Stability of solitary water waves with weak surface tension

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We consider the irrotational flow of a two-dimensional perfect fluid of unit density subject to the forces of gravity and surface tension. The fluid is bounded below by a rigid flat bottom $\{y = 0\}$ and above by a free surface $\{y = 1 + \eta(x, t)\}$. Travelling waves are formally critical points of the energy

$$\mathcal{H}(\eta, \phi) = \int_{\mathbb{R}} \left(\int_0^{1+\eta} \frac{1}{2} |\nabla \phi|^2 \, dy + \frac{1}{2}\eta^2 + \beta[\sqrt{1 + \eta_x^2} - 1] \right) dx,$$

subject to constrained impulse

$$\mathcal{I}(\eta, \phi) = \int_{\mathbb{R}} \eta_x \phi|_{y=1+\eta} \, dx = 2\mu,$$

where ϕ is the velocity potential and $\mu > 0$ a constant. The positive parameter β controls the strength of the surface tension. A natural question is whether one can prove the existence of solitary travelling waves in the form of constrained minimisers. By a standard argument this would imply that the solutions are stable (in a certain sense), since $\mathcal{H}$ and $\mathcal{I}$ are conserved functionals of the evolution problem.

In the case of weak surface tension, $\beta < 1/3$, B. Buffoni ('05) proved that there exists a sequence $\mu_n \to 0$ for which this is the case. The proof is based upon concentration-compactness. The resulting solutions are periodic wave trains modulated by exponentially decaying envelopes, to leading order described by a steady form of the NLS equation. The oscillations are caused by the fact that when $\beta < 1/3$ the minimum linear phase speed is attained at a non-zero wave number.

In this talk we will prove that Buffoni's result actually holds for *all* small μ. The key is to prove the existence of a minimising sequence which has a scaling similar to that of the model equation. This can then be used to prove strict sub-additivity of the function $c(\mu) = \inf\{\mathcal{H}(\eta, \phi) \colon \mathcal{I}(\eta, \phi) = 2\mu\}$, which allows us to use the concentration-compactness method for all $0 < \mu \ll 1$.

This is a joint work with M. Groves, Loughborough/Saarbrücken.