On Characterizing the Relationship between Lower Bound Methods in Communication Complexity

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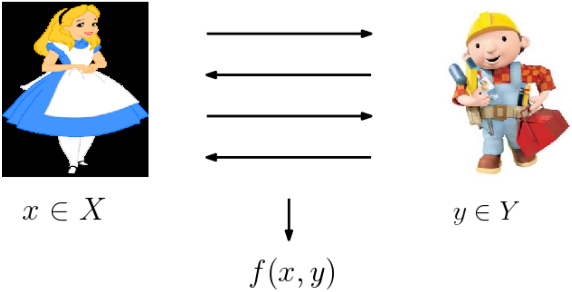
Introduction: Communication Complexity

$$f : X \times Y \to \{0, 1\}$$



- What's the minimum number of bits they need to compute the function correctly?
- We don't care about the running time and space usage of Alice and Bob.

Different Models

- Deterministic: Alice and Bob can only take deterministic actions.
- **Randomized**: Alice and Bob have access to a *joint* (public) source of randomness. Also, they are allowed a small probability of getting the wrong answer.
  - If we restrict their randomness to be private, it turns out that CC is increased by only $O(\log n)$ bits.
- Quantum: Alice and Bob can send qubits.
  - More powerful version: Shared entanglement, but a classical communication channel.

Definition of Communication Complexity

Now, fix some model.

- For a given protocol π, we define $CC(\pi)$ to be the number of bits Alice and Bob communicate, in the worst case.
- For a given function f, we define $CC(f) = \min_{\pi} CC(\pi)$

Examples of Problems

- Equality: Alice and Bob have bit strings x and y, and they want to determine if $x = y$ or $x \neq y$.
  - $DCC(EQ) = \Theta(n)$
  - $RCC^{pub}(EQ) = \Theta(1)$, $RCC(EQ) = \Theta(\log n)$
  - $QCC(EQ) = \Theta(\log n)$
- Disjointness: Alice and Bob have bit strings x and y, and they want to determine if there is an index i s.t. $x_i = y_i = 1$.
  - $DCC(DISJ), RCC(DISJ) = \Theta(n)$
  - $QCC(DISJ) = \Theta(\sqrt{n})$.

Motivation

Communication complexity has applications to

- VLSI circuit design
- Circuit complexity
- Decision tree complexity
- Data structures
- Streaming algorithms
- ...and more ...

Motivation (quantum setting)

Quantum communication complexity has applications to

- Quantum speed-up (unconditional) in communication complexity
- one-shot quantum information theory
- quantum non-locality/Bell inequality

This talk: Lower Bounds

- Since we are working in a restricted model, we will be able to prove *unconditional* hardness results, which can be applied to other problems in computer science.
- In communication complexity, we want to find a lower bound such that for all functions f,

$$\text{bound}(f) \leq CC(f).$$

Characterization of deterministic Protocols

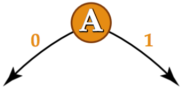
It turns out that any protocol corresponds a partition of M_f into *monochromatic rectangles*, i.e. rectangles (submatrices) that only have 0 or 1 in it.

0	1	1	1	0	1	1	1
0	1	1	1	0	1	1	1
0	1	1	1	0	0	0	0
0	1	1	1	1	1	0	1
0	0	0	1	1	1	0	1
1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1
0	0	0	0	0	1	1	1

$$F\colon \mathcal{X} \times \mathcal{Y} \to \{0,1\}$$

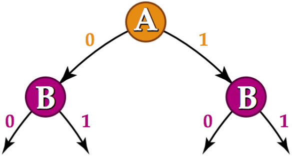
Protocols and Rectangles

0	1	1	1	0	1	1	1
0	1	1	1	0	1	1	1
0	1	1	1	0	0	0	0
0	1	1	1	1	1	0	1
0	0	0	1	1	1	0	1
1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1
0	0	0	0	0	1	1	1



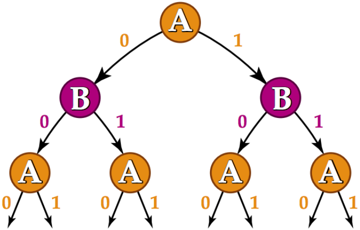
Protocol and Rectangles

0	1	1	1	0	1	1	1
0	1	1	1	0	1	1	1
0	1	1	1	0	0	0	0
0	1	1	1	1	1	0	1
0	0	0	1	1	1	0	1
1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1
0	0	0	0	0	1	1	1

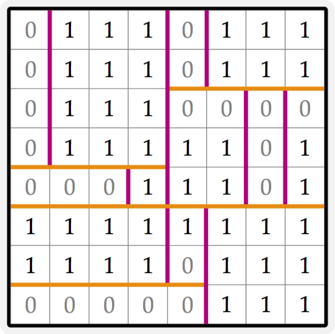


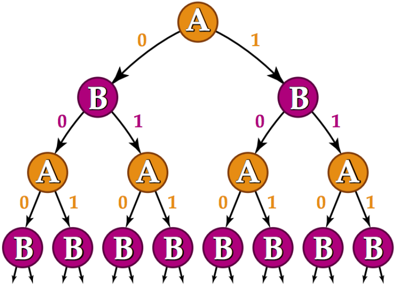
Protocol and Rectangles

0	1	1	1	0	1	1	1
0	1	1	1	0	1	1	1
0	1	1	1	0	0	0	0
0	1	1	1	1	1	0	1
0	0	0	1	1	1	0	1
1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1
0	0	0	0	0	1	1	1



Protocol and Rectangles

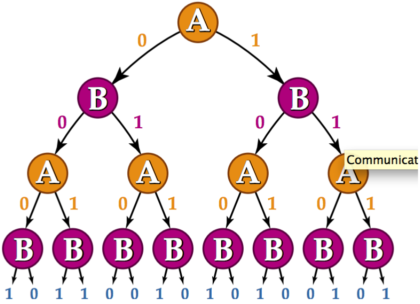




Protocol and Rectangles

When they end up inside a monochromatic rectangle, they don't need to partition any further and can output the answer.

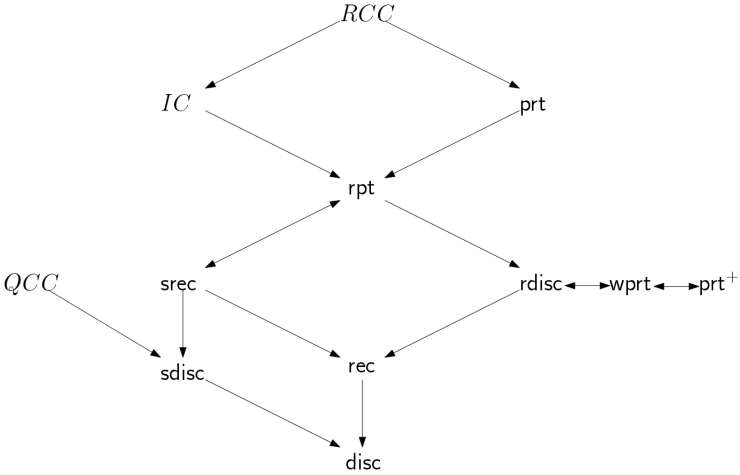
0	1	1	1	0	1	1	1
0	1	1	1	0	1	1	1
0	1	1	1	0	0	0	0
0	1	1	1	1	1	0	1
0	0	0	1	1	1	0	1
1	1	1	1	1	1	1	1
1	1	1	1	0	1	1	1
0	0	0	0	0	1	1	1



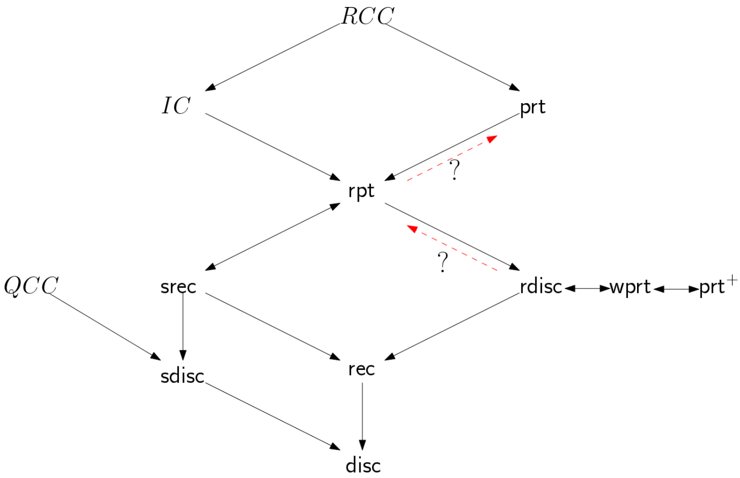
Protocols and Rectangles

- DCC is the height h of the tree. The number of leaves is roughly 2^h, and every leaf corresponds to a rectangle.
- To prove lower bounds on DCC (in fact, 2^{DCC}), we just prove that every partition requires many monochromatic rectangles.
- For RCC, the story is analogous: ϵ-monochromatic rectangles!
- The lower bounds methods we will present here are just different ways of "counting rectangles".

Map of Known Lower Bounds



Our Project



Techniques

- Our Goal: To get the best lower bound, we "optimize" over all possible lower bounds.
- Linear programming: All the lower bounds shown before can be formulated as the optimal value of a linear program. We can then compare these formulations.

Partition bound for RCC

- Idea: Any randomized protocol corresponds to a *convex combination* of partitions.
- For any protocol Π, define the variable, for each rectangle R and output $z \in \{0, 1\}$,

$$w_{z,R} = \Pr[(R, z) \in \Pi]$$

.

- The expected number of rectangles is

$$\sum_z \sum_R w_{z,R} \leq 2^{RCC}$$

Partition bound for RCC

- Accuracy: For every (x, y)

$$1 - \epsilon \leq \Pr[\Pi(x, y) = f(x, y)] = \sum_{R \ni (x,y)} w_{f(x,y),R}$$

- Correctness: For every (x, y)

$$\sum_{z \in Z} \sum_{R \ni (x,y)} w_{z,R} = 1$$

Partition Bound

- Linear program formulation of partition bound:

$$\begin{aligned}\mathrm{prt}_\epsilon(f) = \log\min \quad & \sum_z \sum_R w_{z,R} \\ \text{s.t.} \quad & \forall (x,y) \in X \times Y : \sum_{R \ni (x,y)} w_{f(x,y),R} \geq 1-\epsilon \\ & \forall (x,y) \in X \times Y : \sum_{z \in Z} \sum_{R \ni (x,y)} w_{z,R} = 1 \\ & \forall z \in Z, R \in \mathcal{R} : w_{z,R} \geq 0.\end{aligned}$$

- Fact: $RCC_\epsilon(f) \geq \mathrm{prt}_\epsilon(f)$

Other bounds

- Issue with partition bound: too general, can't actually use it!
- All other bounds can be obtained by relaxing partition bound. (Adding some valid constraint to the LP.)

Relaxed Partition Bound

- Same as partition bound, except that

$$\forall (x,y) \in X \times Y : \sum_{z \in Z} \sum_{R \ni (x,y)} w_{z,R} = 1$$

becomes

$$\forall (x,y) \in X \times Y : \sum_{z \in Z} \sum_{R \ni (x,y)} w_{z,R} \leq 1$$

- Fact: For the Gap-Hamming-Distance (GHD) problem,

$$\mathrm{rpt}_\epsilon(GHD) = \Omega(n).$$

Thus,

$$RCC_\epsilon(GHD) = \Theta(n).$$

Relative Discrepancy Bound

- Similar to prt, rpt, main difference is the following constraint:

$$\forall (x, y) \in X \times Y : \sum_{z \in Z} \sum_{R \ni (x,y)} w_{z,R} \geq 1$$

- Fact: For the Vector-in-Subspace (VSP) problem,

$$\text{rdisc}_\epsilon(VSP) = \Omega(n^{1/3}).$$

The best known upper bound is

$$RCC_\epsilon(VSP) = O(\sqrt{n}).$$

Rectangle Bound

- A one-sided relaxation of partition bound:

$$\begin{aligned} \text{rec}_\epsilon(f) = \log\min \quad & \sum_R w_R \\ \text{s.t.} \quad & \forall (x,y) \in f^{-1}(z) : \sum_{R \ni (x,y)} w_R \geq 1 - \epsilon \\ & \forall R \in \mathcal{R} : w_R \geq 0. \end{aligned}$$

- Fact: For the Disjointness (DISJ) problem,

$$\text{rec}_\epsilon(DISJ) = \Omega(n).$$

Thus,

$$RCC_\epsilon(DISJ) = \Theta(n).$$

Comparing Lower Bounds through linear programs

To show a bound is larger than another bound, for any feasible solution to one LP, we construct a solution to the other LP such that the objective value of the new solution is only better.

What's been proven using this method

- $\mathrm{rpt}(f) \geq \mathrm{rdisc}(f)$
- $\mathrm{rpt}(f) = \mathrm{srec}(f)$

Open problems:

- $\mathrm{rpt}(f) \stackrel{?}{=} \mathrm{rdisc}(f)$
- $\mathrm{rpt}(f) \stackrel{?}{=} \mathrm{prt}(f)$

Open problem for QCC

- Only two lower bound methods known for QCC.
  - In fact, they were originally introduced for RCC.
- For DCC and RCC, we have a simple characterization of protocols: partitions into monochromatic rectangles.
- **Question**: Can we come up with a similar characterization of quantum protocols?
  - Hopefully, this will lead to quantum-specific lower bound methods.

Thanks for listening!

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Questions?