also shows that it is not possible to solve for the TTC cutoffs only from supply-demand equations. In particular, the following equations are equivalent to the condition that for given cutoffs $\{p^e_{bc}\}_{b,c\in\{1,2\}}$, the demand for each school $c$ is equal to the available supply $q_c$ given by the school’s capacity:

$$p \cdot (1 - p^1_1 \cdot p^1_2) = q_1 = q$$

$$(1 - p) \cdot (1 - p^1_1 \cdot p^1_2) + p^1_1 (p^1_2 - p^2_2) = q_2 = q.$$

Any cutoffs $p^1_1 = p^2_1 = x$, $p^1_2 = (1 - q/p)/x$, $p^2_2 = (1 - 2q) x$ with $x \in [1 - q/p, 1]$ solve these equations, but if $x \neq \left(1 - \frac{2}{p}\right)^p$ then the corresponding assignment is different from the TTC assignment. Section 4.2 provides further details as to how the TTC assignment depends on features of the economy that cannot be observed from supply and demand alone. In particular, the TTC cutoffs depend on the relative priority among top-priority students, and not all cutoffs that satisfy supply-demand conditions produce the TTC assignment.

### 3.3 Consistency with the Discrete TTC Model

In this section we first show that any discrete economy can be translated into a continuum economy, and that the cutoffs obtained using Theorem 2 on this continuum economy give the same assignment as discrete TTC. This demonstrates that the continuum TTC model generalizes the standard discrete TTC model. We then show that the TTC assignment changes smoothly with changes in the underlying economy.

To represent a discrete economy $E = (C, S, \succ_c, \succ^S, q)$ by a continuum economy $\Phi(E) = (C, \Theta, \eta, \frac{q}{N})$, we construct a measure $\eta$ over $\Theta$ by placing a mass at $(\succ^s, r^*)$ for each student $s$. To ensure the measure has a bounded density, we spread the mass of each student $s$ over a small region $I^s = \{\theta \in \Theta | \succ^\theta = \succ^s, \ r^\theta \in [r^*_c, r^*_c + \frac{1}{N}] \ \forall c \in C\}$ and identify any point $\theta^s \in I^s$ with student $s$. The following proposition shows that the continuum TTC assigns all $\theta^s \in I^s$ to the same school, which is the assignment of
student $s$ in the discrete model. Moreover, we can directly use the continuum cutoffs for the discrete economy. Further details and a formal definition of the map $\Phi$ are in online Appendix D.4.

**Proposition 2.** Let $E = (C, S, \succ_c, \succ^S, q)$ be a discrete economy, and let $\Phi (E) = (C, \Theta, \eta, \frac{\eta}{N})$ be the corresponding continuum economy. Let $p$ be the cutoffs produced by Theorem 2 for economy $\Phi (E)$. Then the cutoffs $p$ give the TTC assignment for the discrete economy $E$, namely,

$$
\mu_{dTTC} (s \mid E) = \max_{s \succ} \{ c \mid r^s_b \geq p^s_b \text{ for some } b \},
$$

and for every $\theta^s \in I^s$ we have that

$$
\mu_{dTTC} (s \mid E) = \mu_{cTTC} (\theta^s \mid \Phi (E)).
$$

In other words, $\Phi$ embeds each discrete economy into a continuum economy that represents it, and the TTC cutoffs in the continuum embedding give the same assignment as TTC in the discrete model. This shows that the TTC assignment defined in Theorem 2 provides a strict generalization of the discrete TTC assignment to a larger class of economies. We provide an example of an embedding of a discrete economy in Appendix B.

Next, we show that the continuum economy can also be used to approximate sufficiently similar economies. Formally, we show that the TTC allocations for strongly convergent sequences of economies are also convergent.

**Theorem 3.** Consider two continuum economies $\mathcal{E} = (C, \Theta, \eta, q)$ and $\tilde{\mathcal{E}} = (C, \Theta, \tilde{\eta}, q)$, where the measures $\eta$ and $\tilde{\eta}$ have total variation distance $\varepsilon$. Suppose also that both measures have full support. Then the TTC allocations in these two economies differ on a set of students of measure $O(\varepsilon |C|^2)$.

In Section 4.2, we show that changes to the priorities of a set of high priority students can affect the final assignment of other students in a non-trivial manner. This raises the question of what the magnitude of these effects are, and whether the TTC mechanism is robust to small perturbations in student preferences or school priorities. Our convergence result implies that the effects of perturbations are proportional to the total variation distance of the two economies, and suggests that the TTC mechanism is fairly robust to small perturbations in preferences.
3.4 Proper budget sets

The standard definition for a student’s budget set is the set of schools she can be assigned to by reporting some preference to the mechanism. Specifically, let \([E_{-s}; \succ']\) denote the economy where student \(s\) changes her report from \(\succ_s\) to \(\succ'\), and let

\[
B^* (s | E) \triangleq \bigcup_{\succ'} \mu_{\text{TTC}} (s | [E_{-s}; \succ'])
\]

denote the set of schools that student \(s\) can attain by some reported preference, holding fixed the reports of other students. Note that \(s\) cannot misreport her priority.

We observed in Section 2.4 that in the discrete model the budget set \(B(s, p)\) produced by cutoffs \(p = p(E)\) generated by Theorem 1 do not necessarily correspond to the set \(B^* (s | E)\). The analysis in this section can be used to show that the budget sets \(B^* (s | E)\) correspond to the budget sets \(B(s, p^*)\) for appropriate cutoffs \(p^*\).

**Proposition 3.** Let \(E = (C, S, \succ^S, \succ_C, q)\) be a discrete economy, and let

\[
P(E) = \left\{ p \mid p^*_b = \gamma_b \left( t(\cdot) \right) \text{ where } \gamma(\cdot), t(\cdot) \text{ satisfy trade balance and capacity for } \Phi(E) \right\}
\]

be the set of all cutoffs that can be generated by some TTC path \(\gamma(\cdot)\) and stopping times \(\{t(\cdot)\}_{c \in C}\). Then

\[
B^* (s | E) = \bigcap_{p \in P(E)} B(s; p).
\]

Moreover, there exists \(p^* \in P(E)\) such that for every student \(s\)

\[
B^* (s | E) = B(s; p^*).
\]

Proposition 3 allows us to construct proper budget sets for each agent that determine not only their assignment given their current preferences, but also their assignment given any other submitted preferences. This particular budget set representation of TTC makes it clear that it is strategyproof. In the appendix we prove Proposition 3 and constructively find \(p^*\).
4 Applications

4.1 Optimal Investment in School Quality

In this section, we explore how to invest in school quality when students are assigned through the TTC mechanism. School financing has been subject to major reforms, and empirical evidence suggests that increased financing has substantial impact on school quality (Hoxby, 2001; Cellini, Ferreira, and Rothstein, 2010; Jackson, Johnson, and Persico, 2016; Lafortune, Rothstein, and Schanzenbach, 2016). Under school choice, changes in school quality will affect student preferences over schools, and therefore change the assignment of students to schools. This may have a negative welfare effect, as schools that become popular will be excluded from some students’ budget sets. Under heterogeneous preferences (Hastings, Kane, and Staiger, 2009; Abdulkadiroğlu, Agarwal, and Pathak, 2015) welfare depends on whether students can choose a school for which they have an idiosyncratically high preference. Observing students’ budget sets allows us to track the welfare generated by student choices along horizontal dimensions.

We first provide more general comparative statics demonstrating how an increase in school quality affects the TTC assignment. We then examine the question of optimal investment in school quality under a stylized model. Omitted proofs and derivations can be found in online Appendix E.1.

Model with quality dependent preferences and comparative statics

We first enrich our model from Section 3 to allow student preferences to depend on school quality investments. An economy with quality dependent preferences is given by \( E = (C, S, \eta, q) \), where \( C = \{1, 2, \ldots, n\} \) is the set of schools and \( S \) is the set of student types. A student \( s \in S \) is given by \( s = (u^s (\cdot | \cdot), r^s) \), where \( u^s (c | \delta) \) is the utility of student \( s \) for school \( c \) given the quality of each school \( \delta = \{\delta_c\} \) and \( r^s_c \) is the student’s rank at school \( c \). We assume \( u^s (c | \cdot) \) is differentiable, increasing in \( \delta_c \) and non-increasing in \( \delta_b \) for any \( b \neq c \). The measure \( \eta \) over \( S \) specifies the distribution of student types. School capacities are \( q = \{q_c\} \), where \( \sum q_c < 1 \).

For a fixed quality \( \delta \), let \( \eta_\delta \) be the induced distribution over \( \Theta \), and let \( E_\delta = (C, \Theta, \eta_\delta, q) \) denote the induced economy.\(^{17}\) We assume for all \( \delta \) that \( \eta_\delta \) has a Lipschitz

\(^{17}\)To make student preferences strict we arbitrarily break ties in favor of schools with lower indices.
continuous non-negative density \( \nu \) that is bounded below on its support and depends smoothly on \( \delta \). For a given \( \delta \), let \( \mu_\delta \) and \( \{ p^b_\delta(\delta) \}_{c \in \mathcal{C}} \) denote the TTC assignment and associated cutoffs. We omit the dependence on \( \delta \) when it is clear from context.

When there are two schools, we can specify the direction of change of the TTC cutoffs when we slightly increase \( \delta \) for some \( \ell \in \{1, 2\} \). We consider changes that do not change the strict order of school run-out times and without loss of generality assume that schools are numbered in order of their run-out times.

**Proposition 4.** Suppose \( \mathcal{E} = (\mathcal{C} = \{1, 2\}, \mathcal{S}, \eta, q) \) and \( \hat{\delta} \) induces an economy \( \mathcal{E}_{\hat{\delta}} \) such that the TTC cutoffs have a strict runout order \( p^1_{\hat{\delta}}(\hat{\delta}) > p^2_{\hat{\delta}}(\hat{\delta}) \). Suppose \( \hat{\delta} \) has higher school \( 2 \) quality \( \hat{\delta}_2 \geq \delta_2 \), the same quality \( \delta_1 = \hat{\delta}_1 \), and \( \mathcal{E}_{\hat{\delta}} \) has the same runout order, i.e. \( p^1_{\hat{\delta}}(\hat{\delta}) \geq p^2_{\hat{\delta}}(\hat{\delta}) \). When we change from \( \delta \) to \( \hat{\delta} \) the cutoffs \( p^b_\delta(\cdot) \) change as follows:

- \( p^1_{\delta} \) and \( p^1_{\hat{\delta}} \) both decrease, i.e., it becomes easier to trade into school 1; and
- \( p^2_{\delta} \) increases, i.e. higher 2-priority is required to get into school \( \ell = 2 \).

Figure 3 illustrates the effect of improving the quality of school \( \ell = 2 \) when \( \mathcal{C} = \{1, 2\} \). As in Hatfield, Kojima, and Narita (2016), there may be low 2-rank students who will gain assignment to school 2 after the quality change because of the decrease in \( p^2_{\delta} \). Notice also that small changes in the cutoffs can result in individual students’ budget sets growing or shrinking by more than one school. In general, if the TTC cutoffs change slightly then for every pair of schools \( b \neq c \) there will be students whose budget sets switch between \( \mathcal{C}(b) \) and \( \mathcal{C}(c) \).

In general, when \( n \geq 3 \), increasing the quality of a school \( \ell \) can have non-monotone effects on the cutoffs, and it is not possible to specify the direction of change of the cutoffs \( p^b_\delta \). However, with additional structure we can give more descriptive comparative statics. For example, consider the logit economy where students’ utilities for each school \( c \) are randomly distributed as a logit with mean \( \delta_c \), independently of priorities and utilities for other schools. That is, utility for school \( c \) is given by \( u^s(c \mid \delta) = \delta_c + \varepsilon_{cs} \) with \( \eta \) chosen so \( \varepsilon_{cs} \) are i.i.d. EV shifted to have mean 0 (McFadden, 1973). Schools have uncorrelated uniform priorities over the students. This model allows us to capture a fixed utility term \( \delta_c \) that can be impacted by investment together with heterogeneous idiosyncratic taste shocks. The following proposition shows that under the logit economy we have closed form expressions for the TTC cutoffs. This allows us to describe the comparative statics.

We assume the utility of being unassigned is \(-\infty\), so all students find all schools acceptable.
Figure 3: The effect of an increase in the quality of school 2 on TTC cutoffs and budget sets. Dashed lines indicate initial TTC cutoffs, and dotted lines indicate TTC cutoffs given increased school 2 quality. The cutoffs $p_1^c = p_1^1$ and $p_1^c$ decrease and the cutoff $p_2^c$ increases. Students in the colored sections receive different budget sets after the increase. Students in dark blue improve to a budget set of $\{1, 2\}$ from $\emptyset$, students in light blue improve to $\{1, 2\}$ from $\{2\}$, and students in red have an empty budget set $\emptyset$ after the change and $\{2\}$ before.

**Proposition 5.** Under the logit economy with fixed qualities $\delta$ the TTC cutoffs $p_b^c$ for $b \geq c$ are given by

$$p_b^c = \left( \frac{\prod_{c' < c} p_{c'}^{c-1} - \rho_c \pi_c}{\prod_{c' < c} p_{c'}^{c-1}} \right)^{\pi_{b|c}}$$

(3)

where $\pi_{b|c}$ is the probability that a student chooses school $b$ given budget set $C(c)$, $\rho_c = \frac{q_c}{e_c} - \frac{q_{c-1}}{e_{c-1}}$ is the relative residual capacity for school $c$, $\pi_c = \sum_{c' \geq c} e^{\delta_{c'}}$ normalizes $\rho_c$ for when the set of available schools is $C(c)$, and the schools are indexed in the run-out order $\frac{q_1}{e_1} \leq \frac{q_2}{e_2} \leq \cdots \leq \frac{q_n}{e_n}$. Moreover, $p_b^c$ is decreasing in $\delta_c$ for $c < \ell$ and increasing in $\delta_c$ for $b > c = \ell$.

Figure 4 illustrates how the TTC cutoffs change with an increase in the quality of school $\ell$. Using equation (3), we derive closed form expressions for $\frac{dp_b^c}{d\delta_c}$, which can be found in online Appendix E.1.

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18To simplify notation, when $c = 1$ we let $\prod_{c' \geq c} p_{c'}^{c-1} = 1$. 

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Figure 4: The effects of changing the quality $\delta_\ell$ of school $\ell$ on the TTC cutoffs $p_b^\ell$ under the logit economy. If $c < \ell$ then $\frac{dp_b^\ell}{d\delta_\ell} < 0$ for all $b \geq c$, so it becomes easier to get into the more popular schools. If $c > \ell$ then $\frac{dp_b^\ell}{d\delta_\ell} = 0$. If $c = \ell$ then $\frac{dp_b^\ell}{d\delta_\ell} = \frac{dp_b^{\ell-1}}{d\delta_\ell} > 0$ for all $b > \ell$, and $p_b^\ell$ may increase or decrease depending on the specific problem parameters. Note that although $p_b^c$ and $p_b^\ell$ look aligned in the picture, in general it does not hold that $p_b^c = p_b^\ell$ for all $b$.

Optimal investment in school quality

Consider a social planner who selects quality levels $\delta$ for schools in economy $E$. We suppose that the social planner wishes to assign students to schools at which they attain high utility, and for simplicity also assume the social planner wishes to maximize social welfare. For a given assignment $\mu$, the social welfare is given by

$$U(\delta) = \int_{s \in S} u_s(\mu(s) | \delta) \, d\eta.$$  

To illustrate the effects of choice when using TTC, we first consider investment under neighborhood assignment $\mu_{NH}$, which assigns each student to a fixed school regardless of quality and preferences. We assume this assignment fills the capacity of each school. Social welfare for the logit economy is

$$U_{NH}(\delta) = \sum_c q_c \cdot \delta_c,$$

because $E[\varepsilon_{\mu(s)s}] = 0$ under neighborhood assignment. Under neighborhood assignment, the marginal welfare gain from increasing $\delta_\ell$ is $\frac{dU_{NH}}{d\delta_\ell} = q_\ell$, as an increase in the school quality benefits each of the $q_\ell$ students assigned to school $\ell$. 

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When the assignment is determined by TTC we need to use the budget set formulation of TTC to capture student welfare.\(^{19}\) Assume the schools are indexed according to the run-out order given by some fixed \(\delta\). A student who is offered the budget set \(C^{(c)} = \{c, \ldots, n\}\) is assigned to the school \(\ell = \arg \max_{b \in C^{(c)}} \{\delta_b + \varepsilon_{bs}\}\), and the logit distribution implies that their utility is \(U^c = \ln \left( \sum_{b \geq c} e^{\delta_b} \right)\) (Small and Rosen, 1981). Let \(N^c\) be the mass of agents with budget set \(C^{(c)}\). Then social welfare under the TTC assignment given quality \(\delta\) simplifies to

\[
U_{TTC}(\delta) = \sum_c N^c \cdot U^c.
\]

This expression for welfare also allows for a simple expression for the marginal welfare gain from increasing \(\delta_\ell\) under TTC.

**Proposition 6.** For the logit economy, the increase in social welfare \(U_{TTC}(\delta)\) under TTC from a marginal increase in \(\delta_\ell\) is given by

\[
\frac{dU_{TTC}}{d\delta_\ell} = q_\ell + \sum_{c \leq \ell + 1} \frac{dN^c}{d\delta_\ell} \cdot U^c.
\]

Under neighborhood assignment \(\frac{dU_{NH}}{d\delta_\ell} = q_\ell\).

Proposition 6 shows that a marginal increase \(d\delta_\ell\) in the quality of school \(\ell\) will have two effects. It will change the utility of the \(q_\ell\) students assigned to \(\ell\) by \(d\delta_\ell\), which is the same effect as under neighborhood assignment. In addition, the quality increase changes student preferences, and therefore changes the assignment. The second term captures the additional welfare effect of changes in the assignment by looking at the change in the number of students offered each budget set. The additional term can be negative so that \(\frac{dU_{TTC}}{d\delta_\ell} < q_\ell = \frac{dU_{NH}}{d\delta_\ell}\), because an increase in the quality of a school can lead to less efficient sorting of students to schools.

In particular, when there are two schools \(C = \{1, 2\}\) with \(q_1 = q_2\) and \(\delta_1 \geq \delta_2\) we

\(^{19}\)The expected utility of student \(s\) assigned to school \(\mu(s)\) depends on the student’s budget \(B(s, p)\) because of selection on taste shocks. Namely, \(E[u^*(\mu(s) \mid \delta)] = \delta_{\mu(s)} + E[\varepsilon_{\mu(s)s} \mid \delta_{\mu(s)} + \varepsilon_{\mu(s)s} \geq \delta_c + \varepsilon_{cs} \forall c \in B(s, p)]\)
have that
\[
\frac{dU_{TTC}}{d\delta_1} = q_1 + \frac{dN^1}{d\delta_1} \cdot U^1 + \frac{dN^2}{d\delta_1} \cdot U^2 = q_1 + \frac{dN^1}{d\delta_1} \cdot (U^1 - U^2)
\]
\[
= q_1 - (q_1 \cdot e^{\delta_2 - \delta_1}) \left( \ln \left( e^{\delta_1} + e^{\delta_2} \right) - \delta_2 \right) < q_1,
\]
where we use that \( N^c = \left( \frac{q^c e^{\delta^c} - q^c - 1}{e^{\delta^c} - 1} \right) (\sum_{b \geq c} e^{\delta_b}) \).

An increase in the quality of the higher quality school 1 gives higher utility for students assigned to 1, which is captured by the first term. Additionally, it causes some students to switch their preferences to \( 1 \succ 2 \), making school 1 run out earlier in the TTC algorithm, and removing school 1 from the budget set of some students. Students whose budget set did not change and who switched to \( 1 \succ 2 \) are almost indifferent between the schools and hence unaffected. Students who lost school 1 from their budget set can prefer school 1 by a large margin, and incur significant loss. Thus there is a total negative effect from the changes in the assignment, which is captured by the second term, and the derivative is smaller than under neighborhood assignment. Improving the quality of school 1 when \( \delta_1 \leq \delta_2 \) will have the opposite effect, as it enlarges student budget sets. Specifically, \( \frac{dU_{TTC}}{d\delta_1} = q_1 + q_2 \cdot e^{\delta_2 - \delta_1} \left( \ln \left( e^{\delta_1} + e^{\delta_2} \right) - \delta_1 \right) > q_1 \) which is larger than under neighborhood assignment. Note that, holding \( \delta_2 \) fixed, the function \( U_{TTC}(\delta_1) \) has a kink at \( \delta_1 = \delta_2 \).

We now provide an illustrative example of optimal investment with quality constraints under DA, TTC and neighborhood assignment.

**Example 3.** Consider a logit economy with two schools and \( q_1 = q_2 = \frac{3}{8} \), and let \( Q = q_1 + q_2 \) denote the total capacity. The planner is constrained to choose quality levels \( \delta \) such that \( \delta_1 + \delta_2 = 2 \) and \( \delta_1, \delta_2 \geq 0 \).

Under neighborhood assignment \( \frac{U_{NH}}{Q} = 1 \) for any choice of \( \delta_1, \delta_2 \). Under TTC the unique optimal quality is \( \delta_1 = \delta_2 = 1 \), yielding \( \frac{U_{TTC}}{Q} = 1 + E[\max(\varepsilon_1, \varepsilon_2)] = 1 + \ln(2) \approx 1.69 \). This is because any assigned student has the budget set \( B = \{1, 2\} \) and is assigned to the school for which he has higher idiosyncratic taste. Welfare is lower when \( \delta_1 \neq \delta_2 \), because fewer students choose the school for which they have higher idiosyncratic taste. For instance, given \( \delta_1 = 2, \delta_2 = 0 \) welfare is \( \frac{U_{TTC}}{Q} = \frac{1}{2} (1 + e^{-2}) \log (1 + e^2) \approx 1.20 \). Under Deferred Acceptance (DA) the unique optimal quality is also \( \delta_1 = \delta_2 = 1 \), yielding \( \frac{U_{DA}}{Q} = 1 + \frac{1}{3} \ln(2) \approx 1.23 \). This is strictly lower than the welfare under TTC because under DA only students that have sufficiently
(a) TTC, $\delta_1 = \delta_2 = 1$, optimal investment.

(b) TTC, $\delta_1 = 2, \delta_2 = 0$.

(c) Average student welfare under TTC, $\delta_1 + \delta_2 = 2$.

(d) DA, $\delta_1 = \delta_2 = 1$, optimal investment.

(e) DA, $\delta_1 = 2, \delta_2 = 0$.

(f) Average student welfare under DA, $\delta_1 + \delta_2 = 2$.

Figure 5: Illustration for Example 3. Figures (a) and (b) show the budget sets under TTC for different quality levels, and Figure (c) shows the average welfare of assigned students under TTC for quality levels $\delta_1 + \delta_2 = 2$ for different values of $\delta_1 - \delta_2$. Figures (d) and (e) show the budget sets under DA, and Figure (f) shows the average welfare of assigned students under DA.

High priority for both schools have the budget set $B = \{1, 2\}$. The remaining assigned students have a budget set $B = \{1\}$ or $B = \{2\}$, corresponding to the single school for which they have sufficient priority. If $\delta_1 = 2, \delta_2 = 0$ welfare under DA is $U_{DA}/Q \approx 1.11$.

TTC yields higher student welfare by providing all assigned students with a full budget set, thus maximizing each assigned student’s contribution to welfare from horizontal taste shocks. However, the assignment it produces is not stable. In fact, both schools admit students whom they rank at the bottom, and thus virtually all unassigned students can potentially block with either school. Example 3 shows that requiring a stable assignment will constrain many students from efficiently sorting on horizontal taste shocks.

Note that this may not be a concern in a school choice setting where assignments must be authorized by the department of education and blocking pairs cannot deviate and be assigned outside of the mechanism.
We next provide an example where one school has larger capacity. Investment in the larger school yields more direct benefit as it effects more students, but balancing investments in both schools can yield larger budget sets for more students, leading to more welfare from horizontal taste shocks.

**Example 4.** Consider a logit economy with two schools and \( q_1 = \frac{1}{2}, q_2 = \frac{1}{4} \), and let \( Q = q_1 + q_2 \) denote the total capacity. The planner is constrained to choose quality levels \( \delta \) such that \( \delta_1 + \delta_2 = 2 \) and \( \delta_1, \delta_2 \geq 0 \).

Under neighborhood assignment the welfare optimal quality is \( \delta_1 = 2, \delta_2 = 0 \), yielding \( \frac{U_{NH}}{Q} = \frac{4}{3} \approx 1.33 \). Under TTC assignment the unique optimal quality is \( \delta_1 = 1 + \frac{1}{2} \ln(2), \delta_2 = 1 - \frac{1}{2} \ln(2) \), yielding \( \frac{U_{TTC}}{Q} = \ln \left( \frac{3e}{\sqrt{2}} \right) \approx 1.75 \). Under these quality levels any assigned student has the budget set \( B = \{1, 2\} \). Given \( \delta_1 = 2, \delta_2 = 0 \) welfare is \( \frac{U_{TTC}}{Q} \approx 1.61 \). The quality levels that are optimal in Example 3, namely
\( \delta_1 = 1, \delta_2 = 1, \) give welfare \( UTTC/Q \approx 1.46. \) Under DA assignment the unique optimal quality is \( \delta_1 = 2, \delta_2 = 0, \) yielding \( U_{DA}/Q \approx 1.45. \) Given \( \delta_1 = 1, \delta_2 = 1 \) welfare under DA is \( U_{DA}/Q \approx 1.20. \)

Again we find that the optimal quality under TTC provides all assigned students with a full budget set, while the optimal qualities under neighborhood assignment and DA do not. The optimal quality levels under TTC in Example 4 imply that there is a 2/3 chance a student prefers school 1, and therefore both schools run out at the same time and all assigned students are offered a choice between both schools. Increasing \( \delta_1 \) further (and decreasing \( \delta_2 \)) would increase welfare holding the assignment fixed, but would result in worse sorting of students to schools on the horizontal taste shocks.

Finally, consider a central school board with a fixed amount of capital \( K \) to invest in the \( n \) schools. The cost of quality \( \delta_c \) is the convex function \( \kappa_c (\delta_c) = e^{\delta_c}.^{21} \) Using Proposition 6 we solve for optimal investment in school quality. Social welfare is maximized when all assigned students have a full budget set, which occurs when the amount invested in each school is proportional to the number of seats at the school.

**Proposition 7.** Social welfare is uniquely maximized when the amount \( \kappa_c \) invested in school \( c \) is proportional to the capacity \( q_c, \) that is,

\[
\kappa_c (\delta_c) = \frac{q_c}{\sum_b q_b} K
\]

and all assigned students \( \theta \) receive a full budget set, i.e., \( B (\theta, p) = \{1, 2, \ldots, n\} \) for all assigned students \( \theta. \)

Under optimal investment, the resulting TTC assignment is such that every assigned student receives a full budget set and is able to attend their top choice school. More is invested in higher capacity schools, as they provide more efficient investment opportunities, but the investment is balanced across schools to prevent any school from being over-demanded. This allows the TTC mechanism to offer assigned students a choice between all schools.

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21Note that \( \kappa_c \) is the total school funding. This is equivalent to setting student utility of school \( c \) to be to \( \log (\kappa_c) = \log (\kappa_c/q_c) + \log (q_c), \) which is the log of the per-student funding plus a fixed school utility that is larger for bigger schools. Thus, schools with higher capacity also provide more efficient investment opportunities.
4.2 Design of TTC Priorities

To better understand the role of priorities in the TTC mechanism, we examine how the TTC assignment changes with changes in the priority structure. Notice that any student $\theta$ whose favorite school is $c$ and who is within the $q_c$ highest ranked students at $c$ is guaranteed admission to $c$. In the following example, we consider changes to the relative priority of such highly ranked students and find that these changes can have an impact on the assignment of other students, without changing the assignment of any student whose priority changed.

**Example 5.** The economy $E$ has two schools 1, 2 with capacities $q_1 = q_2 = q$, students are equally likely to prefer each school, and student priorities are uniformly distributed on $[0, 1]$ independently for each school and independently of preferences. The TTC algorithm ends after a single round, and the resulting assignment is given by $p^1_1 = p^2_1 = p^1_2 = p^2_2 = \sqrt{1 - 2q}$. The derivation can be found in Appendix E.2.

Consider the set of students $\{\theta \mid r^\theta_c \geq m \ \forall c\}$ for some $m > 1 - q$. Any student in this set is assigned to his top choice, regardless of his rank. Suppose we construct an economy $E'$ by arbitrarily changing the rank of students within the set, subject to the restriction that their ranks must remain in $[m, 1]$.

The range of possible TTC cutoffs for $E'$ is given by $p^1_1 = p^2_1, p^1_2 = p^2_2$ where

$$p^1_1 \in [\underline{p}, \bar{p}], \quad p^2_2 = \frac{1}{p^1_1} (1 - 2q)$$

for $\underline{p} = \sqrt{(1 - 2m + 2m^2)(1 - 2q)}$ and $\bar{p} = \sqrt{\frac{1 - 2q}{1 - 2m + 2m^2}}$. Figure 7 illustrates the range of possible TTC cutoffs for $E'$ and the economy $\bar{E}$ for which TTC obtains one set of extreme cutoffs.

Example 5 has several implications. First, it shows that it is not possible to directly compute TTC cutoffs from student demand. The set of cutoffs such that student demand is equal to school capacity (depicted by the grey curve in Figure 7) are the cutoffs that satisfy $p^1_1 = p^2_1, p^1_2 = p^2_2$ and $p^1_1 p^2_2 = 1 - 2q$. Under any of these cutoffs the students in $\{\theta \mid r^\theta_c \geq m \ \forall c\}$ have the same demand, but the resulting TTC outcomes are different. It follows that the mechanism requires more information to determine the assignment. However, Theorem 3 implies that the changes in TTC outcomes are small if $1 - m$ is small.

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22 The remaining students still have ranks distributed uniformly on the complement of $[1 - r, 1]^2$. 31
A second implication is that the TTC priorities can be ‘bossy’ in the sense that changes in the relative priority of high priority students can affect the assignment of other students, even when all high priority students receive the same assignment. Notice that in all the economies considered in Example 5, we only changed the relative priority within the set \( \{ \theta \mid r_{\theta}^c \geq m \exists c \} \), and all these students were always assigned to their top choice. However, these changes resulted in a different assignment for low priority students. For example, if \( q = 0.455 \) and \( m = 0.4 \), a student \( \theta \) with priority \( r_1^\theta = 0.35, r_2^\theta = 0.1 \) could possibly receive his first choice or be unassigned depending on the choice of \( \mathcal{E}' \). Such changes to priorities may naturally arise when there are many indifferences in student priorities, and tie-breaking is used. Since priorities are bossy, the choice of tie-breaking between high-priority students can have indirect effects on the assignment of low priority students.

### 4.3 Comparing Mechanisms

Both TTC and Deferred Acceptance (DA) (Gale and Shapley, 1962) are strategyproof, but differ in that TTC is efficient whereas DA is stable. In theory, the choice between the mechanisms requires a trade-off between efficiency and stability (this trade-off is evident in Example 3). Kesten (2006); Ehlers and Erdil (2010) show
the two mechanisms are equivalent only under strong conditions that are unlikely to hold in practice. However, Pathak (2016) evaluates the two mechanisms on application data from school choice in New Orleans and Boston, and reports that the two mechanisms produced similar outcomes. In Section 4.1 we compare DA and TTC in terms of welfare and assignment and find potentially large differences. Pathak (2016) conjectures that the neighborhood priority used in New Orleans and Boston led to correlation between student preferences and school priorities that may explain the similarity between the TTC and DA allocations.

To study this conjecture, we consider a simple model with neighborhood priority.23 There are \( n \) neighborhoods, each with one school and a mass \( q \) of students. Schools have capacities \( q_1 \leq \cdots \leq q_n = q \), and each school gives priority to students in their neighborhood. For each student, the neighborhood school is their top ranked choice with probability \( \alpha \); otherwise the student ranks the neighborhood school in position \( k \) drawn uniformly at random from \( \{2, 3, \ldots, n\} \). Student preference orderings over non-neighborhood schools are drawn uniformly at random.

We find that the proportion of students whose assignments are the same under both mechanisms scales linearly with the probability of preference for the neighborhood school \( \alpha \), supporting the conjecture of Pathak (2016).

**Proposition 8.** The proportion of students who have the same assignments under TTC and DA is given by

\[
\alpha \frac{\sum_i q_i}{nq}.
\]

**Proof.** We use the methodologies developed in Section 3.2 and in Azevedo and Leshno (2016) to find the TTC and DA allocations respectively. Students with priority are given a lottery number uniformly at random in \([\frac{1}{2}, 1]\), and students without priority are given a lottery number uniformly at random in \([0, \frac{1}{2}]\), where lottery numbers at different schools are independent. For all values of \( \alpha \), the TTC cutoffs are given by \( p_j^i = p_i^j = 1 - \frac{q_i}{2q} \) for all \( i \leq j \), and the DA cutoffs are given by \( p_i = 1 - \frac{q_i}{2q} \). The derivations of the cutoffs can be found in Appendix E.3.

The students who have the same assignments under TTC and DA are precisely the students at neighborhood \( i \) whose ranks at school \( i \) are above \( 1 - \frac{q_i}{2q} \), and whose

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23Che and Tercieux (2015b) also show that when there are a large number of schools with a single seat per school and preferences are random both DA and TTC are asymptotically efficient and stable and give asymptotically equivalent allocations. As Example 3 shows, these results do not hold when there are many students and a few large schools.
first choice school is their neighborhood school. This set of students comprises an \( \alpha \frac{q_i}{n_q} \) fraction of the entire student population, which scales proportionally with the correlation between student preferences and school priorities.

We can also compare TTC with the Clinch and Trade (C&T) mechanism introduced by Morrill (2015b). The C&T mechanism identifies students who are guaranteed admission to their favorite school \( c \) by having priority \( r_c^\theta \geq 1 - q \) and assigns them to \( c \) by ‘clinching’ without trade. Morrill (2015b) gives an example where the C&T assignment has fewer blocking pairs than the TTC assignment. The fact that allowing students to clinch can change the assignment can be interpreted as another example for the bossiness of priorities under TTC: we can equivalently implement C&T by running TTC on a changed priority structure where students who clinched at school \( c \) have higher rank at \( c \) than any other student.\(^{24}\) The following proposition builds on Example 5 and shows that C&T may produce more blocking pairs than TTC.

**Proposition 9.** The Clinch and Trade mechanism can produce more, fewer or an equal number of blocking pairs compared to TTC.

**Proof.** Morrill (2015b) provides an example where C&T produces fewer blocking pairs than TTC. Both mechanism give the same assignment for the symmetric economy in the beginning of Example 5. It remains to construct an economy \( \mathcal{E}_1 \) for which C&T produces more blocking pairs than TTC. Economy \( \mathcal{E}_1 \) is the same as \( \bar{\mathcal{E}} \), except that school 2 rank is redistributed among students with \( r_2^\theta \leq \bar{p} \) so that students with \( r_1^\theta \geq \bar{p} \) have higher school 2 rank.\(^{25}\) The C&T assignment for \( \mathcal{E}_1 \) is given by \( p_1^1 = p_2^2 = 0.3 \), while TTC gives \( p_1^1 = \bar{p} \) and \( p_2^2 = \bar{p} \) (and under both \( p_1^1 = p_2^1 \), \( p_2^1 = p_2^2 \)). Under TTC unmatched students will form blocking pairs only with school 2, while under C&T all unmatched students will form a blocking pair with either school. See Figure 13 for an illustration.

\(^{24}\)For brevity, we abstract away from certain details of C&T mechanism that are important when not all schools run out at the same round.

\(^{25}\)Specifically, select \( \ell_1 < \ell_2 \). Among students with \( r_2^\theta \leq \bar{p} \) and \( r_1^\theta \geq \bar{p} \) the school 2 rank is distributed uniformly in the range \([\ell_2, \bar{p}]\). Among students with \( r_2^\theta \leq \bar{p} \) and \( r_1^\theta < \bar{p} \) the school 2 rank is distributed uniformly in the range \([0, \ell_1]\). Within each range \( r_1^\theta \) and \( r_2^\theta \) are still independent. See Figure 13 for an illustration.
5 Discussion

We can simplify how the TTC outcome is communicated to students and their families by using the cutoff characterization. The cutoffs $\{p^c_b\}$ are calculated in the course of running the TTC algorithm. The cutoffs can be published to allow parents to verify their assignment, or the budget set structure can be communicated using the language of tokens (see footnote 4). We hope that these methods of communicating TTC will make the mechanism more palatable to students and their parents, and facilitate a more informed comparison with the Deferred Acceptance mechanism, which also has a cutoff structure.

Examples provided in the paper utilized functional form assumptions to gain tractability. The methodology can be used more generally with numerical solvers. This provides a useful alternative to simulation methods that can be more efficient for large economies, or calculating an average outcome for large random economies. For example, most school districts uses tie-breaking rules, and current simulation methods perform many draws of the random tie-breaking lottery to calculate the expected outcomes. Our methodology directly calculates the expected outcome from the distribution. In Section 4.2 we characterize all the possible TTC outcomes for a class of tie-breaking rules, and find that the choice of tie-breaking rule can have significant effect on the assignment. We leave the problem of determining the optimal choice of tie-breaking lottery for future research.

The model assumes for simplicity that all students and schools are acceptable. It can be naturally extended to allow for unacceptable students or schools by erasing from student preferences any school that they find unacceptable or that finds them unacceptable. Type-specific quotas can be incorporated, as in Abdulkadiroğlu and Sönmez (2003), by adding type-specific capacity equations and erasing from the preference list of each type all the schools which do not have remaining capacity for their type.

References


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A Intuition for the Continuum TTC Model

In this section, we provide some intuition for our main results by considering a more direct adaptation of the TTC algorithm to continuum economies. Informally speaking, consider a continuum TTC algorithm in which schools offer seats to their highest priority remaining students, and students are assigned through clearing of trading cycles. This process differs from the discrete TTC algorithm as there is now a set of zero measure of highest priority students at each school, and the resulting trading cycles are also within sets of students of zero measure.

There are few challenges in turning this informal algorithm description into a precise definition. First, each cycle is of zero measure, but the algorithm needs to appropriately reduce school capacities as students are assigned. Second, a school will generally offer seats to multiple types of students at once. This implies each school may be involved in multiple cycles at a given point, a type of multiplicity that leads to non-unique TTC allocations in the discrete setting.

To circumvent the challenges above, we define the algorithm in terms of its aggregate behavior over many cycles. Instead of tracing each cleared cycle, we track the state of the algorithm by looking at the fraction of each school’s priority list that has been cleared. Instead of progressing by selecting one cycle at a time, we determine the progression of the algorithm by conditions that must be satisfied by any aggregation of cleared cycles. These yield equations (1) and (2), which determine the characterization given in Theorem 2.

A.1 Tracking the State of the Algorithm through the TTC Path \( \gamma \)

Consider some point in time during the run of the discrete TTC algorithm before any school has filled its capacity. While the history of the algorithm up to this point includes all previously cleared trading cycles, in order to run the algorithm, it is sufficient to record only the top priority remaining student at each school. This is because knowing the top remaining student at each school allows us to know exactly which students were previously assigned, and which students remain unassigned. Assigned students are relevant for the remainder of the algorithm only insofar as they reduce the number of seats available. Because all schools have remaining capacity, all assigned students are assigned to their top choice, and we can calculate the remaining
capacity at each school.

To formalize this notion, let \( \tau \) be some point during the run of the TTC algorithm before any school has filled its capacity. For each school \( c \), let \( \gamma_c(\tau) \in [0,1] \) be the percentile rank of the remaining student with highest \( c \)-priority. That is, at time \( \tau \) in the algorithm each school \( c \) is offering a seat to students \( s \) for whom \( r^s_c = \gamma_c(\tau) \). Let \( \gamma(\tau) \) be the vector \( (\gamma_c(\tau))_{c \in \mathcal{C}} \). The set of students that have already been assigned at time \( \tau \) is \( \{ s \mid r^s < \gamma(\tau) \} \), because any student \( s \) where \( r^s_c > \gamma_c(\tau) \) for some \( c \) must have already been assigned. Likewise, the set of remaining unassigned students is \( \{ s \mid r^s \leq \gamma(\tau) \} \). See Figure 8 for an illustration. Since all assigned students were assigned to their top choice, the remaining capacity at school \( c \in \mathcal{C} \) is \( q_c - |\{ s \mid r^s < \gamma(\tau) \text{ and } Ch^s(\mathcal{C}) = c \}| \). Thus, \( \gamma(\tau) \) captures all the information needed for the remainder of the algorithm.

![TTC path](image)

**Figure 8:** The set of students assigned at time \( \tau \) is described by the point \( \gamma(\tau) \) on the TTC path. Students in the grey region with rank better than \( \gamma(\tau) \) are assigned, and students in the white region with rank worse than \( \gamma(\tau) \) are unassigned.

This representation can be readily generalized to continuum economies. In the continuum, the algorithm progresses in continuous time. The state of the algorithm at time \( \tau \in \mathbb{R}_+ \) is given by \( \gamma(\tau) \in [0,1]^\mathcal{C} \), where \( \gamma_c(\tau) \in [0,1] \) is the percentile rank of the remaining students with highest \( c \)-priority. By tracking the progression of the algorithm through \( \gamma(\cdot) \) we avoid looking at individual trade cycles, and instead track how many students were already assigned from each school’s priority list.
A.2 Determining the Algorithm Progression through Trade Balance

The discrete TTC algorithm progresses by finding and clearing a trade cycle. This cycle assigns a set of discrete students; for each involved school $c$ the top student is cleared and $\gamma_c(\cdot)$ is reduced. In the continuum each cycle is infinitesimal, and any change in $\gamma(\cdot)$ must involve many trade cycles. Therefore, we seek to determine the progression of the algorithm by looking at the effects of clearing many cycles.

Suppose at time $\tau_1$ the TTC algorithm has reached the state $x = \gamma(\tau_1)$, where $\gamma(\cdot)$ is differentiable at $\tau_1$ and $d = -\gamma'(\tau_1) \geq 0$. Let $\varepsilon > 0$ be a small step size, and assume that by sequentially clearing trade cycles the algorithm reaches the state $\gamma(\tau_2)$ at time $\tau_2 = \tau_1 + \varepsilon$. Consider the sets of students offered seats and assigned seats during this time step from time $\tau_1$ to time $\tau_2$. Let $c \in C$ be some school. For each cycle, the measure of students assigned to school $c$ is equal to the measure of seats offered by school $c$. Therefore, if students are assigned between time $\tau_1$ and $\tau_2$ through clearing a collection of cycles, then the set of students assigned to school $c$ has the same measure as the set of seats offered by school $c$. If $\gamma(\cdot)$ and $\eta$ are sufficiently smooth, the measures of both of these sets can be approximately expressed in terms of $\varepsilon \cdot d$ and the marginal densities $\{H_b^c(x)\}_{b,c \in C}$, yielding an equation that determines $d$. For the sake of clarity, we omit technical details in the ensuing discussion. A rigorous derivation can be found in online Appendix F.

We first identify the measure of students who were offered a seat at a school $b$ or assigned to a school $c$ during the step from time $\tau_1$ to time $\tau_2$. If $d = -\gamma'(\tau_1)$ and $\varepsilon$ is sufficiently small, we have that for every school $b$

$$|\gamma_b(\tau_2) - \gamma_b(\tau_1)| \approx \varepsilon d_b,$$

that is, during the step from time $\tau_1$ to time $\tau_2$ the algorithm clears students with $b$-ranks between $x$ and $x - \varepsilon d_b$. To capture this set of students, let

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26Strictly speaking, the measure of students assigned to each school is equal to the measure of seats at that school which were claimed (not the measure of seats offered). A seat can be offered but not claimed in one of two ways. The first occurs when the seat is offered at time $\tau$ but not yet claimed. The second is when a student is offered two or more seats at the same time, and claims only one of them. Both of these sets are of $\eta$-measure 0 under our assumptions, and thus the measure of seats claimed is equal to the measure of seats offered.
\[ T_b(x, \varepsilon d_b) \overset{\text{def}}{=} \{ \theta \in \Theta \mid r^\theta \leq x, \ r^\theta_b > x - \varepsilon d_b \} \]

denote the set of students with ranks in this range. For all \( \varepsilon \), \( T_b(x, \varepsilon d_b) \) is the set of top remaining students at \( b \), and when \( \varepsilon \) is small, \( T_b(x, \varepsilon d_b) \) is approximately the set of students who were offered a seat at school \( b \) during the step.\(^27\)

To capture the set of students that are assigned to a school \( c \) during the step, partition the set \( T_b(x, \varepsilon d_b) \) according to the top choice of students. Namely, let

\[ T_c^b(x, \varepsilon d_b) \overset{\text{def}}{=} \{ \theta \in T_b(x, \varepsilon d_b) \mid Ch^\theta(C) = c \}, \]

denote the top remaining students on \( b \)'s priority list whose top choice is school \( c \). Then the set of students assigned to school \( c \) during the step is \( \bigcup_a T_c^a(x, \varepsilon d_a) \), the set of students that got an offer from some school \( a \in C \) and whose top choice is \( c \).

We want to equate the measure of the set \( \bigcup_a T_c^a(x, \varepsilon d_a) \) of students who were assigned to \( c \) with the measure of the set of students who are offered\(^28\) a seat at \( c \), which is approximately the set \( T_c(x, \varepsilon d_c) \). By smoothness of the density of \( \eta \), for sufficiently small \( \delta \) we have that

\[ \eta(T_c^b(x, \delta)) \approx \delta \cdot H_c^b(x) . \]

Therefore, we have that\(^29\)

\[ \eta(\bigcup_a T_c^a(x, \varepsilon d_a)) \approx \sum_{a \in C} \eta(T_c^a(x, \varepsilon d_a)) \approx \sum_{a \in C} \varepsilon d_a \cdot H_c^a(x) , \]
\[ \eta(T_c(x, \varepsilon d_c)) = \eta(\bigcup_a T_c^a(x, \varepsilon d_a)) \approx \sum_{a \in C} \varepsilon d_c \cdot H_c^a(x) . \]

In sum, if the students assigned during the step from time \( \tau_1 \) to time \( \tau_2 \) are cleared via a collection of cycles, we must have the following condition on the gradient \( \mathbf{d} = \gamma'(\tau_1) \)

\(^{27}\)The students in the set \( T_b(x, \varepsilon d_b) \cap T_a(x, \varepsilon d_a) \) could have been offered a seat at school \( a \) and assigned before getting an offer from school \( b \). However, for small \( \varepsilon \) the intersection is of measure \( O(\varepsilon^2) \) and therefore negligible.

\(^{28}\)In the continuum model the set of seats offered but not claimed (or traded) is of \( \eta \)-measure 0.

\(^{29}\)These approximations make use of the fact that \( \eta(T_b(x, \varepsilon d_b) \cap T_a(x, \varepsilon d_a)) = O(\varepsilon^2) \) for small \( \varepsilon \).
of the TTC path,
\[ \sum_{a \in C} \varepsilon d_a \cdot H^c_a (x) \approx \sum_{a \in C} \varepsilon d_c \cdot H^a_c (x). \]

Formalizing this argument yields the marginal trade balance equations at \( x = \gamma (\tau_1) \),
\[ \sum_{a \in C} \gamma'_a (\tau_1) \cdot H^c_a (x) = \sum_{a \in C} \gamma'_c (\tau_1) \cdot H^a_c (x). \]

### A.3 Interpretation of the Trade Balance Equations

The previous subsection showed that any small step clearing a collection of cycles must correspond to a gradient \( \gamma' \) that satisfies the trade balance equations. We next characterize the set of solutions to the trade balance equations and explain why any solution corresponds to clearing a collection of cycles.

Let \( \gamma (\tau) = x \), and consider the set of valid gradients \( d = -\gamma' (\tau) \geq 0 \) that solve the trade balance equations for \( x \)
\[ \sum_{a \in C} d_a \cdot H^c_a (x) = \sum_{a \in C} d_c \cdot H^a_c (x). \]

Consider the following equivalent representation. Construct a graph with a node for each school. Let the weight of node \( b \) be \( d_b \), and let the flow from node \( b \) to node \( c \) be \( f_{b \rightarrow c} = d_b \cdot H^c_b (x) \). The flow \( f_{b \rightarrow c} \) represents the flow of students who are offered a seat at \( b \) and wish to trade it for school \( c \) when the algorithm progress down school \( b \)’s priority list at rate \( d_b \). Figure 9 illustrates such a graph for \( C = \{1, 2, 3, 4\} \). The node weights \( d \) solve the trade balance equations if and only if the total flow into a node is equal to the total flow out of a node, i.e. if and only if \( f \) is a zero-sum flow. Standard arguments from network flow theory show that any zero-sum flow can be decomposed into a collection of cycles. In other words, the algorithm can find a collection of cycles that clears each school \( c \)’s priority list at rate \( d_c \) if and only if and only if \( d \) is a solution to the trade balance equations.

To characterize the set of solutions we draw on a connection to Markov chains. Consider a continuous time Markov chain over the states \( C \), and transition rates from state \( b \) to state \( c \) equal to \( H^c_b (x) \). The stationary distributions of the Markov chain are characterized by the balance equations, which state that the total probability flow out of state \( c \) is equal to the total probability flow into state \( c \). Mathematically,
Figure 9: Example of a graph representation for the trade balance equations at $x$. There is an edge from $b$ to $c$ if $H_b^c(x) > 0$. The two communication classes are framed.

these are exactly the trade balance equations. Hence $d$ is a solution to the trade balance equations if and only if $d/\|d\|_1$ is a stationary distribution of the Markov chain.

This connection allows us to fully characterize the set of solutions to the trade balance equations through well known results about Markov chains. We restate them here for completeness. Given a transition matrix $P$, a recurrent communication class is a subset $K \subseteq C$, such that the restriction of $P$ to rows and columns with coordinates in $K$ is an irreducible matrix, and $P_b^c = 0$ for every $c \in K$ and $b \notin K$. See Figure 9 for an example. There exists at least one recurrent communication class, and two different communication classes have an empty intersection. Let the set of communicating classes be $\{K_1, \ldots, K_\ell\}$. For each communicating class $K_i$ there is a unique vector $d_{K_i}$ that is a stationary distribution and $d_{K_i}^c = 0$ for any $c \notin K_i$. The set of stationary distributions of the Markov chain is given by convex combinations of $\{d_{K_1}, \ldots, d_{K_\ell}\}$.

An immediate implication is that a solution to the trade balance equations always exists. Moreover, if $\eta$ has full support\(^{30}\) then the TTC path $\gamma$ is unique (up to rescaling of the time parameter). This is because full support of $\eta$ implies that the matrix $H(x)$ is irreducible for every $x$, i.e. there is a single communicating class. Therefore there is a unique (up to normalization) solution $d = -\gamma'(\tau)$ to the trade balance equations at $x = \gamma(\tau)$ for every $x$ and the path is unique.

**Lemma 1.** Let $\mathcal{E} = (C, \Theta, \eta, q)$ be a continuum economy where $\eta$ has full support. Then there exists a TTC path $\gamma$ that is unique up to rescaling of the time parameter $t$. For $\tau \leq \min_{c \in C} \{t(c)\}$ we have that $\gamma(\cdot)$ is given by

$$\frac{d\gamma(t)}{dt} = d(\gamma(t))$$

\(^{30}\) $\eta$ has full support if for every open set $A \subset \Theta$ we have $\eta(A) > 0$. 

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where \(d(x)\) is the solution to the trade balance equations at \(x\), and \(d(x)\) is unique up to normalization.

In general, there can be multiple solutions to the trade balance equations at \(x\), and therefore multiple TTC paths. The Markov chain and recurrent communication class structure give intuition as to why the TTC assignment is still unique. Each solution \(d^K_i\) corresponds to the clearing of cycles involving only schools within the set \(K\). The discrete TTC algorithm may encounter multiple disjoint trade cycles, and the outcome of the algorithm is invariant to the order in which these cycles are cleared (when preferences are strict). Similarly here, the algorithm may encounter mutually exclusive combinations of trade cycles \(\{d^{K_1}, \ldots, d^{K_\ell}\}\), which can be cleared sequentially or simultaneously at arbitrary relative rates. Theorem 2 shows that just like the outcome of the discrete TTC algorithm does not depend on the cycle clearing order, the outcome of the continuum TTC algorithm does not depend on the order in which \(\{d^{K_1}, \ldots, d^{K_\ell}\}\) are cleared.

As an illustration, consider the unique solution \(d^K\) for the communicating class \(K = \{1, 2\}\), as illustrated in Figure 9. Suppose that at some point \(x\) we have \(H_1^1(x) = 1/2\), \(H_1^2(x) = 1/2\) and \(H_2^1(x) = 1\). That is, the marginal mass of top ranked students at either school is 1, all the top marginal students of school 2 prefer school 1, and half of the top marginal students of school 1 prefer school 1 and half prefer school 2. The algorithm offers seats and goes down the school’s priority lists, assigning students through a combination of two kinds of cycles: the cycle 1 \(\rightsquigarrow\) where a student is offered a seat at 1 and is assigned to 1, and a cycle 1 \(\leftrightarrow\) 2 where a student who was offered a seat at 1 trades her seat with a student who was offered a seat at 2. Given the relative mass of students, the cycle 1 \(\leftrightarrow\) 2 should be twice as frequent as the cycles 1 \(\rightsquigarrow\). Therefore, clearing cycles leads the mechanism to go down school 1’s priority list at twice the speed it goes down school 2’s list, or \(d_1 = 2 \cdot d_2\), which is the unique solution to the trade balance equations at \(x\) (up to normalization).

Figure 10 illustrates the path \(\gamma(\cdot)\) and the solution \(d(x)\) to the trade balance equations at \(x\). Note that for every \(x\) we can calculate \(d(x)\) from \(H(x)\). When there are multiple solutions to the trade balance equations at some \(x\), we may select a solution \(d(x)\) for every \(x\) such that \(d(\cdot)\) is a sufficiently smooth gradient field. The TTC path \(\gamma(\cdot)\) can be generated by starting from \(\gamma(0) = 1\) and following the gradient field.
A.4 When a School Fills its Capacity

So far we have described the progression of the algorithm while all schools have remaining capacity. To complete our description of the algorithm we need to describe how the algorithm detects that a school has exhausted all its capacity, and how the algorithm continues after a school is full.

As long as there is still some remaining capacity, the trade balance equations determine the progression of the algorithm along the TTC path $\gamma(\cdot)$. The mass of students assigned to school $c$ at time $\tau$ is

$$D_c(\gamma(\tau)) = \eta \left( \{ \theta \mid r^\theta \not\in \gamma(\tau), Ch^\theta(C) = c \} \right).$$

Because $\gamma(\cdot)$ is continuous and monotonically decreasing in each coordinate, $D_c(\gamma(\tau))$ is a continuous increasing function of $\tau$. Therefore, the first time during the run of the continuum TTC algorithm at which any reached its capacity is given by $t(c^*)$ that solves the capacity equations

$$D_{c^*}(\gamma(t(c^*))) = q_{c^*}$$
$$D^a(\gamma(t(c^*))) \leq q_a \quad \forall a \in C$$

where $c^*$ is the first school to reach its capacity.
Once a school has filled up its capacity, we can eliminate that school and apply the algorithm to the residual economy. Note that the remainder of the run of the algorithm depends only on the remaining students, their preferences over the remaining schools, and remaining capacity at each school. After eliminating assigned students and schools that have reached their capacity we are left with a residual economy that has strictly fewer schools. To continue the run of the continuum TTC algorithm, we may recursively apply the same steps to the residual economy. Namely, to continue the algorithm after time $t^{(c^*)}$ start the path from $\gamma(t^{(c^*)})$ and continue the path using a gradient that solves the trade balance equations for the residual economy. The algorithm follows this path until one of the remaining schools fills its capacity, and another school is removed.

## A.5 Comparison between Discrete TTC and Continuum TTC

Table 1 summarizes the relationship between the discrete and continuum TTC algorithms, and provides a summary of this section. It parallels the objects that define the continuum TTC algorithm with their counterparts in the discrete TTC algorithm. For example, running the continuum TTC algorithm on the embedding $\Phi(E)$ of a discrete economy $E$ performs the same assignments as the discrete TTC algorithm, except that the continuum TTC algorithm performs these assignments continuously and in fractional amounts instead of in discrete steps.

<table>
<thead>
<tr>
<th>Discrete TTC</th>
<th>$\rightarrow$</th>
<th>Continuum TTC</th>
<th>Expression</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cycle</td>
<td>$\rightarrow$</td>
<td>Valid gradient</td>
<td>$d(x)$</td>
<td>trade balance equations</td>
</tr>
<tr>
<td>Algorithm progression</td>
<td>$\rightarrow$</td>
<td>TTC path</td>
<td>$\gamma(\cdot)$</td>
<td>$\gamma'(\tau) = d(\gamma(\tau))$</td>
</tr>
<tr>
<td>School removal</td>
<td>$\rightarrow$</td>
<td>Stopping times</td>
<td>$t^{(c)}$</td>
<td>capacity equations</td>
</tr>
</tbody>
</table>

*Table 1: The relationship between the discrete and continuum TTC processes.*

Finally, we note that the main technical content of Theorem 2 is that there always exists a TTC path $\gamma$ and stopping times $\{t^{(c)}\}$ that satisfy trade balance and capacity, and that these necessary conditions, together with the capacity equations (2), are sufficient to guarantee the uniqueness of the resulting assignment.
B Example: Embedding a discrete economy in the continuum model

Consider the discrete economy \( E = (C, S, \succ^S, \succ^C, q) \) with two schools and six students, \( C = \{1, 2\} \), \( S = \{a, b, c, u, v, w\} \). School 1 has capacity \( q_1 = 4 \) and school 2 has capacity \( q_2 = 2 \). The school priorities and student preferences are given by

\[
\begin{align*}
1 & : a \succ u \succ b \succ c \succ v \succ w, \quad a, b, c : 1 \succ 2, \\
2 & : a \succ b \succ u \succ v \succ c \succ w, \quad u, v, w : 2 \succ 1.
\end{align*}
\]

In Figure 11, we display three TTC paths for the continuum embedding \( \Phi (E) \) of the discrete economy \( E \). The first path \( \gamma_{\text{all}} \) corresponds to clearing all students in recurrent communication classes, that is, all students in the maximal union of cycles in the pointing graph. The second path \( \gamma_1 \) corresponds to taking \( K = \{1\} \) whenever possible. The third path \( \gamma_2 \) corresponds to taking \( K = \{2\} \) whenever possible. We remark that the third path gives a different first round cutoff point \( p^1 \), but all three paths give the same allocation.

B.1 Calculating the TTC paths

We first calculate the TTC path in the regions where the TTC paths are the same. In the following, we consider only solutions \( d \) to the trade balance equations that have been normalized so that \( d \cdot 1 = -1 \). For brevity we call such solutions valid directions.

At every point \((x_1, x_2)\) with \( \frac{5}{6} < x_1 \leq x_2 \leq 1 \) the \( H \) matrix is
\[
\begin{pmatrix}
x_2 - \frac{5}{6} & 0 \\ x_1 - \frac{5}{6}
\end{pmatrix},
\]
so \( d = [-1, 0] \) is the unique valid direction and the TTC path is defined uniquely for \( t \in [0, \frac{1}{6}] \) by \( \gamma (t) = (1 - t, 1) \). This section of the TTC path starts at \((1, 1)\) and ends at \((\frac{5}{6}, 1)\). At every point \((\frac{5}{6}, x_2)\) with \( \frac{5}{6} < x_2 \leq 1 \) the \( H \) matrix is
\[
\begin{pmatrix}
0 & \frac{1}{6} \\ 0 & 0
\end{pmatrix},
\]
so \( d = [0, -1] \) is the unique valid direction, and the TTC path is defined uniquely for \( t \in \left[\frac{1}{6}, \frac{1}{3}\right] \) by \( \gamma (t) = \left(\frac{5}{6}, \frac{7}{6} - t\right) \). This section of the TTC path starts at \((\frac{5}{6}, 1)\) and ends at \((\frac{5}{6}, \frac{5}{6})\). At every point \((x_1, x_2)\) with \( \frac{2}{3} < x_1, x_2 \leq \frac{5}{6} \) the \( H \) matrix is
\[
\begin{pmatrix}
0 & \frac{1}{6} \\ 1 & 0
\end{pmatrix},
\]
and so \( d = \left[-\frac{1}{2}, -\frac{1}{2}\right] \) is the unique valid direction, the TTC path is defined uniquely
to lie on the diagonal $\gamma_1(t) = \gamma_2(t)$, and this section of the TTC path starts at $(\frac{5}{6}, \frac{5}{6})$ and ends at $(\frac{2}{3}, \frac{2}{3})$. At every point $x = (\frac{1}{3}, x_2)$ with $\frac{1}{3} < x_2 \leq \frac{2}{3}$ the $H$ matrix is
\[
\begin{bmatrix}
0 & 6x_2 - 2 \\
0 & 0
\end{bmatrix},
\]
and so $d = [0, -1]$ is the unique valid direction, and the TTC path is parallel to the $y$ axis. Finally, at every point $(x_1, \frac{1}{3})$ with $0 < x_1 \leq \frac{2}{3}$, the measure of students assigned to school $c_1$ is at most 3, and the measure of students assigned to school $c_2$ is 2, so $c_2$ is unavailable. Hence, from any point $(x_1, \frac{1}{3})$ the TTC path moves parallel to the $x_1$ axis.

We now calculate the various TTC paths where they diverge.

At every point $x = (x_1, x_2)$ with $\frac{1}{2} < x_1, x_2 \leq \frac{2}{3}$ the $H$ matrix is
\[
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix},
\]
(i.e. there are no marginal students), and so $\tilde{H} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Moreover, at every point $x = (x_1, x_2)$ with $\frac{1}{3} < x_1, x_2 \leq \frac{1}{2}$ the $H$ matrix is
\[
\begin{bmatrix}
\frac{1}{6} & 0 \\
0 & \frac{1}{6}
\end{bmatrix},
\]
and so $\tilde{H} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Also, at every point $x = (x_1, x_2)$ with $\frac{1}{3} < x_1 \leq \frac{1}{2}$ and $\frac{1}{2} < x_2 \leq \frac{2}{3}$, the $H$ matrix is
\[
\begin{bmatrix}
\frac{1}{6} & 0 \\
0 & 0
\end{bmatrix},
\]
so again $\tilde{H} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The same argument with the coordinates swapped gives that
\[
\tilde{H} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
\]
when $\frac{1}{2} < x_1 \leq \frac{2}{3}$ and $\frac{1}{3} < x_2 \leq \frac{1}{2}$. Hence in all these regions, both schools are in their own recurrent communication class, and any vector $d$ is a valid direction.

The first path corresponds to taking $d = [-\frac{1}{2}, -\frac{1}{2}]$, the second path corresponds to taking $d = [-1, 0]$ and the third path corresponds to taking $d = [0, -1]$. The first path starts at $(\frac{2}{3}, \frac{2}{3})$ and ends at $(\frac{1}{3}, \frac{1}{3})$ where school 2 fills. The third path starts at $(\frac{2}{3}, \frac{2}{3})$ and ends at $(\frac{2}{3}, \frac{1}{3})$ where school 2 fills. Finally, when $x = (\frac{1}{3}, x_2)$ with $\frac{1}{3} < x_2 \leq \frac{1}{2}$, the $H$ matrix is
\[
\begin{bmatrix}
0 & 1 \\
0 & 1
\end{bmatrix}
\]
and so $d = [0, -1]$ is the unique valid direction, and the second TTC path starts at $(\frac{1}{3}, \frac{1}{2})$ and ends at $(\frac{1}{3}, \frac{1}{3})$ where school 2 fills. All three paths continue until $(0, \frac{1}{3})$, where school 1 fills.

Note that all three paths result in the same TTC allocation, which assigns students $a, b, c, w$ to school 1 and $u, v$ to school 2. All three paths assign the students assigned before $p^1$ (students $a, u, b, c$ for paths 1 and 2 and $a, u, b$ for path 3) to their top choice.
C Proofs for Section 2

C.1 Proof of Theorem 1

For each student $s$ let $B(s,p) = \{c | r^s_b \geq p^c_b \text{ for some } b \}$. It suffices to show that for each student $s$ it holds that $\mu_{\text{ATT}}(s) \in B(s,p)$, and that if $c \in B(s,p)$ then $s$ prefers $\mu_{\text{ATT}}(s)$ to $c$, i.e. $\mu(s) \succeq^s c$. The former is simple to show, since if we let $b$ be the school such that $s$ traded a seat at school $b$ for a seat at school $\mu_{\text{ATT}}(s)$, then by definition $\mu^\mu_{\text{ATT}}(s) \leq r^s_b$ and $\mu_{\text{ATT}}(s) = B(s,p)$. 

Now suppose for the sake of contradiction that $c \in B(s,p)$ and student $s$ strictly prefers $c$ to $\mu_{\text{ATT}}(s)$, i.e. $c \succ^s \mu_{\text{ATT}}(s)$. As $c \in B(s,p)$ there exists a school $b'$ such that $r^s_{b'} \geq p^c_{b'}$. Let $s'$ be the student with rank $r^s_{b'} = p^c_{b'}$ at school $b'$. (Such a student exists since $p^c_{b'} \leq r^s_{b'} < 1$.) Then by definition student $s'$ traded a seat at school $b'$, so since $r^s_{b'} \geq p^c_{b'} = r^s_{b'}$ student $s$ is assigned weakly before student $s'$. Additionally, since $c \succ^s \mu_{\text{ATT}}(s)$ school $c$ must reach capacity before student $s$ is assigned, and so since student $s'$ was assigned to school $c$ student $s$ was assigned strictly before student $s$. This provides the required contradiction.

C.2 Proof of Proposition 1

Let the schools be indexed such that they reach capacity in the order $1, 2, \ldots, |C|$. If a student $s$ was assigned (strictly) after school $\ell - 1$ reached capacity and (weakly) before school $\ell$ reached capacity, we say that the student $s$ was assigned in round $\ell$.

We define new cutoffs $\{p^c_b\}$ by setting $p^c_b = \min_{c \leq c} p^c_b$, so that it evidently holds that $p^1_b \geq p^2_b \geq \cdots \geq p^h_b = p^{h+1}_b = \cdots = p^n_b$ for all $b$. We show that the cutoffs $\{p^c_b\}$ give the same allocation as the cutoffs $\{p^c_b\}$, i.e. for each student $s$ it holds that

$$\max \{c | r^s_b \geq p^c_b \text{ for some } b \} = \mu_{\text{ATT}}(s) = \max \{c | r^s_b \geq p^c_b \text{ for some } b \}.$$ 

For each student $s$ let $B(s,\mathbf{p}) = \{c | r^s_b \geq p^c_b \text{ for some } b \}$. It suffices to show that for each student $s$ it holds that $\mu_{\text{ATT}}(s) \in B(s,\mathbf{p})$, and that if $c \in B(s,\mathbf{p})$ then $s$ prefers $\mu_{\text{ATT}}(s)$ to $c$, i.e. $\mu(s) \succeq^s c$. The former is simple to show, since clearly $\mathbf{p} \preceq \mathbf{p}$ and so $B(s,\mathbf{p}) \supseteq B(s,\mathbf{p}) \supseteq \mu_{\text{ATT}}(s)$ (by Theorem 1).
The rest of the proof can be completed in much the same way as the proof of Theorem 1. Suppose for the sake of contradiction that \( c \in B(s, \bar{p}) \) and student \( s \) strictly prefers \( c \) to \( \mu_{\text{ATTC}}(s) \), i.e. \( c \succ^s \mu_{\text{ATTC}}(s) \). As \( c \in B(s, \bar{p}) \) there exists a school \( b' \) such that \( r_{s'}^{b'} \geq \bar{p}_{b'} \). Let \( s' \) be the student with rank \( r_{s'}^{b'} = \bar{p}_{b'} \) at school \( b' \). (Such a student exists since \( \bar{p}_{b'} \leq r_{s'}^{b'} < 1 \).) Then by definition student \( s' \) traded a seat at school \( b' \), so since \( r_{s'}^{b'} \geq \bar{p}_{b'} = r_{s'}^{b'} \) student \( s \) is assigned weakly before student \( s' \). Additionally, since \( c \succ^s \mu_{\text{ATTC}}(s) \) school \( c \) must reach capacity before student \( s \) is assigned. Finally, by definition there exists some \( c' \leq c \) such that \( \bar{p}_{b'} = p_{b'}^{c'} \) and student \( s' \) was assigned to school \( c' \), and so student \( s \) was assigned weakly before school \( c \) reached capacity, and hence strictly before student \( s \). This provides the required contradiction.
The Cutoff Structure of Top Trading Cycles in School Choice
Supplementary Appendix

D Proofs for Section 3

D.1 Definitions and Notation

We begin with some additional definitions and notation.

Let $x, \bar{x}$ be vectors. We let $(x, \bar{x}] = \{x : x < x \text{ and } x \leq \bar{x} \}$ denote the set of vectors that are weakly smaller than $\bar{x}$ along every coordinate, and strictly larger than $x$ along some coordinate. Let $K \subseteq C$ be a set of schools. For all vectors $x$, we let $\pi_K(x)$ denote the projection of $x$ to the coordinates indexed by schools in $K$.

We now incorporate information about the set of available schools. We denote by $\Theta^{|C|} = \{\theta \in \Theta | Ch^\theta (C) = c\}$ the set of students whose top choice in $C$ is $c$, and denote by $\eta^{|C|}$ the measure of these students. That is, for $S \subseteq \Theta$, let $\eta^{|C|} (S) := \eta (S \cap \Theta^{|C|})$. In an abuse of notation, for a set $A \subseteq [0, 1]^C$, we will often let $\eta (A)$ denote $\eta \left( \{\theta \in \Theta | r^\theta \in A \} \right)$, the measure of students with ranks in $A$, and let $\eta^{|C|} (A)$ denote $\eta \left( \{\theta \in \Theta^{|C|} | r^\theta \in A \} \right)$, the measure of students with ranks in $A$ whose top choice school in $C$ is $c$.

We will also find it convenient to define sets of students who were offered or assigned a seat along some TTC path $\gamma$. These will be useful in considering the result of aggregating the marginal trade balance equations. For each time $\tau$ let $T_c(\gamma; \tau) \overset{\text{def}}{=} \{\theta \in \Theta | \exists \tau' \leq \tau \text{ s.t. } r^\theta_c = \gamma_c(\tau') \text{ and } r^\theta \leq \gamma(\tau')\}$ denote the set of students who were offered a seat by school $c$ before time $\tau$, let $T^c(\gamma; \tau) \overset{\text{def}}{=} \{\theta \in \Theta | r^\theta \not\in \gamma(\tau) \text{ and } Ch^\theta (C(r^\theta)) = c\}$ denote the set of students who were assigned a seat at school $c$ before time $\tau$, and let $T^{c|C}(\gamma; \tau) \overset{\text{def}}{=} \{\theta \in \Theta | r^\theta \not\in \gamma(\tau) \text{ and } Ch^\theta (C) = c\}$ denote the set of students who would be assigned a seat at school $c$ before time $\tau$ if the set of available schools was $C$ and the path followed was $\gamma$.\(^{31}\)

\(^{31}\)Note that $T_c(\gamma; \tau)$ and $T^c(\gamma; \tau)$ include students who were offered or assigned a seat in the school in previous rounds.
For each interval \( T = [t, \bar{t}] \) let \( \mathcal{T}_c (\gamma; T) \stackrel{\text{def}}{=} \mathcal{T}_c (\gamma; \bar{t}) \setminus \mathcal{T}_c (\gamma; t) \) be the set of students who were offered a seat by school \( c \) at some time \( \tau \in T \), and let \( \mathcal{T}^{c|c} (T; \gamma) \stackrel{\text{def}}{=} \mathcal{T}^{c|c} (\gamma; \bar{t}) \setminus \mathcal{T}^{c|c} (\gamma; t) \) be the set of students who were assigned to a school \( c \) at some time \( \tau \in T \), given that the set of available schools was \( C (\gamma (\tau)) = C \) for each \( \tau \in T \). For each union of disjoint intervals \( T = \cup_n T_n \) similar define \( \mathcal{T}_c (\gamma; T) \stackrel{\text{def}}{=} \cup_n \mathcal{T}_c (\gamma; T_n) \) and \( \mathcal{T}^{c|c} (T; \gamma) \stackrel{\text{def}}{=} \cup_n \mathcal{T}^{c|c} (T_n; \gamma) \).

Finally let us set up the definitions for solving the marginal trade balance equations. For a set of schools \( C \) and individual schools \( b, c \in C \), recall that

\[
H^{c|c}_b (x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \eta \left( \{ \theta \in \Theta \mid r^{\theta} \in [x - \varepsilon \cdot e^b, x] \text{ and } Ch^{\theta} (C) = c \} \right)
\]

\[
= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \eta \left( \{ \theta \in \Theta^{c|c} \mid r^{\theta} \in [x - \varepsilon \cdot e^b, x] \} \right)
\]

is the marginal density of students pointed to by school \( b \) at the point \( x \) whose top choice school in \( C \) is \( c \).

Let \( \tilde{H}^{c} (x) \) be the \( |C| \times |C| \) matrix with \( (b,c) \)th entry \( \tilde{H}^{c} (x)_{b,c} = \frac{1}{v_c} H^{c|c}_b (x) + 1_{b=c} \left( 1 - \frac{v_c}{\bar{v}} \right) \), where \( v_c = \sum_{d \in C} H^{d|c}_c (x) \) is the row sum of \( H (x) \), and the normalization \( \bar{v} \) satisfies \( \bar{v} \geq \max_c v_c \).

Let \( M^{c} (x) \) be the Markov chain with state space \( C \), and transition probability from state \( b \) to state \( c \) equal to \( \tilde{H}^{c} (x)_{b,c} \). We remark that such a Markov chain exists, since \( \tilde{H}^{c} (x) \) is a (right) stochastic matrix for each pair \( C, x \).

We will also need the following definitions. For a matrix \( H \) and sets of indices \( I, J \) we let \( H_{I,J} \) denote the submatrix of \( H \) with rows indexed by elements of \( I \) and columns indexed by elements of \( J \). Recall that, by Assumption 1, the measure \( \eta \) is defined by a probability density \( \nu \) that is right-continuous and piecewise Lipschitz continuous with points of discontinuity on a finite grid. Let the finite grid be the set of points \( \{ x \mid x_i \in D_i \forall i \} \), where the \( D_i \) are finite subsets of \( [0,1] \). Then there exists a partition \( \mathcal{R} \) of \( [0,1]^C \) into hyperrectangles such that for each \( R \in \mathcal{R} \) and each face of \( R \), there exists an index \( i \) and \( y_i \in D_i \) such that the face is contained in \( \{ x \mid x_i = y_i \} \).

The following notion of continuity will be useful, given this grid-partition. We say that a multivariate function \( f : \mathbb{R}^n \to \mathbb{R} \) is right-continuous if \( f (x) = \lim_{y \geq x} f (y) \), where \( x, y \) are vectors in \( \mathbb{R}^n \) and the inequalities hold coordinate-wise. For an \( m \times n \)
matrix $A$, let $\mathbf{1}(A)$ be the $m \times n$ matrix with entries

$$
\mathbf{1}(A)_{ij} = \begin{cases} 
1 & \text{if } A_{ij} \neq 0, \\
0 & \text{if } A_{ij} = 0.
\end{cases}
$$

We will also frequently make use of the following lemmas.

**Lemma 2.** Let $\gamma$ satisfy the marginal trade balance equations. Then $\gamma$ is Lipschitz continuous.

*Proof.* By assumption, $\gamma$ is normalized so that $\|d\gamma(t)/dt\|_1 = 1$ a.e., and so since $\gamma(\cdot)$ is monotonically decreasing, for all $c$ it holds that $\gamma_c(\cdot)$ has bounded derivative and is Lipschitz with Lipschitz constant $L_c$. It follows that $\gamma(\cdot)$ is Lipschitz with Lipschitz constant $\max_c L_c$.  

**Lemma 3.** Let $C \subseteq C$ be a set of schools, and let $D$ be a region on which $\tilde{H}^C(x)$ is irreducible for all $x \in D$. For each $x$ let $A(x)$ be given by replacing the $n$th column of $\tilde{H}^C(x) - I_C$ with the all ones vector $\mathbf{1}$.\(^{32}\) Then the function $f(x) = \begin{bmatrix} 0^T & 1 \end{bmatrix} A(x)^{-1}$ is piecewise Lipschitz continuous in $x$.

*Proof.* It suffices to show that the function which, for each $x$, outputs the matrix $A(x)^{-1}$ is piecewise Lipschitz continuous in $x$.

Now

$$
H_{b,c}^x(x) = \lim_{\varepsilon \to 0} \varepsilon \int_{\theta : r_d x \geq x, \theta \not\geq x_b + \varepsilon, c \succ \theta C} \nu(\theta) d\theta,
$$

where $\nu(\cdot)$ is bounded below on its support and piecewise Lipschitz continuous, and the points of discontinuity lie on the grid. Hence $H_{b,c}^x(x)$ is Lipschitz continuous in $x$ for all $b, c$, and $\sum_d H_{d,c}^x(x)$ nonzero and hence bounded below, and so $\tilde{H}^C(x)_{b,c}$ is bounded above and piecewise Lipschitz continuous in $x$, and therefore so is $A(x)$. Finally, since $\tilde{H}^C(x)$ is an irreducible row stochastic matrix for each $x \in D$, it follows that $A(x)$ is full rank and continuous. This is because when $\tilde{H}^C(x)$ is irreducible every choice of $n-1$ columns of $\tilde{H}^C(x) - I_C$ gives an independent set whose span does not contain the all ones vector $\mathbf{1}_C$. Therefore if we let $A(x)$ be given by replacing the $n$th column in $\tilde{H}^C(x) - I_C$ with $\mathbf{1}_C$, then $A(x)$ has full rank.

Since $A(x)$ is full rank and continuous, in each piece $\det(A(x))$ is bounded away from 0, and so $A(x)^{-1}$ is piecewise Lipschitz continuous, as required.  

\(^{32}\) is the identity matrix with rows and columns indexed by the elements in $C$.  


D.2 Connection to Continuous Time Markov Chains

In Section A.3, we appealed to a connection with Markov chain theory to provide a method for solving for all the possible values of $d(x)$. Specifically, we argued that if $\mathcal{K}(x)$ is the set of recurrent communication classes of $\tilde{H}(x)$, then the set of valid directions $d(x)$ is identical to the set of convex combinations of $\{d^K\}_{K \in \mathcal{K}(x)}$, where $d^K$ is the unique solution to the trade balance equations (1) restricted to $K$. We present the relevant definitions, results and proofs here in full.

Let us first present some definitions from Markov chain theory. A square matrix $P$ is a right-stochastic matrix if all the entries are non-negative and each row sums to 1. A probability vector is a vector with non-negative entries that add up to 1. Given a right-stochastic matrix $P$, the Markov chain with transition matrix $P$ is the Markov chain with state space equal to the column/row indices of $P$, and a probability $P_{ij}$ of moving to state $j$ in one time step, given that we start in state $i$. Given two states $i, j$ of a Markov chain with transition matrix $P$, we say that states $i$ and $j$ communicate if there is a positive probability of moving to state $i$ to state $j$ in finite time, and vice versa.

For each Markov chain, there exists a unique decomposition of the state space into a sequence of disjoint subsets $C_1, C_2, \ldots$ such that for all $i, j$, states $i$ and $j$ communicate if and only if they are in the same subset $C_k$ for some $k$. Each subset $C_k$ is called a communication class of the Markov chain. A Markov chain is irreducible if it only has one communication class. A state $i$ is recurrent if, starting at $i$ and following the transition matrix $P$, the probability of returning to state $i$ is 1. A communication class is recurrent if it contains a recurrent state.

The following proposition gives a characterization of the stationary distributions of a Markov chain. We refer the reader to any standard stochastic processes textbook (e.g. Karlin and Taylor (1975)) for a proof of this result.

**Proposition 10.** Suppose that $P$ is the transition matrix of a Markov chain. Let $\mathcal{K}$ be the set of recurrent communication classes of the Markov chain with transition matrix $P$. Then for each recurrent communication class $K \in \mathcal{K}$, the equation $\pi = \pi P$ has a unique solution $\pi^K$ such that $||\pi^K|| = 1$ and $\text{supp}(\pi^K) \subseteq K$. Moreover, the support of $\pi^K$ is equal to $K$. In addition, if $||\pi|| = 1$ and $\pi$ is a solution to the equation $\pi = \pi P$, then $\pi$ is a convex combination of the vectors in $\{\pi^K\}_{K \in \mathcal{K}}$.

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33See standard texts such as Karlin and Taylor (1975) for a more complete treatment.
To make use of this proposition, define at each point \( x \) and for each set of schools \( C \) a Markov chain \( M^C(x) \) with transition matrix \( H^C(x) \). We will relate the valid directions \( d(x) \) to the recurrent communication classes of \( M^C(x) \), where \( C \) is the set of available schools. We will need the following notation and definitions. Given a vector \( v \) indexed by \( C \), a matrix \( Q \) with rows and columns indexed by \( C \) and subsets \( K, K' \subseteq C \) of the indices, we let \( v_K \) denote the restriction of \( v \) to the coordinates in \( K \), and we let \( Q_{K,K'} \) denote the restriction of \( Q \) to rows indexed by \( K \) and columns indexed by \( K' \).

The following lemma characterizes the recurrent communication classes of the Markov chain \( M^C(x) \) using the properties of the matrix \( H^C(x) \), and can be found in any standard stochastic processes text.

**Lemma 4.** Let \( C \) be the set of available school at a point \( x \). Then a set \( K \subseteq C \) is a recurrent communication class of the Markov chain \( M^C(x) \) if and only if \( H^C(x)_{K,K} \) is irreducible and \( H^C(x)_{K,C \setminus K} \) is the zero matrix.

Proposition 10 and Lemma 4 allow us to characterize the valid directions \( d(x) \).

**Theorem 4.** Let \( C \) be the set of available schools, and let \( \mathcal{K}(x) \) be the set of subsets \( K \subseteq C \) for which \( \widetilde{H}^C(x)_{K,K} \) is irreducible and \( \widetilde{H}^C(x)_{K,C \setminus K} \) is the zero matrix. Then the equation \( d = d \cdot \widetilde{H}^C(x) \) has a unique solution \( d^K \) that satisfies \( d^K \cdot 1 = -1 \) and \( \text{supp}(d^K) \subseteq K \), and its projection onto its support \( K \) has the form

\[
(d^K)_K = \begin{bmatrix} 0 & -1 \end{bmatrix} A^K_C(x)^{-1},
\]

where \( A^K_C(x) \) is the matrix obtained by replacing the \((|K| - 1)\)th column of \( \widetilde{H}^C(x)_{K,K} \) with the all ones vector \( 1_K \).

Moreover, if \( d \cdot 1 = -1 \) and \( d \) is a solution to the equation \( d = d \cdot \widetilde{H}^C(x) \), then \( d \) is a convex combination of the vectors in \( \{d^K\}_{K \in \mathcal{K}(x)} \).

**Proof.** Proposition 4 shows that the sets \( K \) are precisely the recurrent sets of the Markov chain with transition matrix \( \widetilde{H}(x) \). Hence uniqueness of the \( d^K \) and the fact that \( d \) is a convex combination of \( d^K \) follow directly from Proposition 10. The form of the solution \( d^K \) follows from Theorem 5.

This has the following interpretation. Suppose that there is a unique recurrent communication class \( K \), such as when \( \eta \) has full support. Then there is a unique
infinitesimal continuum trading cycle of students, specified by the unique direction \( d \) satisfying \( d = d \cdot \tilde{H} (x) \). Moreover, students in the cycle trade seats from every school in \( K \). Any school not in \( K \) is blocked from participating, since there is not enough demand to fill the seats they are offering. When there are multiple recurrent communication classes, each of the \( d^K \) gives a unique infinitesimal trading cycle of students, corresponding to those who trade seats in \( K \). Moreover, these trading cycles are disjoint. Hence the only multiplicity that remains is to decide the order, or the relative rate, at which to clear these cycles. We will show in Section D.3 that, as in the discrete setting, the order in which cycles are cleared does not affect the final allocation.

### D.3 Proof of Theorem 2

We first show that there exist solutions \( p, \gamma, t \) to the marginal trade balance equations and capacity equations. The proof relies on selecting appropriate valid directions \( d (x) \) and then invoking the Picard-Lindelöf theorem to show existence.

Specifically, let \( C \) be the set of available schools, fix a point \( x \), and consider the set of vectors \( d \) such that \( d \cdot H^C (x) = d \). Then it follows from Theorem 4 that if \( d (x) \) is the valid direction from \( x \) with minimal support under the shortlex order, then \( d (x) = d^K (x) \) for the element \( K (x) \in K (x) \) that is the smallest under the shortlex ordering. As the density \( \nu (\cdot) \) defining \( \eta (\cdot) \) is Lipschitz continuous, it follows that \( K (\cdot) \) and \( K (\cdot) \) are piecewise constant. Hence we may invoke Lemma 3 to conclude that \( d (\cdot) \) is piecewise Lipschitz within each piece, and hence piecewise Lipschitz in \([0, 1]^C \). Since \( d (\cdot) \) is piecewise Lipschitz, it follows from the Picard-Lindelöf theorem that there exists a unique function \( \gamma (\cdot) \) satisfying \( \frac{d \gamma (t)}{dt} = d (\gamma (t)) \). It follows trivially that \( \gamma \) satisfies the marginal trade balance equations, and since we have assumed that all students find all schools acceptable and there are more students than seats it follows that there exist runout times \( t^{(\cdot)} \) and cutoffs \( p^{(\cdot)} \).

**Proof of Uniqueness**

In this section, we prove part (ii) of Theorem 2, that any two valid TTC paths give equivalent allocations. The intuition for the result is the following. The connection to Markov chains shows that having multiple possible valid direction in the continuum is parallel to having multiple possible trade cycles in the discrete model. Hence
the only multiplicity in choosing valid TTC directions is whether to implement one set of trades before the others, or to implement them in parallel at various relative rates. We can show that the set of cycles is independent of the order in which cycles are selected, or equivalently that the sets of students who trade with each other is independent of the order in which possible trades are executed. It follows that any pair of valid TTC paths give the same final allocation.

We remark that the crux of the argument is similar to what shows that discrete TTC gives a unique allocation. However, the lack of discrete cycles and the ability to implement sets of trades in parallel both complicate the argument and lead to a rather technical proof.

We first formally define cycles in the continuum setting, and a partial order over the cycles corresponding to the order in which cycles can be cleared under TTC. We then define the set of cycles \( \Sigma(\gamma) \) associated with a valid TTC path \( \gamma \). Finally, we show that the sets of cycles associated with two valid TTC paths \( \gamma \) and \( \gamma' \) are the same, \( \Sigma(\gamma) = \Sigma(\gamma') \).

**Definition 3.** A (continuum) cycle \( \sigma = (K, x, \overline{x}) \) is a set \( K \subseteq C \) and a pair of vectors \( x \leq \overline{x} \) in \([0, 1]^C\). The cycle \( \sigma \) is valid for available schools \( \{C(x)\}_{x \in [0, 1]^C} \) if \( K \in K^C(x) \forall x \in (x, \overline{x}) \).

Intuitively, a cycle is defined by two time points in a run of TTC, which gives a set of students, and the set of schools they most desire. A cycle is valid if the set of schools involved is a recurrent communication class of the associated Markov chains.

We say that a cycle \( \sigma = (K, x, \overline{x}) \) appears at time \( t \) in TTC \( (\gamma) \) if \( K \in K^C(\gamma(t)) \) and \( \gamma_c(t) = \overline{x}_c \) for all \( c \in K \). We say that a student \( \theta \) is in cycle \( \sigma \) if \( r^\theta \in (x, \overline{x}) \), and a school \( c \) is in cycle \( \sigma \) if \( c \in K \).

**Definition 4** (Partial order over cycles). The cycle \( \sigma = (K, x, \overline{x}) \) blocks the cycle \( \sigma' = (K', x', \overline{x}') \), denoted by \( \sigma \triangleright \sigma' \), if at least one of the following hold:

- **(Blocking student)** There exists a student \( \theta \) in \( \sigma' \) who prefers a school in \( K \) to all those in \( K' \), that is, there exists \( \theta \) and \( c \in K \setminus K' \) such that \( c \succ^\theta c' \) for all \( c' \in K' \).

- **(Blocking school)** There exists a school in \( \sigma' \) that prefers a positive measure of students in \( \sigma \) to all those in \( \sigma' \), that is, there exists \( c \in K' \) such that \( \eta(\theta | \theta \in \sigma, r^\theta_c > \overline{x}_c) > 0 \).

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34 The set of students is given by taking the difference between two nested hyperrectangles, one with upper coordinate \( \overline{x} \) and the other with upper coordinate \( x \).

35 Recall that since \( r^\theta, x \) and \( \overline{x} \) are vectors, this is equivalent to saying that \( r^\theta \not\leq x \) and \( r^\theta \leq \overline{x} \).
There exists which exists by Theorem 4. Lemma 5. \( K \) depend only on \( 1 \) bounded away from zero on its support. Hence \( 4 \), since \( M \) Markov chain such that for all \( C \) \( R \) is right-continuous on \( [0,1] \). We may assume that for each interval \( T \) \( C \) disjoint union. We may assume that for each interval \( T \) \( C \) decreasing, it follows from Lemma (5) that \( K, \gamma \) such that for all \( C \subseteq C \) the function \( 1 \left( \bar{H}^C (\cdot) \right) \) is constant on \( R \) for all \( R \in \mathcal{R}' \).

**Definition 5.** The partition \( \mathcal{R}' \) is the minimal rectangular subpartition of \( \mathcal{R} \) such that for all \( C \subseteq C \) the function \( 1 \left( \bar{H}^C (\cdot) \right) \) is constant on \( R \) for all \( R \in \mathcal{R}' \).

For \( x \in [0,1] \) and \( C \subseteq C \), let \( \mathcal{K}^C (x) \) be the recurrent communication classes of the Markov chain \( M^C (x) \). The following lemma follows immediately from Proposition 4, since \( 1 \left( \bar{H}^C (\cdot) \right) \) is constant on \( R \forall R \in \mathcal{R}' \), and recurrent communication classes depend only on \( 1 \left( \bar{H} \right) \).

**Lemma 5.** \( \mathcal{K}^C (\cdot) \) is constant on \( R \) for every \( R \in \mathcal{R}' \).

For each \( K \in \mathcal{K}^C (x) \), let \( d^K (x) \) be the unique vector satisfying \( d = d\bar{H}^C (x) \), which exists by Theorem 4.

Let \( \gamma \) be a TTC path, and assume that the schools are indexed such that for all \( x \) there exists \( \ell \) such that \( C (x) = C^{(\ell)} \text{ def} \{ \ell, \ell + 1, \ldots, |C| \} \). For each set of schools \( K \subseteq C \), let \( T^{(\ell)} (K, \gamma) \) be the set of times \( \tau \) such that \( C (\gamma (\tau)) = C^{(\ell)} \) and \( K \) is a recurrent communication class for \( \bar{H}^{C^{(\ell)}} (\gamma (\tau)) \). Since \( \gamma \) is continuous and weakly decreasing, it follows from Lemma (5) that \( T^{(\ell)} (K, \gamma) \) is the finite disjoint union of intervals of the form \( [t, \bar{t}] \). Let \( \mathcal{I} (T^{(\ell)} (K, \gamma)) \) denote the set of intervals in this disjoint union. We may assume that for each interval \( T \), \( \gamma (T) \) is contained in some hyperrectangle \( R \in \mathcal{R}' \).

For a time interval \( T = [t, \bar{t}] \in \mathcal{I} (T^{(\ell)} (K, \gamma)) \), we define the cycle \( \sigma (T) = (K, \underline{x} (T), \bar{x} (T)) \) as follows. Intuitively, we want to define it simply as \( \sigma (T) = (K, \gamma (\underline{t}), \gamma (\bar{t})) \), but in order to minimize the dependence on \( \gamma \), we define the endpoints \( \underline{x} (T) \) and \( \bar{x} (T) \) of the interval of ranks to be as close together as possible.

---

\(^{36}\) We note that it is necessary but not sufficient that \( \underline{\gamma} > \bar{\gamma}' \).

\(^{37}\) This is without loss of generality, since if \( \gamma (T) \) is not contained we can simply partition \( T \) into a finite number of intervals \( \cup_{R \in \mathcal{R}'} (\gamma (T) \cap R) \), each contained in a hyperrectangle in \( \mathcal{R}' \).
while still describing the same set of students (up to a set of \( \eta \)-measure 0). Define
\[
\underline{x}(T) = \max \{ x : \gamma(t) \leq x \leq \gamma(t) , \eta(\theta : Ch_{\theta}(C(\ell)) \in K, r^\theta \in (x, \gamma(\bar{t}))] = 0 \}, \\
\bar{x}(T) = \min \{ x : \gamma(t) \leq x \leq \gamma(t) , \eta(\theta : Ch_{\theta}(C(\ell)) \in K, r^\theta \in (\gamma(t), x)] = 0 \} ,
\]
to be the points chosen to be maximal and minimal respectively such that the set of students allocated by \( \gamma \) during the time interval \( T \) has the same \( \eta \)-measure as if \( \gamma(t) = \underline{x}(\tau) \) and \( \gamma(\bar{t}) = \bar{x}(\tau) \). In other words \( \eta \left( (\cup_{c \in K} T^c_\ell (\gamma; \bar{t}) \setminus T^c_\ell (\gamma; t)) \setminus \{ \theta : Ch_{\theta}(C(\ell)) \in K, r^\theta \in (\underline{x}(T), \bar{x}(T)) \} \right) = 0 \). In a slight abuse of notation, if \( \sigma = \sigma(T) \) we will let \( \underline{x}(\sigma) \) denote \( \underline{x}(T) \) and \( \bar{x}(\sigma) \) denote \( \bar{x}(T) \).

**Definition 6.** The set of cycles cleared by \( TTC(\gamma) \) in round \( \ell \), denoted by \( \Sigma(\ell)(\gamma) \), is given by
\[
\Sigma(\ell)(\gamma) := \bigcup_{K \subseteq C(\ell)} \bigcup_{T \in I(T(\ell)(K,\gamma))} \sigma(T) .
\]
The set of cycles cleared by \( TTC(\gamma) \), denoted by \( \Sigma(\gamma) \), is the set of cycles cleared by \( TTC(\gamma) \) in some round \( \ell \),
\[
\Sigma(\gamma) := \bigcup_{\ell} \Sigma(\ell)(\gamma) .
\]

For any cycle \( \sigma \in \Sigma(\gamma) \) and time \( \tau \) we say that the cycle \( \sigma \) is clearing at time \( \tau \) if \( \gamma(\tau) \not\leq \underline{x}(\sigma) \) and \( \gamma(\tau) \not\geq \bar{x}(\sigma) \). We say that the cycle \( \sigma \) is cleared at time \( \tau \) or finishes clearing at time \( \tau \) if \( \gamma(\ell)(\tau) \leq \underline{x}(\sigma) \) with at least one equality. We remark that for any TTC path \( \gamma \) there may be multiple cycles clearing at a time \( \tau \), each corresponding to a different recurrent set. For any TTC path \( \gamma \) the set \( \Sigma(\gamma) \) is finite.

Fix two TTC paths \( \gamma \) and \( \gamma' \). Our goal is to show that they clear the same sets of cycles, \( \Sigma(\gamma) = \Sigma(\gamma') \), or equivalently that \( \Sigma(\gamma) \cup \Sigma(\gamma') = \Sigma(\gamma) \cap \Sigma(\gamma') \). We will do this by showing that for every cycle \( \sigma \in \Sigma(\gamma) \cup \Sigma(\gamma') \), if all cycles in \( \Sigma(\gamma) \cup \Sigma(\gamma') \) that block \( \sigma \) are in \( \Sigma(\gamma) \cap \Sigma(\gamma') \), then \( \sigma \in \Sigma(\gamma) \cap \Sigma(\gamma') \). We first show that this is true in a special case, which can be understood intuitively as the case when the cycle \( \sigma \) appears during the run of \( TTC(\gamma) \) and also appears during the run of \( TTC(\gamma') \).

**Lemma 6.** Let \( \mathcal{E} = (\mathcal{C}, \Theta, \eta, q) \) be a continuum economy, and let \( \gamma \) and \( \gamma' \) be two TTC paths for this economy. Let \( K \subseteq \mathcal{C} \) and \( \bar{t} \) be such that at time \( \bar{t} \), \( \gamma(\gamma') \) has available schools \( \mathcal{C}'(\mathcal{C}') \), the paths \( \gamma, \gamma' \) are at the same point when projected onto
the coordinates $K$, i.e. $\gamma(\ell)_K = \gamma'(\ell)_K$, and $K$ is a recurrent communication class of $M^C (\gamma (\ell))$ and of $M^{C'} (\gamma' (\ell))$. Suppose that for all schools $c \in K$ and cycles $\sigma' \triangleright \sigma$ involving school $c$, if $\sigma' \in \Sigma(\gamma)$, then $\sigma'$ is cleared in $TTC(\gamma')$, and vice versa. Suppose also that cycle $\sigma = (K, \underline{x}, \underline{\pi})$ is cleared in $TTC(\gamma)$, $\gamma(t) = \underline{x}$, and at most measure $0$ of $\sigma$ has been cleared by time $t$ in $TTC(\gamma')$. Then $\sigma$ is also cleared in $TTC(\gamma')$.

**Proof.** We define the ‘interior’ of the cycle $\sigma$ by $X = \{x : \underline{x}_c \leq x_c \leq \overline{x}_c \forall c \in K, x_c' \geq \underline{x}_c' \forall c' \not\in K\}$. Fix a time $u$ such that $\gamma'(u) \in X$ and let $D'$ denote the set of available schools at time $u$ in $TTC(\gamma')$. Then we claim that $K$ is a recurrent communication class of $M^{D'} (\gamma'(u))$, and that a similar result is true for $\gamma$ and a similarly defined $D$. The claim for $\gamma, D$ follows from the fact that $\sigma$ is cleared in $TTC(\gamma)$, $\sigma \in \Sigma(\gamma)$. It remains to show that the claim for $\gamma', D'$ is true. Intuitively, $D'$ is some subset of $C'$, where the schools in the set $C' \setminus D'$ only every point to $K$ and are never pointed to by $K$ and so the cycle remains intact. Formally, by Lemma 4 it suffices to show that $H^{D'} (x)_{K,K}$ is irreducible and $H^{D'} (x)_{K,D'\setminus K}$ is the zero matrix.

We first examine the differences between the matrices $H^{C'} (\gamma' (t))$ and $H^{D'} (\gamma' (u))$. Since $K$ is a recurrent communication class of $M^{C'} (\gamma' (u))$, it holds that $\mathbf{1} \left( H^{C'} (\gamma' (u))_{K,C'\setminus K} \right) = 0$, and so since $K \subseteq D' \subseteq C'$ it follows that $\mathbf{1} \left( H^{D'} (\gamma' (u))_{K,D'\setminus K} \right) = 0$. Moreover, since $\mathbf{1} \left( H^{C'} (\gamma' (u))_{K,C\setminus K} \right) = 0$, all students’ top choice schools out of $C'$ or $D'$ are the same (in $K$), and so $H^{C'} (\gamma' (u))_{K,K} = H^{D'} (\gamma' (u))_{K,K} = H^{D'} (\gamma' (u))_{K,K}$ and both matrices are irreducible. Hence $K$ is a recurrent communication class of $M^{D'} (\gamma' (u))$.

We now invoke Theorem 4 to show that in each of the two paths, all the students in the cycle $\sigma$ clear with each other. In other words, there exists a time $\tau$ such that $\gamma(\tau) = \overline{x}_c \forall c \in K$, and similarly there exists a time $\tau'$ such that $\gamma'(\tau') = \overline{x}_c' \forall c \in K$. Specifically, while the path $\gamma$ is in the ‘interior’ of the cycle, that is $\gamma(\tau) \in X$, it follows from Theorem 4 that the projection of the gradient of $\gamma$ to $K$ is a rescaling of some vector $d^K (\gamma(\tau))$, where $d^K (\cdot)$ depends on $H (\cdot)$ but not on $\gamma$. Similarly, while $\gamma'(\tau') \in X$, it holds that the projection of the gradient of $\gamma'$ to $K$ is a rescaling of the vector $d^K (\gamma'(\tau'))$, for the same function $d^K (\cdot)$. Hence if we let $\pi_K (x)$ denote the projection of a vector $x$ to the coordinates indexed by schools in $K$, then $\pi_K (\gamma(\gamma^{-1}(\langle \underline{x}, \overline{x} \rangle))) = \pi_K (\gamma' (\gamma'^{-1}(\langle \underline{x}, \overline{x} \rangle)))$.

Recall that we have assumed that for all schools $c \in K$ and cycles $\sigma' \triangleright \sigma$ involving school $c$, if $\sigma' \in \Sigma(\gamma)$, then $\sigma'$ is cleared in $TTC(\gamma')$, and vice versa. This implies
that for all \( c \in K \), the measure of students assigned to \( c \) in time \([0, t]\) under TTC (\( \gamma \)) is the same as the measure of students assigned to \( c \) in time \([0, t']\) under TTC (\( \gamma' \)). Moreover, we have just shown that for any \( x \in \gamma (\gamma^{-1} ((\bar{x}, x])) \), \( x' \in \gamma' (\gamma'^{-1} ((\bar{x}, x])) \) such that \( x_K = x'_K \), if we let \( \tau = \gamma^{-1} (x) \) and \( \tau' = (\gamma')^{-1} (x') \) then the same measure of students are assigned to \( c \) in time \([t, \tau]\) under TTC (\( \gamma \)) as in time \([t', \tau']\) under TTC (\( \gamma' \)). Since TTC (\( \gamma \)) clears \( \sigma \) the moment it exits the interior of \( \sigma \), this implies that TTC (\( \gamma' \)) also clears \( \sigma \) the moment it exits the interior. \( \square \)

We are now ready to prove that the TTC allocation is unique. As the proof takes several steps, we separate it into several smaller claims for readability.

**Proof of uniqueness.** Let \( \gamma \) and \( \gamma' \) be two TTC paths, and let the sets of cycles associated with TTC (\( \gamma \)) and TTC (\( \gamma' \)) be \( \Sigma = \Sigma (\gamma) \) and \( \Sigma' = \Sigma (\gamma') \) respectively. We will show that \( \Sigma = \Sigma' \).

Let \( \sigma = (K, \bar{x}, \pi) \) be a cycle in \( \Sigma \cup \Sigma' \) such that the following assumption holds:

**Assumption 2.** For all \( \tilde{\sigma} \triangleright \sigma \) it holds that either \( \tilde{\sigma} \) is in both \( \Sigma \) and \( \Sigma' \) or \( \tilde{\sigma} \) is in neither.

We show that if \( \sigma \) is in \( \Sigma \cup \Sigma' \) then it is in \( \Sigma \cap \Sigma' \). Since \( \Sigma \) and \( \Sigma' \) are finite sets, this will be sufficient to show that \( \Sigma = \Sigma' \). Without loss of generality we may assume that \( \sigma \in \Sigma \).

We give here an overview of the proof. Let \( \Sigma_{\triangleright \sigma} = \{ \tilde{\sigma} \in \Sigma : \tilde{\sigma} \triangleright \sigma \} \) denote the set of cycles that are comparable to \( \sigma \) and cleared before \( \sigma \) in TTC (\( \gamma \)). Assumption (2) about \( \sigma \) implies that \( \Sigma_{\triangleright \sigma} \subseteq \Sigma' \). We will show that this implies that no students in \( \sigma \) start clearing under TTC (\( \gamma' \)) until all the students in \( \sigma \) have the same top available school in TTC (\( \gamma' \)) as when they clear in TTC (\( \gamma \)), or in other words, that if some students in \( \sigma \) start clearing under TTC (\( \gamma' \)) at time \( t \), then the cycle \( \sigma \) appears at time \( t \). We will then show that once some of the students in \( \sigma \) start clearing under TTC (\( \gamma' \)) then all of them start clearing. It then follows from Lemma 6 that \( \sigma \) clears under both TTC (\( \gamma \)) and TTC (\( \gamma' \)).

Let \( \ell \) denote the round of TTC (\( \gamma \)) in which \( \sigma \) is cleared, \( C(x) = C^{(\ell)} \forall x \in \sigma \). We define the times in TTC (\( \gamma \)) and TTC (\( \gamma' \)) when all the cycles in \( \Sigma_{\triangleright \sigma} \) are cleared, by

\[
\tilde{t}_{\triangleright \sigma} = \min \left\{ t : \gamma(t) \leq (\bar{x}) \text{ for all } \tilde{\sigma} = (\tilde{K}, \tilde{\bar{x}}, (\tilde{\pi})) \in \Sigma_{\triangleright \sigma} \text{ and } H(\gamma(t)) \neq 0 \right\},
\]

\[
\tilde{t}'_{\triangleright \sigma} = \min \left\{ t : \gamma'(t) \leq (\bar{x}) \text{ for all } \tilde{\sigma} = (\tilde{K}, \tilde{\bar{x}}, (\tilde{\pi})) \in \Sigma_{\triangleright \sigma} \text{ and } H(\gamma'(t)) \neq 0 \right\}.
\]
We define also the times in $TTC(\gamma)$ when $\sigma$ starts to be cleared and finishes clearing,

$$t_\sigma = \max \{ t : \gamma(t) \geq \bar{x} \}, \ \bar{t}_\sigma = \min \{ t : \gamma(t) \leq \bar{x} \}$$

and similarly define the times $t'_\sigma = \max \{ t : \gamma'(t) \geq \bar{x} \}, \ \bar{t}'_\sigma = \min \{ t : \gamma'(t) \leq \bar{x} \}$ for $TTC(\gamma').$

We remark that part of the issue, carried over from the discrete setting, is that these times $t_\sigma$ and $\bar{t}_\sigma$ might not match up, and similarly for $t'_\sigma$ and $\bar{t}'_\sigma.$ In particular, other incomparable cycles could clear at interwoven times. In the continuum model, there may also be sections on the $TTC$ curve at which no school is pointing to a positive density of students. However, all the issues in the continuum case can be addressed using the intuition from the discrete case.

We first show in Claims (1), (2) and (3) that in both $TTC(\gamma)$ and $TTC(\gamma'),$ after all the cycles in $\Sigma_{\triangleright \sigma}$ are cleared and before $\sigma$ starts to be cleared, the schools pointed to by students in $\sigma$ and the students pointed to by schools in $K$ remain constant (up to a set of $\eta$-measure 0).

Claim 1. Let $\sigma = (K, \bar{x}, \bar{\pi}) \in \Sigma$ satisfy Assumption 2. Suppose there is a school $c$ that some student in $\sigma$ prefers to all the schools in $K.$ Then school $c$ is unavailable in $TTC(\gamma)$ at any time $t \geq t_\triangleright \sigma,$ and unavailable in $TTC(\gamma')$ at any time $t \geq t'_\triangleright \sigma.$

Proof. Suppose that school $c$ is available in $TTC(\gamma)$ after all the cycles in $\Sigma_{\triangleright \sigma}$ are cleared. Then there exists a cycle $\tilde{\sigma}$ clearing at time $\tilde{t} \in (t_\triangleright \sigma, \bar{t}_\sigma)$ in $TTC(\gamma)$ involving school $c.$ But this means that $\tilde{\sigma} \triangleright \sigma$ so $\tilde{\sigma} \in \Sigma_{\triangleright \sigma},$ which is a contradiction. Hence the measure of students in $\Sigma_{\triangleright \sigma}$ who are assigned to school $c$ is $q_c,$ and the claim follows.

Claim 2. In $TTC(\gamma),$ let $\tilde{\Theta}$ denote the set of students cleared in time $[t_\triangleright \sigma, \bar{t}_\sigma]$ who are preferred by some school in $c \in K$ to the students in $\sigma,$ that is, $\theta$ satisfying $r_{c\theta} > \bar{x}_c.$ Then $\eta(\tilde{\Theta}) = 0.$

Proof. Suppose $\eta(\tilde{\Theta}) > 0.$ Then, since there are a finite number of cycles in $\Sigma(\gamma),$ there exists some cycle $\tilde{\sigma} = (\tilde{K}, \tilde{\bar{x}}, (\tilde{\bar{\pi}})) \in \Sigma(\gamma)$ containing a positive $\eta$-measure of students in $\tilde{\Theta}.$ We show that $\tilde{\sigma}$ is cleared before $\sigma.$ Since $\tilde{\sigma}$ contains a positive $\eta$-measure of students in $\tilde{\Theta},$ it holds that there exist $t_1, t_2 \in [t_\triangleright \sigma, \bar{t}_\sigma]$ and a school $c \in K$ for which $\tilde{\bar{x}}_c \leq \gamma(t_1)_c < \gamma(t_2)_c \leq (\bar{x})_c.$ Hence $\bar{x}_c \leq \gamma(t_\sigma)_c \leq \gamma(t_1)_c < \gamma(t_2)_c \leq \bar{x}_c,$ so
\( \tilde{\sigma} \triangleright \sigma \) as claimed. But \((\tilde{x})_c \leq \gamma(t_1)_c < \gamma(t_2)_c \leq \gamma(\tilde{t}_{\triangleright \sigma})_c \), so \( \tilde{\sigma} \) is not cleared before \( \tilde{t}_{\triangleright \sigma} \), contradicting the definition of \( \tilde{t}_{\triangleright \sigma} \).

\( \square \)

Claim 3. In TTC (\( \gamma' \)), let \( \tilde{\Theta} \) denote the set of students cleared in time \([\tilde{t}_{\triangleright \sigma}, \tilde{t}'_{\sigma}]\) who are preferred by some school in \( \sigma \) to the students in \( \sigma \), that is, \( \theta \) satisfying \( r^\theta_c > \pi_c \). Then \( \eta(\tilde{\Theta}) = 0 \).

Proof. Suppose \( \eta(\tilde{\Theta}) > 0 \). Then, since there are a finite number of cycles in \( \Sigma(\gamma') \), there exists some cycle \( \tilde{\sigma} = (\tilde{K}, \tilde{x}, (\tilde{\pi})) \in \Sigma(\gamma') \) containing a positive \( \eta \)-measure of students in \( \tilde{\Theta} \). We show that \( \tilde{\sigma} \) is cleared before \( \sigma \). Since \( \tilde{\sigma} \) contains a positive \( \eta \)-measure of students in \( \tilde{\Theta} \), it holds that there exist \( t_1, t_2 \in [\tilde{t}_{\triangleright \sigma}, \tilde{t}'_{\sigma}] \) for which \( \tilde{x}_c \leq \gamma'(t_1)_c < \gamma'(t_2)_c \leq (\tilde{\pi})_c \). Hence \( \pi_c \leq \gamma'(t'_0)_c \leq \gamma'(t_1)_c < \gamma'(t_2)_c \leq \tilde{\pi}_c \), so \( \tilde{\sigma} \triangleright \sigma \) and must be cleared before \( \sigma \). Moreover, \((\tilde{x})_c \leq \gamma'(t_1)_c < \gamma'(t_2)_c \leq (\tilde{\pi})_c \), so it follows from the definition of \( \tilde{t}_{\triangleright \sigma} \) that \( \tilde{\sigma} \notin \Sigma_{\triangleleft \sigma} \), but since we assumed that \( \tilde{\sigma} \in \Sigma' \) it follows that \( \tilde{\sigma} \in \Sigma' \setminus \Sigma \), contradicting assumption (2) on \( \sigma \).

\( \square \)

We now show in Claims (4) and (5) that in both TTC (\( \gamma \)) and TTC (\( \gamma' \)) the cycle \( \sigma \) starts clearing when students in the cycle \( \sigma \) start clearing. We formalize this in the continuum model by considering the coordinates of the paths \( \gamma, \gamma' \) at the time \( t_{\sigma} \) when the cycle \( \sigma \) starts clearing, and showing that, for all coordinates indexed by schools in \( K \), this is equal to \( \pi \).

Claim 4. \( \pi_K(\gamma(t_{\sigma})) = \pi_K(\pi) \).

Proof. The definition of \( t_{\sigma} \) implies that \( \gamma(t_{\sigma})_c \geq \pi_c \) for all \( c \in K \). Suppose there exists \( c \in K \) such that \( \gamma(t_{\sigma})_c > \pi_c \). Since \( \sigma \) starts clearing at time \( t_{\sigma} \), for all \( \varepsilon > 0 \) school \( c \) must point to a non-zero measure of students in \( \sigma \) over the time period \([\varepsilon, t_{\sigma} + \varepsilon] \), whose scores \( r^\theta_c \) satisfy \( \gamma(t_{\sigma})_c \geq r^\theta_c \geq \gamma(t_{\sigma} + \varepsilon)_c \). For sufficiently small \( \varepsilon \) the continuity of \( \gamma(\cdot) \) and the assumption that \( \gamma(t_{\sigma})_c > \pi_c \) implies that \( r^\theta_c \geq \gamma(t_{\sigma} + \varepsilon)_c > \pi_c \), which contradicts the definition of \( \pi_c \).

\( \square \)

Claim 5. \( \pi_K(\gamma'(t'_{\sigma})) = \pi_K(\pi) \).

As in the proof of Claim (4), the definition of \( t'_{\sigma} \) implies that \( \gamma'(t'_{\sigma})_c \geq \pi_c = \gamma(t_{\sigma})_c \) for all \( c \in K \). Since we cannot assume that \( \sigma \) is the cycle that is being cleared at time \( t'_{\sigma} \) in TTC (\( \gamma' \)), the proof of Claim (5) is more complicated than that of the Claim (4) and takes several steps.
We rely on the fact that \( K \) is a recurrent communication class in \( \text{TTC}(\gamma) \), and that all cycles comparable to \( \sigma \) are already cleared in \( \text{TTC}(\gamma') \). The underlying concept is very simple in the discrete model, but is complicated in the continuum by the definition of the TTC path in terms of specific points, as opposed to measures of students, and the need to account for sets of students of \( \eta \)-measure 0.

Let \( K_\neq \) be the set of coordinates in \( K \) at which equality holds, \( \gamma'(\ell'_\sigma)_c = \gamma(\ell_\sigma)_c \), and let \( K_\geq \) be the set of coordinates in \( K \) where strict inequality holds, \( \gamma'(\ell'_\sigma)_c > \gamma(\ell_\sigma)_c \). It suffices to show that \( K_\geq \) is empty. We do this by showing that under \( \text{TTC}(\gamma') \) at time \( \ell'_\sigma \), every school in \( K_\geq \) points to a zero density of students, and some school in \( K_\neq \) points to a non-zero density of students, and so if both sets are non-empty this contradicts the marginal trade balance equations. In what follows, let \( C \) denote the set of available schools in \( \text{TTC}(\gamma) \) at time \( \ell_\sigma \), and let \( C' \) denote the set of available schools in \( \text{TTC}(\gamma') \) at time \( \ell'_\sigma \).

Claim 6. Suppose that \( c \in K_\geq \). Then there exists \( \epsilon > 0 \) such that in \( \text{TTC}(\gamma') \), the set of students pointed to by school \( c \) in time \([\ell'_\sigma, \ell'_\sigma + \epsilon]\) has \( \eta \)-measure 0, i.e. \( \tilde{H}^c(\gamma'(\ell'_\sigma))_{cb} = 0 \).

Proof. Since \( c \in K_\geq \) it holds that \( \gamma'(\ell'_\sigma)_c > \tilde{\pi}_c \), and since \( \gamma' \) is continuous, for sufficiently small \( \epsilon \) it holds that \( \gamma'(\ell'_\sigma + \epsilon)_c > \tilde{\pi}_c \). Hence the set of students that school \( c \) points to in time \([\ell'_\sigma, \ell'_\sigma + \epsilon]\) is a subset of those with score \( r^0_c \) satisfying \( \gamma'(\ell'_\sigma)_c \geq r^0_c \geq \gamma'(\ell'_\sigma + \epsilon)_c > \tilde{\pi}_c \). By assumption (2) and Claim (3) any cycle \( \tilde{\sigma} \) clearing some of these students contains at most measure 0 of them, since \( \tilde{\sigma} \) is cleared after \( \Sigma_{b_\sigma} \) and before \( \sigma \). Since there is a finite number of such cycles the set of students has \( \eta \)-measure 0.

Claim 7. If \( c \in K_\neq \), \( b \in K \) and \( \tilde{H}^c(\gamma(\ell_\sigma))_{cb} > 0 \), then \( \tilde{H}^c(\gamma'(\ell'_\sigma))_{cb} > 0 \).

Proof. Since every \( \tilde{H}^c(\gamma'(\ell'_\sigma))_{cb} \) is a positive multiple of \( H^{b,c}(\gamma'(\ell'_\sigma)) \), it suffices to show that \( H^{b,c}(\gamma'(\ell'_\sigma)) \geq 0 \). Let \( \Sigma_-(\epsilon) \defeq (\gamma'(\ell'_\sigma) - \epsilon \cdot e_c, \gamma'(\ell'_\sigma)] \). We first show that for sufficiently small \( \epsilon \) it holds that \( \eta^{b,c}(\Sigma_-(\epsilon)) = \Omega(\epsilon) \). Let \( \Sigma_-(\epsilon) \defeq (\gamma(\ell_\sigma) - \epsilon \cdot e_c, \gamma(\ell_\sigma)] \). Since \( \tilde{H}^c(\gamma(\ell_\sigma))_{cb} > 0 \), it follows from the definition of \( H^{b,c}(\cdot) \) that \( H^{b,c}(x) \defeq \lim_{\epsilon \to 0} \frac{1}{\epsilon} \eta^{b,c}(\Sigma_-(\epsilon)) > 0 \) and hence \( \eta^{b,c}(\Sigma_-(\epsilon)) = \Omega(\epsilon) \) for sufficiently small \( \epsilon \). Moreover, at most \( \eta \)-measure 0 of the students in \( \Sigma_-(\epsilon) \) are not in the cycle \( \sigma \). Finally, \( \Sigma_-(\epsilon) \supseteq \Sigma_-(\epsilon) \setminus \Sigma_+(\epsilon) \), where \( \Sigma_+(\epsilon) \defeq (\gamma(\ell_\sigma) + \epsilon \cdot e_c, \gamma(\ell_\sigma)] \). If \( \epsilon < \tilde{\pi}_c - \tilde{\pi}_c \) then \( \eta \)-measure 0 of the students in \( \Sigma_+(\epsilon) \) are not cleared by cycle \( \sigma \). Hence \( \eta^{b,c}(\Sigma_-(\epsilon)) \geq \eta^{b,c}(\Sigma_-(\epsilon)) - \eta^{b,c}(\Sigma_+(\epsilon)) = \Omega(\epsilon) \).
Suppose for the sake of contradiction that \( H_c^{b[C']} (\gamma' (t'_a)) = \lim_{\varepsilon \to 0} \frac{1}{2\eta^{b[C']} (\Sigma (\varepsilon))} = 0 \), so that \( \eta^{b[C']} (\Sigma (\varepsilon)) = o (\varepsilon) \) for sufficiently small \( \varepsilon \). Then there is a school \( b' \neq b \) and type \( \theta \in \Theta^{b[C]} \cap \Theta^{b'[C']} \) such that there is an \( \eta \)-measure \( \Omega (\varepsilon) \) of students in \( \sigma \) with type \( \theta \). Since \( b' \in C' \) it is available in \( TTC (\gamma') \) at time \( t'_a \), and by Claim (1) it holds that \( b' \in K \). Moreover, \( \theta \in \Theta^{b[C]} \) implies that \( \theta \) prefers school \( b \) to all other schools in \( K \), so \( b = b' \), contradiction. \( \square \)

Proof of Claim (5). Suppose for the sake of contradiction that \( K_\succ \) is nonempty. Since some students in \( \sigma \) are being cleared in \( TTC (\gamma') \) at time \( t'_a \), by Claim (3) there exists \( c \in K = K_\neq \cup K_\succ \) and \( b \in K \) such that \( \tilde{H}^{C'} (\gamma' (t'_a))_{cb} > 0 \). If \( c \in K_\succ \) this contradicts Claim (6). If \( c \in K_\neq \), then \( \tilde{H}^{C'} (\gamma (t_a))_{cb} > 0 \) and so by Claim (1) \( \tilde{H}^{C'} (\gamma (t_a))_{cb} > 0 \). Moreover, \( K = K_\neq \cup K_\succ \) is a recurrent communication class of \( \tilde{H}^{C} (\gamma (t_a)) \), so there exists a chain \( c = c_0 - c_1 - c_2 - \cdots - c_n \) such that \( \tilde{H}^{C} (\gamma (t_a))_{c_i c_{i+1}} > 0 \) for all \( i < n, c_i \in K \) for all \( i < n-1 \), and \( c_{n-1} \in K_\succ \). By Claim (7) \( \tilde{H}^{C'} (\gamma' (t'_a))_{c_i c_{i+1}} > 0 \) for all \( i < n \). But since \( c_{n-1} \in K_\succ \), by Claim (6) \( \tilde{H}^{C'} (\gamma (t_a))_{c_{n-1} c_n} = 0 \), which gives the required contradiction. \( \square \)

Proof that \( \Sigma = \Sigma' \). We have shown in Claims (4) and (5) that for our chosen \( \sigma = (K, \Xi, \bar{x}) \), it holds that \( \gamma (t_a) = \gamma' (t'_a) = \bar{x}_K \). Invoking Claims (2) and (3) and Lemma 6 shows that \( \sigma \) is cleared under both \( TTC (\gamma) \) and \( TTC (\gamma') \). Hence \( \Sigma = \Sigma' \), as required. \( \square \)

D.4 Proof of Proposition 2

In this section, we show that given a discrete economy, the cutoffs of TTC in a continuum embedding \( \Phi \) give the same assignment as TTC on the discrete model,

\[
\mu_{\text{TTC}} (s \mid E) = \max \{ c : \eta^* \geq \theta^* \text{ for some } b \} = \mu_{\text{TTC}} (\theta^* \mid \Phi (E)) \forall \theta^* \in I^*.
\]

The intuition behind this result is that TTC is essentially performing the same assignments in both models, with discrete TTC assigning students to schools in discrete steps, and continuum TTC assigning students to schools continuously, in fractional amounts. By considering the progression of continuum TTC at the discrete time steps when individual students are fully assigned, we obtain the same outcome as discrete TTC.
Proof. For a discrete economy \( E = (\mathcal{C}, \mathcal{S}, \succ_c, \succ^s, q) \) with \( N = |\mathcal{S}| \) students, we define the continuum economy \( \Phi(E) = (\mathcal{C}, \Theta, \eta, \frac{q}{N}) \) as follows. For each student \( s \in \mathcal{S} \) and school \( c \in \mathcal{C} \), recall that \( r^s_c = \frac{|\{s' | s \succ_c s'\}|}{|\mathcal{S}|} \) is the percentile rank of \( s \) at \( c \). We identify each student \( s \in \mathcal{S} \) with the \( N \)-dimensional cube \( I^s = \succ^s \times \prod_{c \in \mathcal{C}} [r^s_c, r^s_c + \frac{1}{N}] \) of student types, and define \( \eta \) to have constant density \( \frac{1}{N} \cdot N^N \) on \( \cup_s I^s \) and \( 0 \) everywhere else.

We construct a discrete cycle selection rule \( \psi \) and TTC path \( \gamma \) such that TTC on the discrete economy \( E \) with cycle selection rule \( \psi \) gives the same allocation as \( TTC(\gamma) \). Since the assignment of discrete TTC is unique (Shapley and Scarf, 1974), and the assignment in the continuum model is unique (Proposition 2), this proves the theorem.

The discrete cycle selection rule \( \psi \) is defined by taking all available cycles in the pointing graph obtained by having students point to their favorite school, and schools to their favorite student. The TTC path \( \gamma \) is defined by taking valid directions \( d(x) \) that essentially use all available cycles in the pointing graph. Formally, at each point \( x \), let \( C \) be the set of available schools, let \( K(x) \) be the set of all students in recurrent communication classes of \( \tilde{H}(x) \), and let \( d_c(x) = \frac{1}{|K(x)|} \) if \( c \in K \) and \( 0 \) otherwise.

Let \( X \) be the set of points \( x \) such that \( x_c \) is a multiple of \( \frac{1}{N} \) for all \( c \notin K(x) \); we will show that the TTC path stays within this set of points. Note that for each \( x \) the matrix \( N \times \tilde{H}(x) \) is the adjacency matrix of the pointing graph (where school \( b \) points to school \( c \) if some student pointed to by \( b \) wants \( c \) ), and so \( d(x) = d(x) \cdot \tilde{H}(x) \) for all \( x \in X \). Now consider the TTC path \( \gamma \) satisfying \( \gamma'(t) = d(\gamma(t)) \). The path starts at \( \gamma(0) = 1 \in X \). Moreover, at any time \( t \), if \( \gamma(t) \in X \) then \( \gamma'(t) = d(\gamma(t)) \) points along the diagonal in the projection onto the coordinates \( K \), and is \( 0 \) along all other coordinates. Hence \( \gamma(t) \in X \) for all \( t \).

We now show that by considering the progression of continuum TTC at the discrete time steps when individual students are fully assigned, we obtain the same cycles and outcome as discrete TTC. Let \( t_1, t_2, \ldots \) be the discrete set of times when a student \( s \) is first fully assigned, that is \( \{t_i\} = \bigcup_s \{\inf \{t | \exists c \in C \text{ s.t. } \gamma_c(t) \leq r^s_c \forall \theta \in I^s\}\} \). For every two students \( s, s' \) and school \( c \) it holds that the projections \( I^s_c \) and \( I^{s'}_c \) of \( I^s \) and \( I^{s'} \) onto the \( c \)th coordinates are non-overlapping, i.e. either for all \( \theta \in I^s, \theta' \in I^{s'} \) it holds that \( r^s_c < r^{s'}_c \), or for all \( \theta \in I^s, \theta' \in I^{s'} \) it holds that \( r^s_c > r^{s'}_c \). Since all the capacities are multiples of \( \frac{1}{N} \), it follows that \( \gamma_c(t_i) \) is a multiple of \( \frac{1}{N} \) for all \( c, i \) and schools fill at a subset of the set of times \( \{t_i\} \).
In other words, we have shown that for every $i$, if $S$ is the set students who are allocated a seat at time $t_i$, then $S \cup \mu (S)$ are the agents in the maximal union of cycles in the pointing graph at time $t_{i-1}$. Hence $\gamma$ finishes clearing the cubes corresponding to the same set of cycles at time $t_i$ as $\psi$ clears in step $i$. It follows that $\mu_d T T C (s | E) = \mu_{c T T C} (\theta^s | \Phi (E))$, and by definition it holds that $\mu_{c T T C} (\theta^s | \Phi (E)) = \max_{b \in B} \{ c : r^s_b \geq p^c_b \text{ for some } b \}$.

### D.5 Proof of Theorem 3

To prove Theorem 3, we will want some way of comparing two TTC paths $\gamma$ and $\tilde{\gamma}$ obtained under two continuum economies differing only in their measures $\eta$ and $\tilde{\eta}$.

**Definition 7.** Let $\gamma$ and $\tilde{\gamma}$ be increasing continuous functions from $[0, 1]$ to $[0, 1]^C$ with $\gamma (0) = \tilde{\gamma} (0)$. We say that $\gamma (\tau)$ dominates $\tilde{\gamma} (\tau)$ via school $c$ if

$$
\gamma_c (\tau) = \tilde{\gamma}_c (\tau), \quad \text{and} \quad \gamma_b (\tau) \leq \tilde{\gamma}_b (\tau) \text{ for all } b \in C.
$$

We remark that, somewhat unintuitively, the condition $\gamma (\tau) \leq \tilde{\gamma} (\tau)$ implies that more students are offered seats under $\gamma$ than $\gamma'$, since higher ranks give more restrictive sets. We also say that $\gamma$ dominates $\tilde{\gamma}$ via school $c$ at time $\tau$. If $\gamma$ and $\gamma'$ are TTC paths, we can interpret this as school $c$ being more demanded under $\gamma$, since with the same rank at $c$, in $\gamma$ students are competitive with more ranks at other schools $b$.

We now show that any two non-increasing continuous paths $\gamma$, $\gamma'$ starting and ending at the same point can be re-parametrized so that for all $t$ there exists a school $c(t)$ such that $\gamma$ dominates $\gamma'$ via school $c(t)$ at time $t$. We first show that, if $\gamma (0) \leq \tilde{\gamma} (0)$, then there exists a re-parametrization of $\gamma$ such that $\gamma$ dominates $\gamma'$ on some interval starting at 0.

**Lemma 7.** Suppose $\gamma$, $\tilde{\gamma}$ are a pair of non-increasing functions $[0, 1] \to [0, 1]^C$ such that $\gamma (0) \leq \tilde{\gamma} (0)$. Then there exist coordinates $c, b$, a time $\bar{t}$ and an increasing function $g : \mathbb{R} \to \mathbb{R}$ such that $\gamma_b (g(\bar{t})) = \tilde{\gamma}_b (\bar{t})$, and for all $\tau \in [0, \bar{t}]$ it holds that $\gamma_c (g(\tau)) = \tilde{\gamma}_c (\tau)$ and $\gamma (g(\tau)) \leq \tilde{\gamma} (\tau)$.
That is, if we renormalize the time parameter $\tau$ of $\gamma(\tau)$ so that $\gamma$ and $\tilde{\gamma}$ agree along the $c$th coordinate, then $\gamma$ dominates $\tilde{\gamma}$ via school $c$ at all times $\tau \in [0, \tilde{t}]$, and also dominates via school $b$ at time $\tilde{t}$.

Proof. The idea is that if we take the smallest function $g$ such that $\gamma_c(g(\tau)) = \tilde{\gamma}_c(\tau)$ for some coordinate $c$ and all $\tau$ sufficiently small, then $\gamma(g(\tau)) \leq \tilde{\gamma}(\tau)$ for all $\tau$ sufficiently small. The lemma then follows from continuity. We make this precise.

Fix a coordinate $c$. Let $g^{(c)}$ be the renormalization of $\gamma$ so that $\gamma$ and $\tilde{\gamma}$ agree along the $c$th coordinate, i.e. $\gamma_c(g^{(c)}(\tau)) = \tilde{\gamma}_c(\tau)$ for all $\tau$.

For all $\tau$, we define the set $\kappa^\tau_\gamma(\tau) = \{b \mid \gamma_b(g^{(c)}(\tau)) > \tilde{\gamma}_b(\tau)\}$ of schools $b$ along which the $\gamma$ curve renormalized along coordinate $c$ has larger $b$-value at time $\tau$ than $\tilde{\gamma}_b$ has at time $\tau$, and similarly define the set $\kappa^\tau_\gamma(\tau) = \{b \mid \gamma_b(g^{(c)}(\tau)) = \tilde{\gamma}_b(\tau)\}$ where the renormalized $\gamma$ curve is equal to $\tilde{\gamma}$. It suffices to show that there exists $b, c$ and a time $\tilde{t}$ such that $\kappa^\tau_\gamma(\tau) = \emptyset$ for all $\tau \in [0, \tilde{t}]$ and $b \in \kappa^\tau_\gamma(\tilde{t})$.

Since $\gamma$ and $\tilde{\gamma}$ are continuous, there exists some maximal $\tilde{t}^{(c)} > 0$ such that the functions $\kappa^\tau_\gamma(\tau)$ are constant over the interval $\left(0, \tilde{t}^{(c)}\right)$. If there exists $c$ such that $\kappa^\tau_\gamma(\tau) = \emptyset$ for all $\tau \in (0, \tilde{t}^{(c)})$ then by continuity there exists some time $\tilde{t} \leq \tilde{t}^{(c)}$ and school $b$ such that $b \in \kappa^\tau_\gamma(\tilde{t})$ and we are done. Hence we may assume that for all $c$ it holds that $\kappa^\tau_\gamma(\tau) = C^{(c)}_\gamma(\tilde{t})$ for all $\tau \in \left(0, \tilde{t}^{(c)}\right)$ for some fixed non-empty set $C^{(c)}_\gamma$. We will show that this leads to a contradiction.

We first claim that if $b \in C^{(c)}_\gamma$, then $g^{(b)}(\tau) > g^{(c)}(\tau)$ for all $\tau \in (0, \tilde{t})$. This is because $\gamma$ is increasing and $\gamma_b(g^{(b)}(\tau)) = \tilde{\gamma}_b(\tau) > \gamma_b(g^{(c)}(\tau))$ for all $\tau \in (0, \tilde{t})$, where the equality follows from the definition of $g^{(b)}$ and the inequality since $b \in C^{(c)}_\gamma$. But this completes the proof, since it implies that for all $c$ there exists $b$ such that $g^{(b)}(\tau) > g^{(c)}(\tau)$ for all $\tau \in (0, \tilde{t})$, which is impossible since there are a finite number of schools $c \in \mathcal{C}$.

We are now ready to show that there exists a re-parametrization of $\gamma$ such that $\gamma$ always dominates $\tilde{\gamma}$ via some school.

Lemma 8. Suppose $\tilde{t} \geq 0$ and $\gamma, \tilde{\gamma}$ are a pair of non-increasing functions $[0, \tilde{t}] \to [0, 1]^C$ such that $\gamma(0) \leq \tilde{\gamma}(0) = 1$ with equality on at least one coordinate, and $0 = \gamma(1) \leq \tilde{\gamma}(1)$ with equality on at least one coordinate. Then there exists an increasing function $g : [0, \tilde{t}] \to \mathbb{R}$ such that for all $\tau \geq 0$, there exists a school $c(\tau)$ such that $\gamma(g(\tau))$ dominates $\tilde{\gamma}(\tau)$ via school $c(\tau)$.
\textbf{Proof.} Without loss of generality let us assume that $\mathcal{I} = 1$. Fix a coordinate $c$. We define $g^{(c)}$ to be the renormalization of $\gamma$ so that $\gamma$ and $\tilde{\gamma}$ agree along the $c$th coordinate. Formally, let $\tilde{t}^{(c)} = \min \{ \tau \mid \gamma_c (0) \geq \tilde{\gamma}_c (\tau) \}$ and define $g^{(c)}$ so that $\gamma_c (g^{(c)} (\tau)) = \tilde{\gamma}_c (\tau)$ for all $\tau \in \lbrack \tilde{t}^{(c)}, 1 \rbrack$. Let $A^{(c)}$ be the set of times $\tau$ such that $\gamma (g^{(c)} (\tau))$ dominates $\tilde{\gamma} (\tau)$. The idea is to pick $g$ to be equal to $g^{(c)}$ in $A^{(c)}$. In order to do this formally, we need to show that the sets $A^{(c)}$ cover $[0, 1]$, and then turn (a suitable subset of) $A^{(c)}$ into a union of disjoint closed intervals, on each of which we can define $g(\cdot) \equiv g^{(c)} (\cdot)$.

We first show that $\cup_c A^{(c)} = [0, 1]$. Suppose not, so there exists some time $\tau$ such that for all $c$ such that $\tau \geq \tilde{t}^{(c)}$ there exists $b$ such that $\gamma_b (g^{(c)} (\tau)) > \tilde{\gamma}_b (\tau)$. This implies that $\tilde{\gamma}_b (\tau) \leq \gamma_b (0)$, and so there exists $g^{(b)}$ such that $\tilde{\gamma}_b (\tau) = \gamma_b (g^{(b)} (\tau))$. Since $\gamma$ is increasing, this implies that for all $c$ such that $\tau \geq \tilde{t}^{(c)}$ there exists $b$ such that $g^{(c)} (\tau) < g^{(b)} (\tau)$, which is a contradiction since the set of such schools is finite but non-empty (since $\gamma (0) \leq \tilde{\gamma} (0) = 1$, with equality on at least one coordinate).

We now turn (a suitable subset of $A^{(c)}$) into a union of disjoint closed intervals. By continuity, $A^{(c)}$ is closed. Consider the closure of the interior of $A^{(c)}$, which we denote by $B^{(c)}$. Since the interior of $A^{(c)}$ is open, it is a countable union of open intervals, and hence $B^{(c)}$ is a countable union of disjoint closed intervals. To show that $\cup_c B^{(c)} = [0, 1]$, fix a time $\tau \in [0, 1]$. As $\cup_c A^{(c)} = [0, 1]$, there exists $c$ such that $\gamma (g^{(c)} (\tau)) \leq \tilde{\gamma} (\tau)$. Hence we may invoke Lemma 7 to show that there exists some school $b$, time $\tau > \tau$ and an increasing function $g$ such that $\gamma_b (g (g^{(c)} (\tau'))) = \tilde{\gamma}_b (\tau')$ and $\gamma (g (g^{(c)} (\tau'))) \leq \tilde{\gamma} (\tau')$ for all $\tau' \in [\tau, \overline{\tau}]$. But by the definition of $g^{(b)} (\cdot)$ this means that $\gamma_b (g (g^{(c)} (\tau'))) = \tilde{\gamma}_b (\tau') = \gamma_b (g^{(b)} (\tau'))$ for all $\tau' \in [\tau, \overline{\tau}]$, and so $g \circ g^{(c)} = g^{(b)}$ and we have shown that $[\tau, \overline{\tau}] \subseteq B^{(b)}$. Hence we may write $[0, 1] = \cup_n \{ T_n \}$ as a countable union of closed intervals such that any pair of intervals intersects at most their endpoints, and each interval $T_n$ is a subset of $B^{(c)}$ for some $c$. For each $T_n$, fix some $c(n) = c$ so that $T_n \subseteq B^{(c)}$. Intuitively, this means that at any time $\tau \in T_n$, it holds that $\gamma (g^{(c(n))} (\tau))$ dominates $\tilde{\gamma} (\tau)$ via school $c (n)$.

We now construct a function $g$ that satisfies the required properties as follows. If $\tau \in T_n \subseteq B^{(i)}$, let $g (\tau) = g^{(c)} (\tau)$. Now $g$ is well-defined despite the possibility that $T_n \cap T_m \neq \emptyset$. This is because if $\tau$ is in two different intervals $T_n, T_m$, then $\gamma_{c(n)} (g^{(c(n))} (\tau)) = \tilde{\gamma}_{c(n)} (\tau) \geq \gamma_{c(n)} (g^{(c(m))} (\tau))$ (by domination via $c (n)$ and $c (m)$ respectively), and $\gamma_{c(m)} (g^{(c(m))} (\tau)) = \tilde{\gamma}_{c(m)} (\tau) \geq \gamma_{c(m)} (g^{(c(n))} (\tau))$ (by domination via $c (m)$ and $c (n)$ respectively), and so $g^{(c(n))} (\tau) \leq g^{(c(m))} (\tau) \leq g^{(c(n))} (\tau)$ and we can pick one value for $g$ that satisfies all required properties. Now by definition
\( \gamma(g(\tau)) \) dominates \( \tilde{\gamma}(\tau) \) via school \( c(\tau) = c(n) \), and moreover \( g \) is defined on all of \([0,1]\) since \( \cup_{c \in C} B^{(c)} = [0,1] \). This completes the proof. \( \square \)

Consider two continuum economies \( E = (C, \Theta, \eta, q) \) and \( \tilde{E} = (C, \Theta, \tilde{\eta}, q) \), where the measures \( \eta \) and \( \tilde{\eta} \) satisfy the assumptions given in Section 3. Suppose also that the measure \( \eta \) and \( \tilde{\eta} \) have total variation distance \( \varepsilon \) and have full support. Let \( \gamma \) be a TTC path for economy \( E \), and let \( \tilde{\gamma} \) be a TTC path for economy \( \tilde{E} \). Consider any school \( c \) and any points \( x \in Im(\gamma), \tilde{x} \in Im(\tilde{\gamma}) \) such that \( x_c = \tilde{x}_c \), and both are cleared in the first round of their respective TTC runs, \( C(x|\gamma) = C, \ C(c|\tilde{\gamma}) = \tilde{C} \). We show that the set of students allocated to school \( c \) when running \( TTC(\gamma) \) up to \( x \) differs from the set of students allocated to school \( c \) when running \( TTC(\tilde{\gamma}) \) up to \( \tilde{x} \) by a set of measure \( O(\varepsilon|C|) \).

**Proposition 11.** Suppose that \( \gamma, \tilde{\gamma} \) are TTC paths in one round of the continuum economies \( E \) and \( \tilde{E} \) respectively, where the set of available schools \( C \) is the same in these rounds of \( TTC(\gamma) \) and \( TTC(\tilde{\gamma}) \). Suppose also that \( \gamma \) starts and ends at \( x, y \) and \( \tilde{\gamma} \) starts and ends at \( \tilde{x}, \tilde{y} \), where there exist \( b,c \in C \) such that \( x_b = \tilde{x}_b, y_c = \tilde{y}_c \), and \( x_a \leq \tilde{x}_a, y_a \leq \tilde{y}_a \) for all \( a \in C \). Then for all \( c \in C \), the set of students with ranks in \( (y,x] \) under \( E \) and ranks in \( (\tilde{y},\tilde{x}] \) under \( \tilde{E} \) who are assigned to \( c \) under \( TTC(\gamma) \) and not under \( TTC(\tilde{\gamma}) \) has measure \( O(\varepsilon|C|) \).

**Proof.** By Lemma 8, we may assume without loss of generality that \( \gamma \) and \( \tilde{\gamma} \) are parametrized such that \( x = \gamma(0), y = \gamma(1) \) and \( \tilde{x} = \tilde{\gamma}(0), \tilde{y} = \tilde{\gamma}(1) \), and for all times \( \tau \leq 1 \) there exists a school \( c(\tau) \) such that \( \gamma(\tau) \) dominates \( \tilde{\gamma}(\tau) \) via school \( c(\tau) \).

Let \( T_c = \{ \tau \leq 1 : c(\tau) = c \} \) be the times when \( \gamma \) dominates \( \tilde{\gamma} \) via school \( c \). We remark that, by our construction in Lemma 8, we may assume that \( T_c \) is the countable union of disjoint closed intervals, and that if \( c \neq c' \) then \( T_c \) and \( T_{c'} \) have disjoint interiors.

Since \( \gamma \) is a TTC path for \( E \) and \( \tilde{\gamma} \) is a TTC path for \( \tilde{E} \), by integrating over the marginal trade balance equations we can show that the following trade balance equations hold,

\[
\eta(T_c(\gamma; T_c)) = \eta(T^{c}_{|C}(\gamma; T_c)) \quad \text{for all } c \in C. \tag{4}
\]
\[
\tilde{\eta}(T_c(\tilde{\gamma}; T_c)) = \tilde{\eta}(T^{c}_{|C}(\tilde{\gamma}; T_c)) \quad \text{for all } c \in C. \tag{5}
\]

\(^{38}\)This is according to both measures \( \eta \) and \( \tilde{\eta} \).
Since \( \gamma \) dominates \( \tilde{\gamma} \) via school \( b \) at all times \( \tau \in T_b \), we have that
\[
T_b (\gamma; T_b) \subseteq T_b (\tilde{\gamma}; T_b). \tag{6}
\]

Moreover, by the choice of parametrization, \( \cup_b T_b = [0, 1] \) and so, since \( x \leq \tilde{x} \),
\[
\cup_{b,c} T_b (\gamma; T_b) \supseteq \cup_{b,c} T_b (\tilde{\gamma}; T_b). \tag{7}
\]

Now since \( \eta, \tilde{\eta} \) have total variation \( \varepsilon \), for every school \( c \) it holds that
\[
\eta \left( T^c \cap (\gamma; T_b) \right) \cap T^c \cap (\tilde{\gamma}; T_b) \right) \leq \eta \left( T^c \cap (\gamma; T_b) \right) - \eta \left( T^c \cap (\tilde{\gamma}; T_b) \right) + \varepsilon \text{ (by (7))}
= \eta \left( T_c (\gamma; T_b) \right) - \eta \left( T_c (\tilde{\gamma}; T_b) \right) + \varepsilon \text{ (by (4) and (5))}
\leq 2 \varepsilon \text{ (by (6))}, \tag{8}
\]

Also, for all schools \( b \neq c \), since \( \eta \) has full support, it holds that
\[
\eta \left( T^c \cap (\gamma; T_b) \right) \cap T^c \cap (\tilde{\gamma}; T_b) \right) \leq \frac{M}{m} \eta \left( T^b \cap (\gamma; T_b) \right) \cap T^b \cap (\tilde{\gamma}; T_b) \right). \tag{9}
\]

Hence, as \( T_b \) have disjoint interiors,
\[
\eta \left( T^c \cap (\gamma; 1) \right) \cap T^c \cap (\tilde{\gamma}; 1) \right) = \sum_{b \in C} \left( \eta \left( T^c \cap (\gamma; T_b) \right) - \eta \left( T^c \cap (\tilde{\gamma}; T_b) \right) \right) \text{ (by (7))}
\leq \sum_{b \in C} \eta \left( T^c \cap (\gamma; T_b) \right) \cap T^c \cap (\tilde{\gamma}; T_b) \right)
\leq \sum_{b \in C} \frac{M}{m} \eta \left( T^c \cap (\gamma; T_b) \right) \cap T^c \cap (\tilde{\gamma}; T_b) \right) \text{ (by (9))}
\leq 2 |C| \varepsilon \frac{M}{m} \text{ (by (8))}.
\]

That is, given a school \( c \), the set of students assigned to school \( c \) with score \( r^c \leq x \) under \( \gamma \) and not assigned to school \( c \) with score \( r^c \leq \tilde{x} \) under \( \tilde{\gamma} \) has \( \eta \)-measure \( O (\varepsilon |C|) \). The result for \( \tilde{\eta} \) follows from the fact that the total variation distance of \( \eta \) and \( \tilde{\eta} \) is \( \varepsilon \).

We are now ready to prove Theorem 3.

**Proof of Theorem 3.** Assume without loss of generality that the schools are indexed
such that the stopping times $t^{(e)}$ for $TTC(\gamma)$ satisfy $t^{(1)} \leq t^{(2)} \leq \cdots \leq t^{(|C|)}$ i.e. school $c_e$ is the $\ell$th school to reach capacity under $TTC(\gamma)$. Let $\sigma$ be a permutation such that the stopping times $\bar{t}^{(e)}$ for $TTC(\bar{\gamma})$ satisfy $t^{(\sigma(1))} \leq \cdots \leq t^{(\sigma(|C|))}$. We show by induction on $\ell$ that $\sigma(\ell) = \ell$ and that for all schools $c$, the set of students assigned to $c$ under $TTC(\gamma)$ by time $t^{(\ell)}$ and not under $TTC(\bar{\gamma})$ by time $\bar{t}^{(\sigma(\ell))}$ has $\eta$-measure $O(\varepsilon \ell |C|)$. This will prove the theorem.

We first consider the base case $\ell = 1$. Let $x = \bar{x} = \gamma(0)$ and $y = \gamma(t^{(1)})$. Define $\bar{y} \in Im(\bar{\gamma})$ to be the minimal point such that $y \leq \bar{y}$ and there exists $c$ such that $y_c = \bar{y}_c$. We show that $\bar{y}$ is near $\bar{\gamma}(\bar{t}^{(1)})$, i.e. $|\bar{y} - \bar{\gamma}(\bar{t}^{(1)})|_2 = O(\varepsilon)$. Now by Proposition 11 the set of students with ranks in $(y, \gamma(0)]$ under $\mathcal{E}$ and ranks in $(\bar{y}, \gamma(0)]$ under $\bar{\mathcal{E}}$ who are assigned to $c$ under $TTC(\gamma)$ and not under $TTC(\bar{\gamma})$ has $\bar{\eta}$-measure $O(\varepsilon |C|)$. Hence the residual capacity of school 1 at $\bar{y}$ under $TTC(\bar{\gamma})$ is $O(\varepsilon |C|)$, and so since $\bar{\eta}$ has full support and has density bounded from above and below by $M$ and $m$, it holds that $|\bar{y} - \bar{\gamma}(\bar{t}^{(1)})|_2 = O\left(\frac{M}{m} \varepsilon |C| \right)$. (If the residual capacity is negative we can exchange the roles of $\gamma$ and $\bar{\gamma}$ and argue similarly.)

Let us now show that the inductive assumption holds. Fix a school $c$. Then by Proposition 11 the set of students with ranks in $(y, \gamma(0)]$ under $\mathcal{E}$ and ranks in $(\bar{y}, \gamma(0)]$ under $\bar{\mathcal{E}}$ who are assigned to $c$ under $TTC(\gamma)$ and not under $TTC(\bar{\gamma})$ has $\bar{\eta}$-measure $O(\varepsilon |C|)$. Moreover, since $|\bar{y} - \bar{\gamma}(\bar{t}^{(1)})|_2 = O\left(\frac{M}{m} \varepsilon |C| \right)$ and $\bar{\eta}$ has full support and has density bounded from above and below by $M$ and $m$, the set of students with ranks in $(\bar{y}, \bar{\gamma}(\bar{t}^{(1)}))$ assigned to school $c$ by $TTC(\bar{\gamma})$ has $\bar{\eta}$-measure $O(\varepsilon |C|)$. Hence the set of students assigned to $c$ under $TTC(\gamma)$ by time $t^{(1)}$ and not under $TTC(\bar{\gamma})$ by time $\bar{t}^{(1)}$ has $\eta$-measure $O(\varepsilon |C|)$. Moreover, if $t^{(1)} < t^{(2)}$ then for sufficiently small $\varepsilon$ it holds that $\bar{t}^{(1)} = \min_c \bar{t}^{(e)}$, and otherwise there exists a relabeling of the schools such that this is true, and so $\sigma(1) = 1$.

We now show the inductive step, proving for $\ell + 1$ assuming true for $1, 2, \ldots, \ell$. By inductive assumption, for all $c$ the measure of students assigned to $c$ under $TTC(\gamma)$ and not under $TTC(\bar{\gamma})$ by the points $\gamma\left(t^{(\ell)}\right)$, $\bar{\gamma}\left(\bar{t}^{(\ell)}\right)$ is $O(\varepsilon \ell |C|)$ for all $c$.

Let $x = \gamma\left(t^{(\ell)}\right)$ and $y = \gamma\left(t^{(\ell+1)}\right)$. Define $\bar{x} \in Im(\bar{\gamma})$ to be the minimal point such that $x \leq \bar{x}$ and there exists $b$ such that $x_b = \bar{x}_b$. We show that $\bar{x}$ is near $\bar{\gamma}(\bar{t}^{(\ell)})$, i.e. $|\bar{x} - \bar{\gamma}(\bar{t}^{(\ell)})|_2 = O(\varepsilon)$. Now by inductive assumption $\eta\left\{ \theta \mid \sigma(\theta) \in (x = \gamma\left(t^{(\ell)}\right), \bar{\gamma}(\bar{t}^{(\ell)})]\right\} = O(\varepsilon |C|)$ and so $|x - \bar{\gamma}(\bar{t}^{(\ell)})|_2 = O(\varepsilon)$. Moreover $|\bar{x}_b - \gamma_b\left(t^{(\ell)}\right)|_2 = |x_b - \gamma_b(\bar{t}^{(\ell)})|_2$ which we have just shown is $O(\varepsilon)$. Finally, since $\eta$ has full support and has density bounded from above and below by $M$ and $m$, it holds that $\max_{b,c,\tau} \frac{\gamma_b(\tau)}{\gamma^c(\tau)} = O\left(\frac{M}{m}\right)$.
and so for all \( c \) it holds that \(|\tilde{x}_c - \tilde{\gamma}_c(\tilde{t}^{(c)})| \leq O(\frac{M}{m} \varepsilon)\).

The remainder of the proof runs much the same as in the base case, with slight adjustments to account for the fact that \( x \neq \tilde{x} \). Define \( \bar{y} \in Im(\bar{\gamma}) \) to be the minimal point such that \( y \leq \bar{y} \) and there exists \( c \) such that \( y_c = \bar{y}_c \). We show that \( \bar{y} \) is near \( \tilde{\gamma}(t^{(c)+1}) \), i.e. \(|\bar{y} - \tilde{\gamma}(t^{(c)+1})|_2 = O(\varepsilon)\). Now by Proposition 11 the set of students with ranks in \((y, x]\) under \( E \) and ranks in \((\bar{y}, \tilde{x}] \) under \( \tilde{E} \) who are assigned to \( \ell+1 \) under \( TTC(\gamma) \) and not under \( TTC(\tilde{\gamma}) \) has \( \tilde{\eta} \)-measure \( O(\varepsilon |C|) \). This, together with the inductive assumption that the difference in students assigned to school \( \ell \) is \( O(\varepsilon |C|) \), shows that the residual capacity of school \( \ell+1 \) at \( \bar{y} \) under \( TTC(\tilde{\gamma}) \) is \( O(\varepsilon (\ell+1) |C|) \), and so since \( \tilde{\eta} \) has full support and has density bounded from above and below by \( M \) and \( m \), it holds that \(|\bar{y} - \tilde{\gamma}(t^{(c)+1})|_2 = O(\frac{M}{m} \varepsilon (\ell+1) |C|)\). (If the residual capacity is negative we can exchange the roles of \( \gamma \) and \( \tilde{\gamma} \) and argue similarly.)

Let us now show that the inductive assumption holds. Fix a school \( c \). Then by Proposition 11 the set of students with ranks in \((y, x]\) under \( E \) and ranks in \((\bar{y}, \tilde{x}] \) under \( \tilde{E} \) who are assigned to \( c \) under \( TTC(\gamma) \) and not under \( TTC(\tilde{\gamma}) \) has \( \tilde{\eta} \)-measure \( O(\varepsilon |C|) \). Moreover, since \(|\bar{y} - \tilde{\gamma}(t^{(c)+1})|_2 = O(\frac{M}{m} \varepsilon (\ell+1) |C|)\) and \( \tilde{\eta} \) has full support and has density bounded from above and below by \( M \) and \( m \), the set of students with ranks in \((\bar{y}, \tilde{\gamma}(t^{(c)+1})\] alternated to school \( c \) by \( TTC(\tilde{\gamma}) \) has \( \tilde{\eta} \)-measure \( O(\varepsilon (\ell+1) |C|) \). Hence the set of students assigned to \( c \) under \( TTC(\gamma) \) by time \( t^{(c)+1} \) and not under \( TTC(\tilde{\gamma}) \) by time \( t^{(c)+1} \) has \( \eta \)-measure \( O(\varepsilon (\ell+1) |C|) \). Moreover if \( t^{(c)+1} < t^{(c)+2} \) then for sufficiently small \( \varepsilon \) it holds that \( \tilde{t}^{(c)+1} = \min_{c > \ell} \tilde{t}^{(c)} \), and otherwise there exists a relabeling of the schools such that this is true, and so \( \sigma(\ell+1) = \ell+1 \). \( \square \)

### D.6 Proof of Proposition 3

Throughout the proof, we omit the dependence on \( E \). We show that there exist TTC cutoffs \( \{(p^*)^c_b = \tilde{\gamma}_b(t^{(c)})\} \) such that the TTC path and stopping times \( \gamma, \{t^{(c)}\} \) satisfy trade balance and capacity for \( \Phi(E) \) and \( B^*(s) \subseteq B(s) \subseteq B(s; p^*) \subseteq B^*(s) \). For brevity, for a TTC path \( \gamma \) and discrete economy \( E' \), we say that \( \gamma \) is a TTC path for \( E' \) if there exist stopping times \( \{t^{(c)}\} \) such that \( \gamma(\cdot), \{t^{(c)}\} \) satisfy trade balance and capacity and write \( p = \{\gamma_b(t^{(c)})\} \in \mathcal{P}(E') \).

We first show that \( B^*(s) \subseteq B(s) \). Suppose \( c \not\in B(s) \). Then there exists a TTC path \( \gamma \) for \( E \) such that \( r^a + \frac{1}{|S|} 1 \leq \gamma(t^{(c)}) \). Hence for all \( \tilde{\gamma} \) there exists a TTC path \( \tilde{\gamma} \in \mathcal{P}([E-s; \tilde{\gamma}]) \) such that \( r^a + \frac{1}{|S|} 1 \leq \tilde{\gamma}(t^{(c)}) \). By Proposition 2 and Theorem
2 for all $\tilde{\succ}$ it holds that $\mu_{d_{TTC}}(s \mid [E_{-s}; \tilde{\succ}]) = \max_{\tilde{\gamma}} \{c : r^s_\tilde{\gamma} \geq \tilde{\gamma}(t^{(c)})_b \text{ for some } b\}$. Hence for all $\tilde{\succ}$ it holds that $\mu_{d_{TTC}}(s \mid [E_{-s}; \tilde{\succ}]) \neq c$ and so $c \not\in B^*(s)$.

We next show that $B(s) \subseteq B(s; p^*) \subseteq B^*(s)$. Intuitively, we construct the special TTC path $\tilde{\gamma}$ for $E$ by clearing as many cycles as possible that do not involve student $s$. Formally, let $\succ$ be an ordering over subsets of $C$ where: (1) all subsets involving student $s$’s top choice available school $c$ (under the preferences $\succ^s$ in $E$) come after all subsets not involving $c$; and (2) subject to this, subsets are ordered via the shortlex order. Let $\tilde{\gamma}$ be the TTC path for $E$ obtained by selecting valid directions with minimal support under the order $\succ$. (Such a path exists since the resulting valid directions $d$ are piecewise Lipschitz continuous.)

It follows trivially from the definition of $B(s)$ that $B(s) \subseteq B(s; p^*)$. We now show that $B(s; p^*) \subseteq B^*(s)$. For suppose $c \in B(s; p^*)$. Consider the preferences $\succ'$ that put school $c$ first, and then all other schools in the order given by $\succ^s$. We show that $\mu_{d_{TTC}}(s \mid [E_{-s}; \succ']) = c$. Now since $c \in B(s; p^*)$, it holds that $r^s \not\leq \tilde{\gamma}(t^{(c)})$. In other words, if we let $\tau^s = \inf \{\tau \mid \tilde{\gamma}(\tau) \not\geq r^s\}$ be the time that the cube $I^s$ corresponding to student $s$ starts clearing, then school $c$ is available at time $\tau^s$. Moreover if we let $\tilde{\gamma}'$ be the TTC path for $[E_{-s}; \succ']$ obtained by selecting valid directions with minimal support under the order $\succ'$, then for all $\tau \leq \tau^s$ it holds that $\tilde{\gamma}'(\tau) = \tilde{\gamma}(\tau)$, and so school $c$ is again available at time $\tau^s$. Hence by Proposition 2 and Theorem 2 it holds that $\mu_{d_{TTC}}(s \mid [E_{-s}; \succ']) = c$ and so $c \in B^*(s)$.

## E Proofs for Applications (Section 4)

Throughout this section, we will say that a vector $d$ is a valid direction at point $x$ if $d$ satisfies the marginal trade balance equations at $x$, and $d \cdot 1 = -1$.

### E.1 Optimal Investment in School Quality

In this section, we prove the results stated in Section 4.1. We will assume that the total measure of students is 1, and speak of student measures and student proportions interchangeably.
Proofs for Section 4.1

Proof of Proposition 4. Let $\gamma, p, \{t^{(1)}, t^{(2)}\}$ be the TTC path, cutoffs and stopping times with quality $\delta$, and let $\hat{\gamma}, \hat{p}, \{\hat{t}^{(1)}, \hat{t}^{(2)}\}$ be the TTC path, cutoffs and stopping times with quality $\hat{\delta}$. When we change $\delta_\ell$ to $\hat{\delta}_\ell$, this increases the relative popularity of school $\ell$.

Consider first when $\ell = 1$. As there are only two schools, $|d_1(x)|$ decreases and $|d_2(x)|$ increases for all $x$. It follows that if $\gamma_1(t) = \hat{\gamma}_1(\hat{t})$ then $\gamma_2(t) \geq \hat{\gamma}_2(\hat{t})$, and if $\gamma_2(t) = \hat{\gamma}_2(\hat{t})$ then $\gamma_1(t) \leq \hat{\gamma}_1(\hat{t})$. Suppose that $p^1_2 \leq \hat{p}^1_2$. Then there exists $t \leq t^{(1)}$ such that $p^1_1 = \gamma_2(t^{(1)}) \leq \gamma_2(t) = \hat{\gamma}_2(\hat{t}^{(1)})$, and so

$$p^1_1 = \gamma_1(t^{(1)}) \leq \gamma_1(t) \leq \hat{\gamma}_1(\hat{t}^{(1)}) = \hat{p}^1_1$$

as required. Hence it suffices to show that $p^1_2 \leq \hat{p}^1_2$.

Suppose for the sake of contradiction that $p^1_2 > \hat{p}^1_2$. Then there exists $t < \hat{t}^{(1)}$ such that $p^1_2 = \gamma_2(t^{(1)}) = \hat{\gamma}_2(t) > \hat{\gamma}_2(\hat{t}^{(1)})$, and so $T_2(\gamma, t^{(1)}) \subseteq T_2(\hat{\gamma}, t) \subset T_2(\hat{\gamma}, \hat{t}^{(1)})$ and similarly $T_1(\gamma, t^{(1)}) \supseteq T_1(\hat{\gamma}, \hat{t}^{(1)})$. It follows that

$$\eta \left( \left\{ \theta \in T_2(\gamma; t^{(1)}) \mid \max_{\gamma_\theta} \{1, 2\} = 1 \right\} \right) < \hat{\eta} \left( \left\{ \theta \in T_2(\hat{\gamma}; \hat{t}^{(1)}) \mid \max_{\gamma_\theta} \{1, 2\} = 1 \right\} \right),$$

since the set increased and more students want school 1, and similarly

$$\eta \left( \left\{ \theta \in T_1(\gamma; t^{(1)}) \mid \max_{\gamma_\theta} \{1, 2\} = 2 \right\} \right) > \hat{\eta} \left( \left\{ \theta \in T_1(\hat{\gamma}; \hat{t}^{(1)}) \mid \max_{\gamma_\theta} \{1, 2\} = 2 \right\} \right),$$

However, integrating over the marginal trade balance equations gives that

$$\eta \left( \left\{ \theta \in T_2(\gamma; t^{(1)}) \mid \max_{\gamma_\theta} \{1, 2\} = 1 \right\} \right) = \eta \left( \left\{ \theta \in T_1(\gamma; t^{(1)}) \mid \max_{\gamma_\theta} \{1, 2\} = 2 \right\} \right)$$

and

$$\hat{\eta} \left( \left\{ \theta \in T_2(\hat{\gamma}; \hat{t}^{(1)}) \mid \max_{\gamma_\theta} \{1, 2\} = 1 \right\} \right) = \hat{\eta} \left( \left\{ \theta \in T_1(\hat{\gamma}; \hat{t}^{(1)}) \mid \max_{\gamma_\theta} \{1, 2\} = 2 \right\} \right),$$

which gives the required contradiction. The fact that $p^2_2$ decreases follows from the fact that $p^1_1$ increases, since the total number of assigned students is the same. \qed

Proof of Proposition 5.
TTC Cutoffs We calculate the TTC cutoffs under the logit model for different student choice probabilities by using the TTC paths and trade balance equations. In round 1, the marginals $\tilde{H}^c_b(x)$ for $b, c \in C$ at each point $x \in [0, 1]$ are given by $\tilde{H}^c_b(x) = e^{\delta_b} \prod_{c' \neq b} x_{c'}$. Hence $v_b = \sum_c \tilde{H}^c_b(x) = (\sum_b e^{\delta_c}) \prod_{c' \neq b} x_{c'} = \prod_{c' \neq b} x_{c'}$, so $v = \prod_c x_c$ and the matrix $H(x)$ is given by

$$H_{b,c}(x) = e^{\delta_c} \frac{\min_{c'} x_{c'}}{x_b} + 1_{b-c} \left( 1 - \frac{\min_{c'} x_{c'}}{x_b} \right) = \begin{cases} 1 - \left( 1 - e^{\delta_c} \right) \frac{\min_{c'} x_{c'}}{x_b} & \text{if } b = c, \\ e^{\delta_c} \frac{\min_{c'} x_{c'}}{x_b} & \text{otherwise,} \end{cases}$$

which is irreducible and gives a unique valid direction $d(x)$ satisfying $d(x) H(x) = d(x)$. To solve for this, we observe that this equation is the same as $d(x) (H(x) - I) = 0$, where $I$ is the $n$-dimensional identity matrix, and and $[H(x) - I]$ has $(b, c)$th entry

$$[H(x) - I]_{b,c} = \begin{cases} - \left( 1 - e^{\delta_b} \right) \frac{\min_{c'} x_{c'}}{x_b} & \text{if } b = c, \\ e^{\delta_c} \frac{\min_{c'} x_{c'}}{x_b} & \text{otherwise.} \end{cases}$$

Since this has rank $n - 1$, the nullspace is easily obtained by replacing the last column of $H(x) - I$ with ones, inverting the matrix and left multiplying it to the vector $e^{|C|}$ (the vector with all zero entries, other than a 1 in the $|C|$th entry). This yields the valid direction $d(x)$ with $c$th component

$$d_c(x) = -\frac{e^{\delta_c} x_c}{\sum_b e^{\delta_b} x_b}.$$

We now find a valid TTC path $\gamma$ using the trade balance equations 1. Since the ratios of the components of the gradient $\frac{d \gamma}{d(x)}$ only depend on $x_b, x_c$ and the $\delta_{c'}$, we solve for $x_c$ in terms of $x_1$, using the fact that the path starts at $(1, 1)$. This gives the path $\gamma$ defined by $\gamma_c \left( \gamma^{-1}_c(x_1) \right) = x_1^{\delta_{c-\delta_1}}$ for all $c$.

Recall that the schools are indexed so that school $c_1$ is the most demanded school, that is, $\frac{\delta_1}{q_1} = \max_c \frac{\delta_c}{q_c}$. Since we are only interested in the changes in the cutoffs $\gamma \left( t^{(1)} \right)$ and not in the specific time, let us assume without loss of generality that $\gamma_1(t) = 1 - t$. Then school $c_1$ fills at time $t^{(1)} = 1 - \left( 1 - \frac{q_1}{e^{\delta_1}} \left( \sum_{c'} e^{\delta_{c'}} \right) \frac{\delta_{c_1}}{\sum_{c'} \delta_{c'}} \right)$.
\[ 1 - (1 - \rho_1 \left( \sum_{c'} e^{\delta_{c'}} \right)) \frac{\Delta_1}{\sum_{c'} e^{\delta_{c'}}} \]. Hence the round 1 cutoffs are

\[ p_b^1 = \left( 1 - t^{(1)} \right) e^{\delta_b - \delta_1} = \left( 1 - \rho_1 \left( \sum_{c'} e^{\delta_{c'}} \right) \right) \frac{\Delta_b}{\sum_{c'} e^{\delta_{c'}}} = \left( 1 - \rho_1 \left( \sum_{c'} e^{\delta_{c'}} \right) \right)^{\pi_{b|c}}. \] (10)

It can be shown by projecting onto the remaining coordinates and using induction that the round \( i \) cutoffs are given by

\[ p_{b^i} = \begin{cases} \left( \prod_{c' < c} \frac{1}{p_{c'}^{i-1}} \right)^{\pi_{b|c}} \left( \prod_{c' \geq c} p_{c'}^{i-1} - \rho_c \left( \sum_{c' \geq c} e^{\delta_{c'}} \right) \right)^{\pi_{b|c}} & \text{if } b \geq c \\ p_b & \text{if } b \leq c. \end{cases} \]

**TTC Cutoffs - Comparative Statics**

We perform some comparative statics calculations for the TTC cutoffs under the logit model. For \( b \neq \ell \) it holds that the TTC cutoff \( p_b^1 \) for using priority at school \( b \) to receive a seat at school 1 is decreasing in \( \delta_{\ell} \),

\[
\frac{\partial p_b^1}{\partial \delta_{\ell}} = \frac{\partial}{\partial \delta_{\ell}} \left[ \left( 1 - \frac{q_1}{e^{\delta_{\ell}}} \left( \sum_{c'} e^{\delta_{c'}} \right) \right) \frac{\Delta_b}{\sum_{c'} e^{\delta_{c'}}} \right]
= -p_b^1 \left( \frac{e^{\delta_{\ell} + \delta_b}}{\Delta^1} \right) \left[ -\ln \left( \frac{1}{1 - \frac{q_1}{e^{\delta_{\ell}}} \frac{\Delta_b}{\Delta^1}} \right) + \frac{1}{\left( 1 - \frac{q_1}{e^{\delta_{\ell}}} \right) \Delta^1} - 1 \right]
\]

is negative, since \( 0 < \frac{1}{1 - \frac{q_1}{e^{\delta_{\ell}}} \frac{\Delta_b}{\Delta^1}} < 1 \), where for brevity we define \( \Delta^c = \sum_{b \geq c} e^{\delta_b} \).

We can decompose this change as

\[
\frac{\partial p_b^1}{\partial \delta_{\ell}} = -p_b^1 \left( \frac{e^{\delta_{\ell} + \delta_b}}{\Delta^1} \right) \left[ \ln \left( 1 - \left( \frac{q_1}{e^{\delta_{\ell}}} \right) \Delta^1 \right) \right] - p_b^1 \left( \frac{e^{\delta_{\ell} + \delta_b}}{\Delta^1} \right) \left[ \frac{1}{\left( 1 - \frac{q_1}{e^{\delta_{\ell}}} \right) \Delta^1} - 1 \right] < 0,
\]

where the first term is the increase in \( p_b^1 \) due to the fact that relatively fewer students are pointed to and cleared by school \( b \) for every marginal change in rank, and the second term is the decrease in \( p_b^1 \) due to the fact that school 1 is relatively less popular now, and so more students need to be given a budget set of \( C^{(1)} \) in order for school 1 to reach capacity.
For $b = \ell$ the TTC cutoff $p_\ell^1$ is again decreasing in $\delta_\ell$,

$$\frac{\partial p_\ell^1}{\partial \delta_\ell} = \frac{\partial}{\partial \delta_\ell} \left[ \left( 1 - \frac{q_1}{e^{\delta_1}} \left( \sum_{c'} e^{\delta_{c'}} \right) \right)^{\frac{e^{\delta_1}}{\sum_{c'} e^{\delta_{c'}}}} \right]$$

$$= -p_\ell^1 \left( \frac{e^{\delta_1} (\Delta_1 - e^{\delta_1})}{(\Delta_1)^2} \right) \ln \left( \frac{1}{1 - \left( \frac{q_1}{e^{\delta_1}} \right) \Delta_1} \right) - p_\ell^1 \left( \frac{e^{2\delta_1 + \delta_b}}{(\Delta_1)^2} \right) \left( \frac{1}{1 - \left( \frac{q_1}{e^{\delta_1}} \right) \Delta_1} - 1 \right)$$

is negative since both terms are negative.

Similarly, for $c < \ell$ and $b \neq \ell$ the TTC cutoff $p_c^b$ is decreasing in $\delta_\ell$, since (if we let $\tilde{q}_c = \frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}}$)

$$\frac{\partial p_c^b}{\partial \delta_\ell} = \frac{\partial}{\partial \delta_\ell} \left[ \left( 1 - \left( \prod_{c' < c} \frac{1}{p_{c'}^{c'}} \right) \tilde{q}_c \left( \sum_{c' \geq c} e^{\delta_{c'}} \right) \right)^{\frac{e^{\delta_1}}{\sum_{c' \geq c} e^{\delta_{c'}}}} \right]$$

$$= -p_c^b \left( \frac{e^{\delta_1 + \delta_b}}{(\Delta_c)^2} \right) \left[- \ln \left( \frac{1}{1 - P^c \tilde{q}_c \Delta_c} \right) + \frac{1}{1 - P^c \tilde{q}_c \Delta_c} - 1 \right] - p_c^b \left[ \frac{e^{\delta_b} \tilde{q}_c \frac{\partial P^c}{\partial \delta_\ell}}{(1 - P^c \tilde{q}_c \Delta_c)} \right]$$

is negative, where $P^c = \prod_{c' < c} \frac{1}{p_{c'}^{c'}}$, since $0 < 1 - P^c \tilde{q}_c \Delta_c < 1$ and $\frac{\partial P^c}{\partial \delta_\ell} = P^c \left( \sum_{c' < c} - \frac{\partial p_{c'}^c}{\partial \delta_\ell} \frac{p_{c'}^c}{p_{c'}^{c'}} \right) > 0$ so both terms are negative.

We can decompose this change as follows. Let $P^c = \prod_{c' < c} \frac{1}{p_{c'}^{c'}}$. For $c < \ell$ and $b \geq c$, $b \neq \ell$ it holds that

$$\frac{\partial p_b^c}{\partial \delta_\ell} = -p_b^c \left[ \frac{e^{\delta_1 + \delta_b}}{(\Delta_c)^2} \right] \left[ \ln \left( 1 - P^c \left( \frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) \Delta_c \right) \right]$$

$$- p_b^c \left[ \frac{e^{\delta_1 + \delta_b}}{(\Delta_c)^2} \right] \left[ \frac{1}{1 - P^c \left( \frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) \Delta_c} - 1 \right] - p_b^c \left[ \frac{e^{\delta_b} \left( \frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) \frac{\partial P^c}{\partial \delta_\ell}}{(1 - P^c \left( \frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) \Delta_c)} \right]$$

which is negative. The first term is the increase in $p_b^c$ due to the fact that relatively fewer students are pointed to and cleared by school $j$ for every marginal change in rank, and the second and third terms are the decrease in $p_b^c$ due to the fact that schools 1 through $c$ are relatively less popular now, and so more students need to be given a budget set of $C^{(1)}, C^{(2)}, \ldots, C^{(c)}$ in order for schools 1 through $c$ to reach capacity.

For $c < \ell$ and $b = \ell$ the TTC cutoff $p_c^b$ is also decreasing in $\delta_\ell$, since (if we let
\[
\tilde{q}_c = \frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}}
\]

\[
\frac{\partial p^f_{\ell}}{\partial \delta^f_{\ell}} = \frac{\partial}{\partial \delta^f_{\ell}} \left( 1 - \left( \prod_{c' < c} \frac{1}{p^f_{c'}} \right) \tilde{q}_c \left( \sum_{c' \geq c} e^{\delta_{c'}} \right) \right) \)

\[
= -p^f_{\ell} \left( \frac{e^{\delta_{\ell}} (\Delta_c - e^{\delta_c})}{(\Delta_c^2)} \ln \left( \frac{1}{1 - P^c \tilde{q}_c \Delta_c} \right) \right) - p^f_{\ell} \left( \frac{e^{\delta_{\ell}}}{\Delta_c} \left[ (P^c \tilde{q}_c e^{\delta_{\ell}} + \tilde{q}_c \Delta_c \frac{\partial p^c_{\ell}}{\partial \delta^f_{\ell}}) / (1 - P^c \tilde{q}_c \Delta_c) \right] \right)
\]

which is negative, since \( \frac{\partial p^c_{\ell}}{\partial \delta^f_{\ell}} = P^c \left( \sum_{c' < c} \frac{\partial p^c_{\ell}}{\partial \delta^f_{\ell}} \cdot \frac{1}{p^f_{c'}} \right) > 0 \) so both terms are negative.

When \( c = \ell \), the effects of changing \( \delta^f_{\ell} \) on the cutoffs required the obtain a seat at school \( \ell \) are a little more involved. For \( c = \ell \) and \( b \neq \ell \),

\[
\frac{\partial p^f_{b}}{\partial \delta^f_{\ell}} = \frac{\partial}{\partial \delta^f_{\ell}} \left( 1 - \left( \prod_{c' < \ell} \frac{1}{p^f_{c'}} \right) \left( \frac{q_{\ell}}{e^{\delta_{\ell}}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \left( \sum_{c' \geq \ell} e^{\delta_{c'}} \right) \right) \)

\[
= p^f_{\ell} \left( \frac{e^{\delta_{\ell}}}{\Delta^\ell} \ln \left( \frac{1}{1 - P^\ell \left( \frac{q_{\ell}}{e^{\delta_{\ell}}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell} \right) + \frac{\partial}{\partial \delta^f_{\ell}} \left( 1 - P^\ell \left( \frac{q_{\ell}}{e^{\delta_{\ell}}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell \right) \right)
\]

where \( P^\ell = \prod_{c' < \ell} \frac{1}{p^f_{c'}} \), the first term is positive, and the second term has the same sign as its numerator \( \frac{\partial}{\partial \delta^f_{\ell}} \left( 1 - P^\ell \left( \frac{q_{\ell}}{e^{\delta_{\ell}}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell \right) \). Similarly for \( c = \ell \) and \( b = \ell \),

\[
\frac{\partial p^f_{\ell}}{\partial \delta^f_{\ell}} = \frac{\partial}{\partial \delta^f_{\ell}} \left( 1 - P^\ell \left( \frac{q_{\ell}}{e^{\delta_{\ell}}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \left( \sum_{c' \geq \ell} e^{\delta_{c'}} \right) \right) \)

\[
= p^f_{\ell} \left( \frac{e^{\delta_{\ell}}}{\Delta^\ell} \ln \left( \frac{1}{1 - P^\ell \left( \frac{q_{\ell}}{e^{\delta_{\ell}}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell} \right) + \frac{\partial}{\partial \delta^f_{\ell}} \left( 1 - P^\ell \left( \frac{q_{\ell}}{e^{\delta_{\ell}}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell \right) \right)
\]

where \( P^\ell = \prod_{c' < \ell} \frac{1}{p^f_{c'}} \), the first term is negative, and the second term has the same sign as its numerator \( \frac{\partial}{\partial \delta^f_{\ell}} \left( 1 - P^\ell \left( \frac{q_{\ell}}{e^{\delta_{\ell}}} - \frac{q_{\ell-1}}{e^{\delta_{\ell-1}}} \right) \Delta^\ell \right) \). Since \( \frac{\partial}{\partial \delta^f_{\ell}} \left( \prod_{b > \ell} p^f_{b} \right) > 0 \), it follows that \( \frac{\partial p^f_{\ell}}{\partial \delta^f_{\ell}} > 0 \) for all \( b \neq \ell \), and there are regimes in which \( \frac{\partial p^f_{\ell}}{\partial \delta^f_{\ell}} \) is positive, and regimes where it is negative. \( \square \)
Proofs for Section 4.1

Proof of Proposition 6.

Welfare Expressions  We derive the welfare expressions corresponding to these cutoffs. Let $C^{(c)} = \{c, c+1, \ldots, n\}$. Since the schools are ordered so that $\frac{q_1}{c^{\delta_1}} \leq \frac{q_2}{c^{\delta_2}} \leq \cdots \leq \frac{q_n}{c^{\delta_n}}$, it follows that the schools also fill in the order 1, 2, \ldots, n.

Suppose that the total mass of students is 1. Then the mass of students with budget set $C^{(1)}$ is given by

$$N^1 = q_1 \left( \sum_{b} e^{\delta_b} \right),$$

and the mass of students with budget set $C^{(2)}$ is given by

$$N^2 = \left( q_2 - \frac{e^{\delta_2}}{\sum_{b} e^{\delta_b}} N^1 \right) \left( \sum_{b \geq 2} e^{\delta_b} \right) = \left( \frac{q_2}{e^{\delta_2}} - \frac{q_1}{e^{\delta_1}} \right) \left( \sum_{b \geq 2} e^{\delta_b} \right).$$

An inductive argument shows that the proportion of students with budget set $C^{(c)}$ is

$$N^c = \left( \frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) \left( \sum_{b \geq c} e^{\delta_b} \right),$$

which depends only on $\delta_b$ for $b \geq c - 1$.

Moreover, each such student with budget set $C^{(c)}$, conditional on their budget set, has expected utility Small and Rosen (1981)

$$U^c = \mathbb{E} \left[ \max_{c' \in C^{(c)}} \{ \delta_{b'} + \varepsilon_{\theta c'} \} \right] = \ln \left( \sum_{b \geq c} e^{\delta_b} \right),$$

which depends only on $\delta_b$ for $b \geq c$. Hence the expected social welfare from fixed qualities $\delta_c$ is given by

$$U_{TTC} = \sum_c N^c \cdot U^c = \sum_c \left( \frac{q_c}{e^{\delta_c}} - \frac{q_{c-1}}{e^{\delta_{c-1}}} \right) \Delta^c \ln \Delta^c,$$

where $\Delta^c = \sum_{b \geq c} e^{\delta_b}$.

Welfare - Comparative Statics  Taking derivatives, we obtain that

$$\frac{dU_{TTC}}{d\delta_\ell} = \sum_c \left( \frac{dN^c}{d\delta_\ell} \cdot U^c + N^c \cdot \frac{dU^c}{d\delta_\ell} \right) = \sum_{c \leq \ell+1} \frac{dN^c}{d\delta_\ell} \cdot U^c + \sum_{c \leq \ell} N^c \cdot \frac{dU^c}{d\delta_\ell},$$

30
where \( \sum_{c \leq \ell} N_c \cdot \frac{dU_c}{d\delta} = e^{\delta} \sum_{c \leq \ell} \left( \frac{q_c}{e^{\delta c}} - \frac{q_{c-1}}{e^{\delta c-1}} \right) = q_\ell. \) It follows that
\[
\frac{dU_{TTC}}{d\delta} = q_\ell + \sum_{c \leq \ell+1} \frac{dN_c}{d\delta} \cdot U_c.
\]

\[\square\]

**Proof of Proposition 7.** We solve for the social welfare maximixing budget allocation. For a fixed runout ordering (i.e. \( \frac{q_1}{e^{\delta_1}} \leq \frac{q_2}{e^{\delta_2}} \leq \ldots \leq \frac{q_n}{e^{\delta_n}} \)), the central school board’s investment problem is given by the program

\[
\max_{\kappa_1, \kappa_2, \ldots, \kappa_n} \sum_i \left( \frac{q_i}{\kappa_i} - \frac{q_{i-1}}{\kappa_{i-1}} \right) \left( \sum_{j \geq i} \kappa_j \right) \ln \left( \sum_{j \geq i} \kappa_j \right) \tag{11}
\]

s.t. \( \frac{q_{i-1}}{\kappa_{i-1}} \leq \frac{q_i}{\kappa_i} \forall i \)

\[\sum_i \kappa_i = K \]

\[q_0 = 0.\]

We can reformulate this as the following program,

\[
\max_{\kappa_2, \ldots, \kappa_n} \left( \frac{q_1}{K - \sum_i \kappa_i} \right) K \ln K + \left( \frac{q_2}{K - \sum_i \kappa_i} - \frac{q_1}{K - \sum_i \kappa_i} \right) U_2 \ln U_2 + \sum_{i \geq 2} \left( \frac{q_i}{\kappa_i} - \frac{q_{i-1}}{\kappa_{i-1}} \right) U_i \ln U_i \tag{12}
\]

s.t. \( \frac{q_{i-1}}{\kappa_{i-1}} \leq \frac{q_i}{\kappa_i} \forall i \geq 3 \)

\[\frac{q_1}{K - \sum_i \kappa_i} \leq \frac{q_2}{\kappa_2},\]

where \( U_i = \sum_{j \geq i} \kappa_j. \)

The reformulated problem (12) has objective function

\[
U(\kappa) = \left( \frac{q_1}{K - \sum_i \kappa_i} \right) K \ln K + \left( \frac{q_2}{K - \sum_i \kappa_i} - \frac{q_1}{K - \sum_i \kappa_i} \right) \Delta^2 \ln \Delta^2 + \sum_{i \geq 2} \left( \frac{q_i}{\kappa_i} - \frac{q_{i-1}}{\kappa_{i-1}} \right) \Delta^i \ln \Delta^i,
\]

where \( \Delta^i = \sum_{j \geq i} \kappa_j. \)

Taking the derivatives with respect to the budget allocations \( \kappa_k \) gives

\[
\frac{\partial U}{\partial \kappa_k} = \left( \frac{q_1}{(K - \sum_i \kappa_i)^2} \right) \left( K \ln \frac{K}{\Delta^2} - (K - \Delta^2) \right) + \sum_{2 \leq j < k} \frac{q_j}{\kappa_j} \ln \frac{\Delta^j}{\Delta^{j+1}} + \frac{q_k}{(\kappa_k)^2} \left( \kappa_k - \Delta^{k+1} \ln \frac{\Delta^k}{\Delta^{k+1}} \right),
\]

where \( K \ln \frac{K}{\Delta^2} - (K - \Delta^2) \geq 0, \ln \frac{\Delta^j}{\Delta^{j+1}} \geq 0, \) and \( \kappa_k - \Delta^{k+1} \ln \frac{\Delta^k}{\Delta^{k+1}} \geq 0 \) and so
\[ \frac{\partial U}{\partial \kappa_k} \geq 0 \forall k. \]

Moreover, if \( \frac{q_{i-1}}{\kappa_{i-1}} = \frac{q_i}{\kappa_i} \), then defining a new problem with \( n - 1 \) schools, and capacities \( \tilde{q} \) and budget \( \tilde{\kappa} \)

\[ \tilde{q}_j = \begin{cases} 
q_j & \text{if } j < i - 1 \\
q_{i-1} + q_i & \text{if } j = i - 1, \kappa_j = \begin{cases} 
\kappa_{i-1} & \text{if } j = i - 1 \\
\kappa_i & \text{if } j > i - 1 
\end{cases} 
\end{cases} \]

leads to a problem with the same objective function, since

\[
\left( \frac{q_{i-1}}{\kappa_{i-1}} - \frac{q_{i-2}}{\kappa_{i-2}} \right) \Delta^{i-1} \ln \Delta^{i-1} + \left( \frac{q_i}{\kappa_i} - \frac{q_{i-1}}{\kappa_{i-1}} \right) \Delta^i \ln \Delta^i + \left( \frac{q_{i+1}}{\kappa_{i+1}} - \frac{q_i}{\kappa_i} \right) \Delta^{i+1} \ln \Delta^{i+1}
\]

\[
= \left( \frac{q_{i-1} + q_i}{\kappa_{i-1} + \kappa_i} - \frac{q_{i-2} + q_i}{\kappa_{i-2} + \kappa_i} \right) \Delta^{i-1} \ln \Delta^{i-1} + \left( \frac{q_{i+1} + q_i}{\kappa_{i+1} + \kappa_i} - \frac{q_{i-1} + q_i}{\kappa_{i-1} + \kappa_i} \right) \Delta^{i+1} \ln \Delta^{i+1}. \]

Hence if there exists \( i \) for which \( \frac{q_i}{\kappa_i} \neq \frac{q_{i-1}}{\kappa_{i-1}} \), we may take \( i \) to be minimal such that this occurs, decrease each of \( \kappa_1, \ldots, \kappa_{i-1} \) proportionally so that \( \kappa_1 + \cdots + \kappa_{i-1} \) decreases by \( \varepsilon \) and increase \( \kappa_i \) by \( \varepsilon \) and increase resulting value of the objective. It follows that the objective is maximized when \( \frac{q_1}{\kappa_1} = \frac{q_2}{\kappa_2} = \cdots = \frac{q_n}{\kappa_n} \), i.e. when the money assigned to each school is proportional to the number of seats at the school. \( \square \)

### E.2 Design of TTC Priorities

We demonstrate how to calculate the TTC cutoffs for the two economies in Figure 7 by using the TTC paths and trade balance equations.

Consider the economy \( \mathcal{E}_0 \), where the top priority students have ranks uniformly distributed in \([m, 1]^2\). If \( x = (x_1, x_1) \) is on the diagonal, then \( H_i^j(x) = \frac{q_i}{2} \) for all \( i, j \in \{1, 2\} \), and so there is a unique valid direction \( d(\vec{x}) = \left[ \begin{array}{c} -\frac{1}{2} \\ -\frac{1}{2} \end{array} \right] \). Moreover, \( \gamma(t) = (\frac{t}{2}, \frac{t}{2}) \) satisfies \( \frac{d\gamma(t)}{dt} = d(\gamma(t)) \) for all \( t \) and hence Theorem 2 implies that \( \gamma(t) = (\frac{t}{2}, \frac{t}{2}) \) is the unique TTC path, and the cutoff points \( p_k^j = \sqrt{1 - 2q} \) give the unique TTC allocation.

Consider now the economy \( \mathcal{E}_1 \), where top priority students have ranks uniformly distributed in the \( \bar{r} \times \bar{r} \) square \((1 - \bar{r}, 1] \times (m, m + \bar{r})\) for some small \( \bar{r} \).

If \( x \) is in \((1 - \bar{r}, 1] \times [m + \bar{r}, 1] \) then \( H_1^j(x) = \frac{1}{2} \left( m + (1 - m) \frac{1-m}{\bar{r}} \right) \) for all \( j \) and
If $x$ is in $(m, 1 − ̄r) \times (m, 1]$ then $H^j_i(x) = \frac{m}{2}$ for all $i, j$ and there is a unique valid direction $d(x) = \left[\begin{array}{c} -\frac{1}{2} \\ -\frac{1}{2} \end{array}\right]$.

Finally, if $x = (x_1, x_2)$ is in $[0, 1] \setminus (m, 1]^2$ then $H^j_1(x) = \frac{1}{2}x_2$ and $H^j_2 = \frac{1}{2}x_1$ for all $j$ and there is a unique valid direction $d(x) = \frac{1}{x_1 + x_2} \left[\begin{array}{c} -x_1 \\ -x_2 \end{array}\right]$. Hence the TTC path $\gamma(t)$ has gradient proportional to $\left[\begin{array}{c} -1 \\ -1 - (1-m)^2 \end{array}\right]$ from the point $(1, 1)$ to the point $(1 − ̄r, 1 − ̄r − \frac{r^2}{1−r}), \text{ to } \left[\begin{array}{c} -\frac{1}{2} \\ -\frac{1}{2} \end{array}\right]$ from the point $(1 − ̄r, 1 − ̄r − \frac{r^2}{1−r})$ to the point $(m + \frac{r^2}{1−r}, m)$ and to $\left[\begin{array}{c} -1 \\ -1 - (1-m)^2 \end{array}\right]$ from the point $(m + \frac{1-m}{m}, m)$ to the point $(\sqrt{\frac{1-2q}{1-2m+2m^2}}, \sqrt{(1-2q)(1-2m+2m^2)}) = (\bar{p}, p)$.

Finally, we show that if economy $E_2$ is given by perturbing the relative ranks of students in $\{\theta \mid r^c \geq m \forall c\}$, then the TTC cutoffs for $E_2$ are given by $p_1^1 = p_2^1 = x$, $p_1^2 = p_2^2 = y$ where $x \leq \bar{p} = \sqrt{\frac{1-2q}{1-2m+2m^2}}$ and $y \geq p = \sqrt{(1-2q)(1-2m+2m^2)}$. (By symmetry, it follows that $p \geq x, y \leq \bar{p}$.) Let $\gamma_1$ and $\gamma_2$ be the TTC paths for $E_1$ and $E_2$ respectively. Then the TTC path $\gamma_2$ for $E_2$ has gradient $\frac{1}{x_{\text{bound}} + m} \left[\begin{array}{c} -x_{\text{bound}} \\ -m \end{array}\right]$ from $(x_{\text{bound}}, m)$ to $(x, y)$.

Consider the aggregate trade balance equations for students assigned before the TTC path reaches $(x_{\text{bound}}, m)$. They stipulate that the measure of students in $[0, m] \times [m, 1)$ who prefer school 1 is at most the measure of students who are either perturbed, or in $[x_{\text{bound}}, 1] \times [0, m]$, and who prefer school 2. This means that $\frac{1}{2}m(1 − m) \leq \frac{1}{2}((1 − m)^2 + m(1 − x_{\text{bound}}))$, or $x_{\text{bound}} \leq m + (1-m)^2$, and hence $\gamma_2$ lies above $\gamma_1^{39}$ and so $x \leq \bar{p}$ and $y \geq \frac{1-2q}{p} = p$.

\[^{39}\text{That is, for each } x_1, \text{ if } (x_1, y_1) \text{ lies on } \gamma_1 \text{ and } (x_1, y_2) \text{ lies on } \gamma_2, \text{ then } y_2 \geq y_1.\]
E.3 Comparing Top Trading Cycles and Deferred Acceptance

In this section, we derive the expressions for the TTC and DA cutoffs given in Section 4.3.

Consider the TTC cutoffs for the neighborhood priority setting. We prove by induction on \( \ell \) that \( p^\ell_j = 1 - \frac{q}{2q} \) for all \( \ell, j \) such that \( j \geq \ell \).

**Base case:** \( \ell = 1 \).
For each school \( i \), there are measure \( q \) of students whose first choice school is \( i \), \( \alpha q \) of whom have priority at \( i \) and \( (1 - \alpha)q \) of whom have priority at school \( j \), for all \( j \neq i \).
The TTC path is given by the diagonal, \( \gamma(t) = (1 - \frac{t}{\sqrt{n}}, 1 - \frac{t}{\sqrt{n}}, \ldots, 1 - \frac{t}{\sqrt{n}}) \).
At the point \( \gamma(t) = (x, x, \ldots, x) \) (where \( x \geq \frac{1}{2} \)) a fraction \( 2(1 - x) \) of students from each neighborhood have been assigned. Since the same proportion of students have each school as their top choice, this means that the quantity of students assigned to each school is \( 2(1 - x)q \). Hence the cutoffs are given by considering school 1, which has the smallest capacity, and setting the quantity assigned to school 1 equal to its capacity \( q_1 \). It follows that \( p^1_j = x^* \) for all \( j \), where \( 2(1 - x^*)q = q_1 \), which yields

\[
p^1_j = 1 - \frac{q_1}{2q} \quad \text{for all } j.
\]

**Inductive step.**
Suppose we know that the cutoffs \( \{p^\ell_j\}_{i,j:i\leq\ell} \) satisfy \( p^\ell_j = 1 - \frac{q}{2q} \). We show by induction that the \((\ell + 1)\)th set of cutoffs \( \{p^{\ell+1}_j\}_{j>\ell} \) are given by \( p^{\ell+1}_j = 1 - \frac{q_{\ell+1}}{2q} \).

The TTC path is given by the diagonal when restricted to the last \( n - \ell \) coordinates, \( \gamma(t^{\ell} + t) = (p^1, p^2, \ldots, p^\ell, p^\ell - \frac{t}{\sqrt{n-\ell}}, p^\ell - \frac{t}{\sqrt{n-\ell}}, \ldots, p^\ell - \frac{t}{\sqrt{n-\ell}}) \).

Consider a neighborhood \( i \). If \( i > \ell \), at the point \( \gamma(t) = (p^1, p^2, \ldots, p^\ell, x, x, \ldots, x) \) (where \( x \geq \frac{1}{2} \)) a fraction \( 2(p^\ell - x) \) of (all previously assigned and unassigned) students from neighborhood \( i \) have been assigned in round \( \ell + 1 \). If \( i \leq \ell \), no students from neighborhood \( i \) have been assigned in round \( \ell + 1 \).

Consider the set of students \( S \) who live in one of the neighborhoods \( \ell + 1, \ell + 2, \ldots, n \). The same proportion of these students have each remaining school as their top choice out of the remaining schools. This means that for any \( i > \ell \), the quantity of students assigned to school \( i \) in round \( \ell + 1 \) by time \( t \) is a \( \frac{1}{n-\ell} \) fraction of the total number of students assigned in round \( \ell + 1 \) by time \( t \), and is given by \((n-\ell)q\frac{1}{n-\ell} = \)
2 \left( p_{\ell}^j - x \right) q. Hence the cutoffs are given by considering school \( \ell + 1 \), which has the smallest residual, and setting the quantity assigned to school \( \ell + 1 \) equal to its residual capacity \( q_{\ell+1} - q_{\ell} \). It follows that \( p_{j}^{\ell+1} = x* \) for all \( j > \ell \) where \( 2 \left( p_{\ell}^j - x* \right) q = q_{\ell+1} - q_{\ell} \), which yields

\[
p_{j}^{\ell+1} = p_{\ell}^j - \frac{q_{\ell+1} - q_{\ell}}{2q} = 1 - \frac{q_{\ell}}{2q} - \frac{q_{\ell+1} - q_{\ell}}{2q} = 1 - \frac{q_{\ell+1}}{2q} \quad \text{for all } j > \ell.
\]

This completes the proof that the TTC cutoffs are given by \( p_j^i = p_i^j = 1 - \frac{q_i}{2q} \) for all \( i \leq j \).

Now consider the DA cutoffs. We show that the cutoffs \( p_i = 1 - \frac{q_i}{2q} \) satisfy the supply-demand equations. We first remark that the cutoff at school \( i \) is higher than all the ranks of students without priority at school \( i \), \( p_i \geq \frac{1}{2} \). Since every student has priority at exactly one school, this means that every student is either above the cutoff for exactly one school and is assigned to that school, or is below all the cutoffs and remains unassigned. Hence there are \( 2q \left( 1 - p_i \right) = q_i \) students assigned to school \( i \) for all \( i \), and the supply-demand equations are satisfied.

\section{Proofs for Section A}

\subsection{Derivation of Marginal Trade Balance Equations}

In this section, we show that the marginal trade balance equations (1) hold,

\[
\gamma' (\tau) = \gamma' (\tau) \tilde{H} (\gamma (\tau)) \quad \text{for all } \tau.
\]

The idea is that the measure of students who trade into a school \( c \) must be equal to the measure of students who trade out of \( c \).

In particular, suppose that at some time \( \tau \) the TTC algorithm has assigned exactly the set of students with rank better than \( x = \gamma (\tau) \), and the set of available schools is \( C \). Consider the incremental step of a TTC path \( \gamma \) from \( \gamma(\tau) = x \) over \( \epsilon \) units of time. The process of cycle clearing imposes that for any school \( c \in C \), the total amount of seats offered by school \( c \) from time \( \tau \) to \( \tau + \epsilon \) is equal to the amount of students assigned to \( c \) plus the amount of seats that were offered but not claimed over that same time period. In the continuum model the set of seats offered but not

35
claimed is of $\eta$-measure 0. Hence the set of students assigned to school $c$ from time $\tau$ to $\tau + \epsilon$ has the same measure as the set of students that were offered a seat at school $c$ in that time,

$$\eta\left(\{\theta \in \Theta \mid r^\theta \in [\gamma(\tau + \epsilon), \gamma(\tau)) \text{ and } Ch^\theta(C) = c\}\right)$$

$$= \eta\left(\{\theta \in \Theta \mid \exists r' \in [\tau, \tau + \epsilon] \text{ s.t. } r^\theta_c = \gamma_c(\tau') \text{ and } r^\theta \leq \gamma(\tau')\}\right), \quad (13)$$

or more compactly, $\eta(T^c(\gamma; [\tau, \tau + \epsilon])) = \eta(T_c(\gamma; [\tau, \tau + \epsilon]))$.

Let us divide equation (13) by $\delta_c(\epsilon) = \gamma_c(\tau) - \gamma_c(\tau + \epsilon)$ and take the limit as $\epsilon \to 0$. We will show that the resulting left hand side expression is equal to

$$\sum_{b \in C} \lim_{\epsilon \to 0} \frac{\delta_b(\epsilon)}{\delta_c(\epsilon)} \cdot \frac{1}{\delta_b(\epsilon)} \eta\left(\{\theta \in \Theta \mid r^\theta \in [x - \delta_b \cdot e^b, x) \text{ and } Ch^\theta(C) = c\}\right) \quad (14)$$

where $e^b$ denotes the unit vector in the direction of coordinate $b$, and $\delta_b(\epsilon) = (\gamma_b(\tau) - \gamma_b(\tau + \epsilon))$. Similarly, we will show that the resulting right hand side expression is equal to

$$\lim_{\delta \to} \frac{1}{\delta} \eta\left(\{\theta \in \Theta \mid r^\theta \in [x - \delta \cdot e^c, x]\}\right) = \sum_{a \in C} H^a_c(x). \quad (15)$$

After equating equations (14) and (15), a little algebra shows that this is equivalent to the marginal trade balance equations (1),

$$\gamma'(\tau) = \gamma'(\tau) \tilde{H}(\gamma(\tau)).$$

Let us now formally prove that the marginal trade balance equations follow from equation (13). For $b, c \in C$, $x \in [0, 1]^C$, $\alpha \in \mathbb{R}$ we define the set.

---

40 A student can have a seat that is offered but not claimed in one of two ways. The first is the seat is offered at time $\tau$ and not yet claimed by a trade. The second is that the student that got offered two or more seats at the same time $\tau' \leq \tau$ (and was assigned through a trade involving only one seat). Both of these sets of students are of $\eta$-measure 0 under our assumptions.

41 The fact that the quantities in equation (13) are equal to the quantities in equations (14) and (15) follows from our assumption that the density is bounded, since in both cases we double count a set of students whose ranks have Lebesgue measure tending to 0.

42 We use the notation $[\underline{x}, \bar{x}] = \{z \in \mathbb{R}^n \mid \underline{x}_i \leq z_i < \bar{x}_i \forall i\}$ for $\underline{x}, \bar{x} \in \mathbb{R}^n$, and $e^c \in \mathbb{R}^C$ is a vector
\[ T^c_b (x, \alpha) \doteq \{ \theta \in \Theta \mid r^\theta \in [x - \alpha \epsilon^b, x) \text{ and } Ch^\theta \left( C (r^\theta) \right) = c \}. \]

We may think of \( T^c_b (x, \alpha) \) as the set of the next \( \alpha \) students on school \( b \)'s priority list who are unassigned when \( \gamma (\tau) = x \), and want school \( c \). We remark that the sets used in the definition of the \( H^c_b (x) \) are precisely the sets \( T^c_b (x, \alpha) \).

We can use the sets \( T^c_b (x, \alpha) \) to approximate the expressions in equation (13) involving \( T^c_c (\gamma; \cdot) \) and \( T^c (\gamma; \cdot) \).

**Lemma 9.** Let \( \gamma (\tau) = x \) and for all \( \epsilon > 0 \) let \( \delta (\epsilon) = \gamma (\tau) - \gamma (\tau + \epsilon) \). For sufficiently small \( \epsilon \), during the interval \([\tau, \tau + \epsilon]\), the set of students who were assigned to school \( c \) is

\[ T^c (\gamma; [\tau, \tau + \epsilon]) = \bigcup_b T^c_b (x, \delta_b (\epsilon)) \]

and the set of students who were offered a seat at school \( c \) is

\[ T^c_c (\gamma; [\tau, \tau + \epsilon]) = \bigcup_d T^d_c (x, \delta_c (\epsilon)) \cup \Delta = \bigcup_b T^c_b (x, \delta_b (\epsilon)) \]

for some small set \( \Delta \subset \Theta \). Further, it holds that \( \lim_{\tau \to 0^+} \frac{1}{\tau} \cdot \eta (\Delta) = 0 \), and for any \( c \neq c', d \neq d' \in C \) we have \( \lim_{\tau \to 0^+} \frac{1}{\tau} \cdot \eta (T^d_c (x, \delta_c (\epsilon)) \cap T^d_{c'} (x, \delta_{c'} (\epsilon))) = 0 \) and \( T^d_c (x, \delta_c (\tau)) \cap T^d_{c'} (x, \delta_{c'} (\epsilon)) = \phi \).

**Proof.** The first two equations are easily verified, and the fact that the last intersection is empty is also easy to verify. To show the bound on the measure of \( \Delta \), we observe that it is contained in the set \( \bigcup_c \bigcup_d (T^d_c (x, \delta_c (\epsilon)) \cap T^d_{c'} (x, \delta_{c'} (\epsilon))) \), so it suffices to show that \( \lim_{\tau \to 0^+} \frac{1}{\tau} \cdot \eta (T^d_c (x, \delta_c (\epsilon)) \cap T^d_{c'} (x, \delta_{c'} (\epsilon))) = 0 \). This follows from the fact that the density defining \( \eta \) is upper bounded by \( M \), so \( \eta \left( T^d_c (x, \delta_c (\epsilon)) \cap T^d_{c'} (x, \delta_{c'} (\epsilon)) \right) \leq M |\gamma_c (\tau) - \gamma_{c'} (\tau + \epsilon)| \leq \gamma_c (\tau) - \gamma_{c'} (\tau + \epsilon) \). Since for all schools \( c \) the function \( \gamma_c \) is continuous and has bounded derivative, it is also Lipschitz continuous, so

\[ \frac{1}{\tau} \eta (\Delta) \leq \frac{1}{\tau} \eta (T^d_c (x, \delta_c (\epsilon)) \cap T^d_{c'} (x, \delta_{c'} (\epsilon))) \leq ML_c L_{c'} \epsilon \]

for some Lipschitz constants \( L_c \) and \( L_{c'} \) and the lemma follows. \( \square \)

We now now ready to take limits and verify that equation (13) implies that the marginal trade balance equations hold. Let us divide equation (13) by \( \delta_c (\epsilon) = \) whose \( c \)-th coordinate is equal to 1 and all other coordinates are 0.
\(\gamma_c(\tau) - \gamma_c(\tau + \epsilon)\) and take the limit as \(\epsilon \to 0\). Then on the left hand side we obtain

\[
\lim_{\epsilon \to 0} \frac{1}{\delta_c(\epsilon)} \eta(T_c^\epsilon(\gamma; [\tau, \tau + \epsilon])) = \lim_{\epsilon \to 0} \frac{1}{\delta_c(\epsilon)} \eta \left( \bigcup_b T_b^\epsilon(x, \delta_b(\epsilon)) \right) \quad \text{(Lemma 9)}
\]

\[
= \lim_{\epsilon \to 0} \left[ \sum_{b \in C} \frac{1}{\delta_c(\epsilon)} \eta(T_b^\epsilon(x, \delta_b(\epsilon))) + O \left( \frac{||\gamma(\tau) - \gamma(\tau + \epsilon)||_\infty^2}{\delta_c(\epsilon)} \right) \right] \quad (\nu < M)
\]

\[
= \lim_{\epsilon \to 0} \left[ \sum_{b \in C} \frac{\delta_b(\epsilon)}{\delta_c(\epsilon)} \cdot \frac{1}{\delta_b(\epsilon)} \eta \left( \{ \theta \in \Theta \mid r^\theta \in [x - \delta_b(\epsilon) \cdot e^b, x) \text{ and } Ch^\theta(C) = c \} \right) \right]
\]

\[
= \sum_{b \in C} \frac{\gamma_b'(\tau)}{\gamma_c'(\tau)} \cdot H_b^{|C}(x) \quad \text{(by definition of } \delta \text{ and } H) \]

as required. Similarly, on the right hand side we obtain

\[
\lim_{\epsilon \to 0} \frac{1}{\delta_c(\epsilon)} \eta(T_c(\gamma; [\tau, \tau + \epsilon])) = \lim_{\epsilon \to 0} \left[ \sum_{a \in C} \frac{1}{\delta_c(\epsilon)} \eta(T_a^c(x, \delta_c(\epsilon))) + O \left( \frac{||\gamma(\tau + \epsilon) - \gamma(\tau)||_\infty^2}{\delta_c(\epsilon)} \right) \right] \quad \text{(Lemma 9)}
\]

\[
= \lim_{\epsilon \to 0} \left[ \sum_{a \in C} \frac{1}{\delta_c(\epsilon)} \eta(T_a^c(x, \delta_c(\epsilon))) \right] \quad (\gamma \text{ is Lipschitz continuous})
\]

\[
= \lim_{\epsilon \to 0} \left[ \sum_{a \in C} \frac{1}{\delta_c(\epsilon)} \eta \left( \{ \theta \in \Theta \mid r^\theta \in [x - \delta_c(\epsilon) \cdot e^c, x) \text{ and } Ch^\theta(C) = a \} \right) \right]
\]

\[
= \sum_{a \in C} H_a^{|C}(x) \quad \text{(by definition of } \delta \text{ and } H) \]

as required. This completes the proof.

### F.2 Proof of Lemma 1

We prove the following slightly more general theorem.

**Theorem 5.** Let \( \mathcal{E} = (C, \Theta, \eta, q) \) be a continuum economy such that \( \tilde{H}(x) \) is irreducible for all \( x \) and \( C \). Then there exists a unique valid TTC path \( \gamma \). Within each round \( \gamma(\cdot) \) is given by

\[
\frac{d\gamma(t)}{dt} = d(\gamma(t))
\]

where \( d(x) \) is the unique valid direction from \( x = \gamma(t) \) that satisfies \( d(x) = d(x) \tilde{H}(x) \).

Moreover, if we let \( A(x) \) be obtained from \( \tilde{H}(x) - I \) by replacing the \( n \)th column

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with the all ones vector $\mathbf{1}$, then

$$d(x) = \begin{bmatrix} \mathbf{0}^T & -1 \end{bmatrix} A(x)^{-1}.$$

**Proof.** It suffices to show that $d(\cdot)$ is unique. The existence and uniqueness of $\gamma(\cdot)$ satisfying $\frac{d\gamma(t)}{dt} = d(\gamma(t))$ follows by invoking Picard-Lindelöf as in the proof of Theorem 2.

Consider the equations,

$$d(x) \tilde{H}(x) = d(x)$$
$$d(x) \cdot \mathbf{1} = -1.$$

When $\tilde{H}(x)$ is irreducible, every choice of $n - 1$ columns of $\tilde{H}(x) - I$ gives an independent set whose span does not contain $\mathbf{1}$. Therefore if we let $A(x)$ be given by replacing the $n$th column in $\tilde{H}(x) - I$ with $\mathbf{1}$, then $A(x)$ has full rank, and the above equations are equivalent to

$$d(x) A(x) = \begin{bmatrix} \mathbf{0}^T & -1 \end{bmatrix},$$

i.e. $d(x) = \begin{bmatrix} \mathbf{0}^T & -1 \end{bmatrix} A(x)^{-1}$.

Hence $d(x)$ is unique for each $x$, and hence $\gamma(\cdot)$ is uniquely determined. \[\square\]
TTCA path $\gamma_{all}$ clears all students in recurrent communication classes.

TTCA path $\gamma_1$ clears all students who want school 1 before students who want school 2.

TTCA path $\gamma_2$ clears all students who want school 2 before students who want school 1.

Figure 11: Three TTCA paths and their cutoffs and allocations for the discrete economy in example B. In each set of two squares, students in the left square prefer school 1 and students in the right square prefer school 2. The first round TTCA paths are solid, and the second round TTCA paths are dotted. The cutoff points $p^1$ and $p^2$ are marked by filled circles. Students shaded light blue are assigned to school 1 and students shaded dark blue are assigned to school 2.
Figure 12: The valid directions $d(x)$ for the continuum embedding $\Phi(E)$. Valid directions $d(x)$ are indicated for points $x$ in the grey squares (including the upper and right boundaries but excluding the lower and left boundaries), as well as for points $x$ on the black lines. Any vector $d(x)$ is a valid direction in the lower left grey square. The borders of the squares corresponding to the students are drawn using dashed grey lines.

Figure 13: Economy $E_1$ from Example 9. The black borders partition the space of students into four regions. The density of students is zero on white areas, and constant on each of the shaded areas within a bordered region. In each of the four regions, the total measure of students within is equal to the total area (white and shaded) within the borders of the region.