Information Complexity vs. Communication Complexity: Hidden Layers Game

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Final Project Presentation for Information Theory in TCS

Introduction

- Review of IC vs CC
- Hidden Layers Game
- Upper Bound of IC
- Intuition for lower bound of CC

Communication Complexity and Information Complexity

Communication Complexity

$CC(f;\epsilon)$ is the smallest number of bits that Alice and Bob need to exchange to compute f with error probability ϵ.

Information Complexity

- The protocol π on the pair of (random) inputs $(X;Y) \sim \mu$ gives the transcript $\Pi = \Pi(X;Y)$
- The information cost of a protocol Π is the amount of information that the protocol reveals to Alice and Bob about their input.
- the amount revealed to Alice – who knows X – about Y is given by the conditional mutual information $I(Y;\Pi|X)$.

information cost of Π is given by:

$$IC_\mu(\pi) = I(Y;\Pi|X) + I(X;\Pi|Y)$$

Information Complexity

(continued) The task of finding information complexity of f is the task of minimizing the information complexity of the protocol for f:

$$IC_\mu = inf_{\text{protocol } \pi \text{ performing f}} IC_\mu(\pi)$$

IC vs CC

Information complexity can be viewed as the interactive analogue of Shannon's entropy.
Equality between information and communication complexity is equivalent to compression theorem in the interactive setting: whether it is possible to compress an interactive conversation into its information content like we compress a single message.

A problem with large CC and small IC

Hidden Layers Game problem

Conjectured in [Bra13] with lower bound proved in[GKR16]

- k is a parameter used in the problem
- IC upper bound $= O(logk)$
- CC Lower Bound $= \Omega(k)$

Exponential separation!
In fact proved separation with external IC(stronger result)
Using embedding of set disjointness inputs.

Hidden Layers Game

Hidden Layers Game is a sampling problem.
Parameters:

- strings over an alphabet Σ of size k
- another parameter $N = 2^n$. fix $N = 2^n = 2^{2^k}$.

Input:

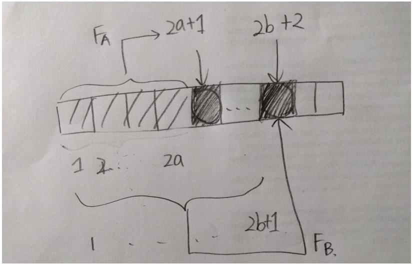
- Alice and Bob are given a pair of numbers $a, b \sim \{0, ..., N-1\}$ (i.e. two n-bit numbers that take $\Omega(logn)$ communication to compare)
- Alice is given a uniformly random function $F_A : \Sigma^{2a} \to \Sigma$ (the function is only known to Alice)
- Bob is given a uniformly random function $F_B : \Sigma^{2b+1} \to \Sigma$ (the function is only known to Bob)

Hidden Layers Game

Alice and Bob need to sample a uniformly random string $s \in \Sigma^{2N}$ subject to the constraints:

- $s_{2a+1} = F_A(s_{1\dots 2a})$
- $s_{2b+2} = F_B(s_{1\dots 2b+1})$

In other words, they want a $2N$-symbol string over the alphabet, where its first $2a$-symbols substring can be mapped to its $(2a+1)$th symbol by Alice's function and its first $(2b+1)$-symbols substring can be mapped to its $(2b+2)$th symbol by Bob's function.



Naive Protocol and IC upper bound

Naive Protocol π_0 for hidden layers game H

- in odd rounds Alice samples the next symbol of s and in even rounds Bob does.
- In rounds $i \neq 2a+1$, Alice just sends a uniformly random $s_i \sim \Sigma$. In round $i = 2a+1$, Alice computes and sends $si = F_A(s_{1..2a})$.
- Bob does the similar thing for even rounds

Communication complexity $= \Theta(N \log k) = \Theta(2^{2^k} \log k)$ ($2N$ rounds, we can view the encoding of each char s_i as $\log k$ since the alphabet has size k).

Naive Protocol and IC upper bound

s is sampled uniformly from the subset S of strings which satisfy the two constraints.
The size of S, $|S| = k^{2N-2}$; uniformly random except $(2a+1)$th and $(2b+2)$th symbols.
Thus the KL-divergence between s and the uniform distribution on Σ^{2N} is $2\log k$.

Naive Protocol and IC upper bound

The transcript of π_0 is distributed exactly as the output s of H given a, F_A, b, F_B.
Denote μ as the distribution of the inputs a, F_A, b, F_B to H.

$$IC_\mu(H) = IC_\mu(\pi_0) = I_\mu(s; a, F_A, b, F_B)$$
$$= \mathbf{E}\mathbb{D}(s|a; F_A, b, F_B||s) = 2logk$$

CC Lower Bound Intuition 1: Disjointness

- A randomly selected string $t \in \Sigma^{2N}$ has a probability of exactly $1/k$ of being consistent with Alice's input:
  t just needs its $(2a+1)$th symbol happen to satisfy
  $t_{2a+1} = F_A(t_{1..2a})$
  Same for Bob.
- So t has probability $1/k^2$ to be consistent with both Alice and Bob

CC Lower Bound Intuition 1: Disjointness

Protocol 1:

- Using public randomness and no communication;
  sample k^2 strings $s_1, ... s_{k^2}$ drawn uniformly at random from Σ^{2N}
- Let A be the subset (of approximately k) strings satisfying Alice's constraint
- Let B be the subset satisfying Bob's constraint
- Alice and Bob communicate to determine whether $A \cap B = \emptyset$, if not, they output the first element of $A \cap B$; otherwise they repeat the entire process.

CC Lower Bound Intuition 1: Disjointness

Correctness

- the first string in the intersection between A and B must satisfy distribution of s
- The probability that $A \cap B \neq \emptyset$ is approximately $1 - 1/e$, and the process will terminate after an expected constant number of iterations.

Lower bound

- CC upper bound for disjointness of two sets with size k is $O(k)$.
- if we can reduce to Disjointness...
- CC lower bound for disjointness of two sets with size k is $\Omega(k)$.

CC Lower Bound Intuition 2: Greater Than(GT)

Protocol 2:

- find a c such that $a \leq c \leq b$ or $a \geq c \geq b$,
  wlog assume that $a \leq c \leq b$
- Use public randomness and no communication, generate strings $t_1, t_2.. \in \Sigma^{2c+1}$ uniformly randomly
- Alice sends Bob the index of the fist string s_0 satisfying her constraint
- Using public randomness and no communication, generate strings $r_1, r_2.. \in \Sigma^{2N}$ which extend s_0 with uniformly random symbols
- Bob sends Alice the index of the first string s satisfying his constraint. This s is the output.

CC Lower Bound Intuition 2: Greater Than(GT)

The first step of comparing a, b requires CC of the Greater Than function.
As discussed above, the probability of a string to be acceptable is $1/k$, and therefore communicating the index of the first acceptable string requires $O(\log k)$ bits.

Lower bound for GT

a and b are $n - bit$ numbers.
$n = \log N$. Therefore the lb for GT here is
$\Omega(\log n) = \Omega(\log \log N) = \Omega(k)$.

Sampling vs Decision?

- Can this sampling problem be generalized to to a decision problem?
- if no such decision problem exists, what property makes protocols for decision problems easier to compress?
- Sampling problems require a lot of public randomness. Decision problems protocols: the answer is determined by messages sent by Alice and Bob.

Sampling vs Decision?

- Exponential separation of IC and CC for Boolean functions has been shown in [GKR15].
- With the introduction of a lower bound method: Relative discrepancy method.
- But is Relative discrepancy method to separate all boolean functions?
- Actually No! More in my report.

Questions?

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