Visual Noise Due to Quantum Indeterminacies

John Morrison and David Anderson

November 17, 2009

In "Color in a Physical World" Morrison argues that certain assumptions imply:

> NOISE
> For any observer and any context: If two foundational colors reliably produce different experiences, then there is a foundational color between them that does not reliably produce any particular experience (even if it will always produce some experience or another).

For concreteness, Morrison's argument focuses on the noise produced by internal, biological processes. We will establish that, given four assumptions, NOISE is true even for observers whose internal, biological processes do not produce any noise. In particular, we will establish that certain quantum indeterminacies by themselves suffice for NOISE.[1]

The four assumptions:

i. The relevant contexts are governed by the laws of quantum mechanics.

ii. The visual systems of the relevant observers cannot take infinitely long measurements.

iii. The visual systems of the relevant observers produce at least three different color experiences.

iv. Foundational colors are as fine-grained as spectral dispositions.

Our argument focuses on an observer with an **ideal visual system**, which is a visual system that does not have any computational limits, has perfect information about the illuminant and has internal processes that do not contribute any noise. If we can show that NOISE is true for all observers with ideal visual systems then that will show that NOISE is true for all observers. Consider that, if an observer with a non-ideal visual system were a counter-example to NOISE, then an observer with an idealization of that visual system would also be

[1] We are using 'quantum indeterminacies' so that it includes any probability distribution on a set of measurements that is due to the laws of quantum mechanics.

a counter-example because better measurements, better mathematics and better information would make the visual system even more sensitive to small differences between foundational colors.

Visual systems take measurements of the spectral power distribution of light rays. These measurements involve measurements of both wavelengths and intensities. There are many ways for a visual system to take these measurements. For example, like the common spectrophotometer, a visual system might use a prism to separate light by wavelength and then measure the intensity of each separated signal.

Our strategy is to first consider an observer whose visual system *only* measures the intensity of a light wave, and then consider an observer whose visual system *only* measures the wavelength of a single photon. In each case we will show that quantum indeterminacies by themselves suffice for NOISE. We will then generalize to an observer who measures both wavelengths and intensities.

To simplify, we will focus on light *after* it has been reflected, emitted or transmitted from a surface. Because reflection, emission and transmission introduce additional indeterminacies, we are effectively setting aside additional reasons why NOISE is true.

We are assuming that the foundational colors are as fine-grained as spectral dispositions (see (iv) above). Spectral dispositions are functions that, for a given content and location, specify the properties of the light reflected, emitted or transmitted by an object to that location. Therefore, given our assumption, if two objects produce light rays that are characterized by different wave functions then those objects have different foundational colors.

Intensity

The mean intensity of a light wave will be $x \in \Re^+$. Any measurement of the intensity of that light wave will be a nonnegative integer $m \in \mathbb{N}^+$. It will be a nonnegative integer because light is composed of indivisible photons. Due to quantum indeterminacies, for any measurement of finite length, the probability of measuring m photons from a coherent light with mean intensity x is governed by the Poisson distribution:

$$prob(m|x) = \frac{e^{-x}x^m}{m!} \tag{1}$$

Incoherent light is noisier, so we'll focus on coherent light. For more about the physics behind the Poisson distribution see Fox [3, p.76-80] and Orszag [5, p.29-32].

Let $E1$, $E2$ and $E3$ be three different color experiences that the visual system produces. Let Ω_{E1} be the set of all measurements that lead that visual system to produce $E1$, Ω_{E2} be the set of all measurements that lead it to produce $E2$, and Ω_{E3} be the set of all measurements that lead it to produce $E3$. How the ideal visual system decides which measurements should produce which experience is an interesting question. No doubt it will rely on perfect information about the illuminant. But we don't need to worry about these details.

Building on (1), the following function gives the probability that a light wave with mean intensity $x \in \Re^{+}$ will produce experience $E1$ in the relevant observer:

$$g_{E1}(x) = \sum_{m \in \Omega_{E1}} \frac{e^{-x} x^m}{m!}. \tag{2}$$

Likewise for $g_{E2}(x)$ and $g_{E3}(x)$.

Because $E1$, $E2$ and $E3$ are different experiences, each measurement will produce only one of them. That is:

$$\Omega_{E1} \cap \Omega_{E2} = \emptyset \tag{3}$$

$$\Omega_{E1} \cap \Omega_{E3} = \emptyset \tag{4}$$

$$\Omega_{E2} \cap \Omega_{E3} = \emptyset \tag{5}$$

As is evident from the fact that the Poisson distribution is a probability distribution, for any value $x \in \Re^{+}$:

$$\sum_{m=0}^{\infty} \frac{e^{-x} x^m}{m!} = 1 \tag{6}$$

It follows from (2)-(6) that for any value $x \in \Re^{+}$:

$$g_{E1}(x) + g_{E2}(x) + g_{E3}(x) \leq 1. \tag{7}$$

Next, we're assuming that the the relevant visual system produces each experience ($E1$, $E2$ and $E3$), at least sometimes. It follows that:

$$\Omega_{E1}, \Omega_{E2}, \Omega_{E3} \neq \emptyset \tag{8}$$

Building on (1), (8) and the fact that $prob(m|x) > 0$ for any $x \in \Re^+, m \in \mathbb{N}^+$, it follows that for any value x:

$$g_{E1}(x), g_{E2}(x), g_{E3}(x) \neq 0. \tag{9}$$

Finally, to make things concrete, we will assume that the threshold for reliably production is at 0.5 so that, for instance, a light with mean intensity value x reliably causes experience $E1$ if and only if $prob(E1|x) \geq 0.5$. The proof straightforwardly generalizes to all thresholds above 0.5.

Given these assumptions, let's now prove that NOISE is true for an observer who only measures the intensity of a light:

Theorem 1. *If for some value of* $x, y \in \Re^+, g_{E1}(x) \geq 0.5$ *and* $g_{E2}(x+y) \geq 0.5$ *then there is an* i *such that* $0 < i < y$ *and* $g_1(x+i), g_2(x+i), g_3(x+i) < 0.5$.

Proof. The main point is that g_{E1}, g_{E2}, and g_{E3} are continuous for all $x \in \Re^+$. We will prove this for g_{E1}, the proof for the other two functions being analogous.

For the purposes of this proof, write:

$$p_m(x) = \frac{e^{-x}x^m}{m!}$$

for the Poisson distribution function. Also, let

$$\chi_m = \begin{cases} 1 & \text{if } m \in \Omega_{E1}; \\ 0 & \text{otherwise} \end{cases}$$

be the indicator function for Ω_{E1}. Using this notation, we can write

$$g_{E1}(x) = \sum_{m=0}^{\infty} \chi_m \, p_m(x). \tag{10}$$

By the uniform convergence theorem, it suffices to show the series on the right-hand side of (10) converges uniformly. For this, we will first prove that another, related function is continuous. Let $N > 0$. We claim that the function

$$\widetilde{g}_{E1}(x) = \sum_{m=N}^{\infty} \chi_m \, p_m(x) \tag{11}$$

is continuous for all x in the interval $[0, N]$. We will use the Weierstrass M-test. Here we need an easily verified property of the functions $p_m(x)$: For $m \geq N$, the maximum value of $p_m(x)$ on the interval $[0, N]$ is $p_m(N) = \frac{e^{-N} N^m}{m!}$. Now apply the M-test with $M_m = p_m(N)$, and observe that the series

$$\sum_{m=N}^{\infty} M_m = \sum_{m=N}^{\infty} \frac{e^{-N} N^m}{m!}$$

converges, since it is dominated by $\sum_{m=0}^{\infty} \frac{e^{-N} N^m}{m!} = 1$. Therefore the series in (11) converges uniformly for x in $[0, N]$, so $\widetilde{g}_{E1}$ is continuous on $[0, N]$.

We will now prove that g_{E1} is continuous. First note that g_{E1} is the sum of $\widetilde{g}_{E1}$ and the (finitely many) continuous functions $\chi_m \, p_m(x)$ for $m < N$. Therefore, g_{E1} is continuous on the interval $[0, N]$. Because N is chosen arbitrarily, it follows that g_{E1} is continuous for all $x \in \Re^+$.

Similarly, g_{E2} and g_{E3} are continuous for all $x \in \Re^+$

Now suppose that for some values $x, y \in \Re^+, g_{E1}(x) \geq 0.5$ and $g_{E2}(x + y) \geq 0.5$. By (7) and (9), it follows that $g_{E1}(x + y) < 0.5$. Further, because $g_{E1}(x)$ is continuous for all $x \in \Re^+$, it follows that there is a $0 \leq k < y$ such that $g_{E1}(x + k) = 0.5$. By (7) and (9), it follows that $g_{E2}(x + k) < 0.5$. Further, because $g_{E2}(x)$ is continuous for all $x \in \Re^+$, there must be a value $0 \leq \delta$ such that $g_{E2}(x + k + \delta) = 0.5$. Therefore, we can establish the theorem just by letting $x + k < i < x + k + \delta$. At that point $g_{E1}(x), g_{E2}(x), g_{E3}(x) < 0.5$. In fact, the consequent of the theorem will not just be true at a single point, it will be true for all points in the interval $(x + k, x + k + \delta)$. □

This proof establishes that, due to the quantum indeterminacies that are described by (1), NOISE is true for an ideal observer who is measuring only intensity. In particular, if two foundational colors each reliably produce a different color experience ($E1$, $E2$ or $E3$), and those foundational colors correspond to light with intensities x and $x + y$ then there will be a foundational color that corresponds to light with intensity $x + i$ that does not reliably produce $E1$, does not reliably produce $E2$ and does not reliable produce $E3$.

Wavelength

Let's now consider an observer whose visual system is only measuring the wavelength of a single photon. We will establish NOISE in two ways. The first way will appeal to the quantum indeterminacies that help produce thermal Doppler shifting. The second way will appeal to indeterminacies in measurements of photons with certain wave functions.

First way: thermal Doppler shifting

Let's start with the simplest case for the visual system: when the photon is monochromatic. This is the simplest case because, as we will see later, photons with mixed wavelengths contribute additional indeterminacies. The wave function for a monochromatic photon in free space is:

$$\psi(x) = \sin(2\pi\omega x) \tag{12}$$

where its wavelength, $\lambda \in [400, 700]$, is the quotient of the speed of light, c, and its frequency, ω. That is, $\lambda = \frac{c}{\omega}$.

Due to quantum indeterminacies, when any photon passes through a gas its wavelength is thermally Doppler shifted. The probability of different shifts depends, in part, on the value of the self-diffusion constant of the gas, D, which depends on temperature and pressure. For details see chapter eight of Jeans [4]. To simplify, we treat D as a constant; perhaps the visual system has some way of determining the temperature and pressure of the gas. Due to quantum indeterminacies, the probability that an ideal visual system will measure the photon's wavelength within some interval of wavelengths can be determined by integrating the following function:

$$p(\alpha, \lambda) = \frac{\frac{2\pi D}{\lambda^2}}{(\alpha - \frac{c}{\lambda})^2 + (\frac{2\pi D}{\lambda^2})^2} \tag{13}$$

For details, see Dickie [2].

As before, for an arbitrary visual system that is ideal, let $E1$, $E2$ and $E3$ be different experiences. Also, let Ω_{E1}, Ω_{E2} and Ω_{E3} be measurable subsets such that Ω_{E1} is the set of all measurements that lead it to produce $E1$, Ω_{E2} is the set of all measurements that lead it to produce $E2$, and Ω_{E3} is the set of all measurements that lead it to produce $E3$.

Where

$$\chi_{E1}(\alpha) = \begin{cases} 1 & \text{if } \alpha \in \Omega_{E1}; \\ 0 & \text{otherwise} \end{cases}$$

the probability that a photon that is monochromatic with wacelength $\lambda \in [400, 700]$ will produce an $E1$ experience is:

$$f_{E1}(\lambda) = \int_{400}^{700} \chi_{E1}(\alpha) p(\alpha, \lambda)\, d\alpha \tag{14}$$

Likewise for f_{E2} and f_{E3}.

Without rehearsing all the details, in the same way that we established (7) and (9) in our discussion of intensity measurements, we can also establish that for all $\lambda \in [400, 700]$:

$$f_{E1}(x) + f_{E2}(x) + f_{E3}(x) \leq 1 \tag{15}$$

$$f_{E1}(x), f_{E2}(x), f_{E3}(x) \neq 0 \tag{16}$$

Finally, let's again assume that the relevant threshold is at 0.5.

Given these assumptions, NOISE is true for an observer whose visual system only measures the wavelength of single monochromatic photons:

Theorem 2. *If for some λ and λ', $f_{E1}(\lambda) \geq 0.5$ and $f_{E2}(\lambda') \geq 0.5$ then there is a λ'' such that $f_1(\lambda''), f_2(\lambda''), f_3(\lambda'') < 0.5$.*

Proof. $p(\alpha, \lambda)$ is continuous for a fixed α. Just note that it is the quotient of two continuous functions and the denominator is never zero. Therefore, $f_{E1}(\lambda)$ is also continuous. Likewise for f_{E2} and f_{E3}. Accordingly, the same reasoning employed at the end of the proof of Theorem 1 suffices for Theorem 2. □

This proof establishes that, due to the quantum indeterminacies that produce thermal Doppler shifting, NOISE is true for an idea observer who is measuring only the wavelength of a single photon.

Second way: mixed waves

One might try to avoid the indeterminacies inherent in Doppler shifting by appealing to observers who not only know the average temperature and pressure of the gas but who also know the position and momentum of every molecule. Such an observer would be equivalent to Maxwell's demon and therefore might involve a violation of the laws of thermodynamics (see Brillouin [1]). However, even if it were possible, there are still quantum indeterminacies that suffice for NOISE.

Let's broaden our perspective to include mixed waves. For any assignment of a_i's, the following describes a wave equation for a possible photon:

$$\psi(x) = \sum_i a_i \sin(2\pi\omega_i x). \tag{17}$$

The probability of measuring $\lambda_i = \frac{c}{\omega_i}$ is proportional to a_i^2. Without affecting wavelength measurements, we may assume ψ is *normalized*, so the probability of measuring λ_i is equal to a_i^2. Therefore, for a given ψ, the probability of measuring a wavelength in Ω_{E1} is equal to:

$$f_{E1}(\psi) = \sum_i \chi_{E1}(a_i)(\lambda_i)a_i^2 \tag{18}$$

Similarly for Ω_{E2} and Ω_{E3}, yielding f_{E2} and f_{E3}.

Because the visual system produces each of the experiences ($E1$, $E2$ and $E3$), at least sometimes:

$$\Omega_{E1}, \Omega_{E2}, \Omega_{E3} \neq \emptyset, \tag{19}$$

We will assume that these are the only experiences produced by the relevant visual system, and that, in addition, every measurement produces some experience, in which case:

$$f_{E1}(\psi) + f_{E2}(\psi) + f_{E3}(\psi) \;=\; 1 \qquad (20)$$

Our proof trivially generalizes to visual experiences that produce four, five, etc., different experiences.

Finally, we will again assume that the threshold for reliably production is at 0.5.

Given these assumptions, let's now prove that NOISE is true for an observer whose visual system is only measuring the wavelength of single photons:

Theorem 3. *If for some* ψ *and* ψ', $f_{E1}(\psi) \geq 0.5$ *and* $f_{E2}(\psi') \geq 0.5$ *then there is a* ψ'' *such that* $f_1(\psi''), f_2(\psi''), f_3(\psi'') < 0.5$.

Proof. Write

$$\psi(x) = \sum_i a_i \sin(2\pi\omega_i x)$$

and

$$\psi'(x) = \sum_i b_i \sin(2\pi\omega_i x).$$

Define

$$\psi''(x) = N_1 \sum_{i:\lambda_i \in \Omega_{E1}} b_i \sin(2\pi\omega_i) + N_2 \sum_{i:\lambda_i \in \Omega_{E2}} a_i \sin(2\pi\omega_i) + N_3 \sum_{i:\lambda_i \in \Omega_{E3}} \left(\frac{a_i + b_i}{2}\right) \sin(2\pi\omega_i),$$

where

$$N_1 = \frac{1}{\sqrt{3 \sum_{j:\lambda_j \in \Omega_{E1}} b_j^2}}$$

is a normalization factor, and N_2, N_3 are defined similarly.

Now we see

$$f_{E1}(\psi'') = \sum_{i:\lambda_i \in \Omega_{E1}} N_1^2 b_i^2 = \frac{1}{3},$$

and similarly for $f_{E2}(\psi'')$ and $f_{E3}(\psi'')$. Note that by adjusting N_1, N_2 and N_3 we can identify other wave fuctions that also establish the theorem. □

The preceding proves that, due to quantum indeterminacies, NOISE is true for an ideal observer who is measuring only the wavelength of a single photon and who knows the position and momentum of each particle in the gas.

Intensity and Wavelength: spectral power distributions

It is straightforward generalize to an ideal observer who is measuring both intensity and wavelength. Indeed, given pairs (x, ψ) and $(x+y, \psi')$, with x and $x+y$ as in the hypothesis of Theorem 1 and ψ and ψ' as in the hypothesis of Theorem 2 (or 3), the pair $(x+i, \psi'')$ chosen as in the conclusions of Theorems 1 and 2 (or 3) will do. It also worth noting that when multiple photons are involved, there might be entanglements, which will introduce additional indeterminacies.

We conclude that, given our assumptions, certain quantum indeterminacies by themselves suffice for NOISE.

References

[1] Brillouin. Maxwell's demon cannot operate: information and entropy. *Journal of Applied Physics*, 22:334–337, 1951.

[2] Dicke. The effect of collisions upon the doppler width of spectral lines. *Physical Review*, 89:472–473, 1953.

[3] Fox. *Quantum Optics: an introduction.* Oxford University Press, 2006.

[4] Jeans. *An introduction to the kinetic theory of gases.* Cambridge University Press, 1967.

[5] Orszag. *Quantum Optics: including noise reduction, trapped ions, quantum trajectories, and decoherence.* Harvard University Press, 1981.