Homework #4

1. a. What is the optimal production mix? What contribution can the firm anticipate by producing this mix?

   The optimal production mix is:

$$\begin{aligned} x_{chair} &= 700 \\ x_{bench} &= 0 \\ x_{table} &= 133.333 \end{aligned} \tag{1}$$

   The firm can anticipate a contribution of 2766.67 from this mix.

   b. What is the value of one unit more of tube-bending time? of welding time? of metal tubing?
   This correspond to the shadow prices associated with each constraint. Then the value of one unit more of tube-bending time, welding time and metal tubing are 1.16, 0 and 0.8 respectively.

   c. A local distributor has offered to sell Outdoors, Inc. some additional metal tubing for $0.60\backslash lb$. Should Outdoors but it? If yes, how much would the firm's contribution increase if they bought 500lbs. and used it in an optimal fashion?

   Yes, Outdoors Inc. should by it because the price per pound offered by the local distributor is less than 0.8 which is the shadow price for such constraint. If they buy 500lbs then the RHS of third constraint increase by 500, which is still in the range $[1666.67, 2555.6]$ of values of RHS limit for the third constraint. which implies that current basis is still optimal and then the contribution of the firm will increase by $500 * (0.8 - 0.6) = 100$.

   d. If Outdoors, Inc. feels that it must produce at least 100 benches to round out its product line, what effect will that have on its contribution.

   This will reduce the contribution of the company by: $1.38333 \cdot 100 = 138.333$.

   e. The $R\&D$ department has been redesign the bench to make it more profitable. The new design will require 1.1 hours of tube-bending time, 2 hours of welding time and 2 lbs of metal tubing. If it can sell one unit of this bench with a unit contribution of 3, what effect will it have on overall contribution?
   To answer this question we must check that the reduced cost of bench is still negative:

$$\bar{c}_{bench} = c_{bench} - p^T \cdot A_{bench} = 3 - [1.16667, 0, 0.8] \cdot \begin{bmatrix} 1.1 \\ 2 \\ 2 \end{bmatrix} = 3 - 2.8833 = 0.1167 \tag{2}$$

   Given that the reduced cost become positive, if we are able to sell one unit of this bench we will increase the overall contribution by 0.1167.

f. Marketing has suggested a new patio awning that would require 1.8 hours of tube-bending time, 0.5 hours of welding time, and 1.3 lbs. of metal tubing. What contribution must this new product have to make it attractive to produce this season?

$$c > p^T \cdot \begin{bmatrix} 1.8 \\ 0.5 \\ 1.3 \end{bmatrix} = [1.16667; 0; 0.8]^T \cdot \begin{bmatrix} 1.8 \\ 0.5 \\ 1.3 \end{bmatrix} = 3.14 \tag{3}$$

Then the contribution need to be greater or equal to 3.14 to be attractive to produce this season.

g. Outdoors, Inc. has a chance to sell some of its capacity in tube bending at cost $+1.5/hour$. If it sells 200 hours at that price, how will this affect contribution.

Given that $1000 - 200 > 533.33$ current basis is still optimal. Then the change in the contribution is:

$$p^T \cdot b + 1.5 \cdot 200 - 2766.67 = [1.16667; 0; 0.8] \cdot \begin{bmatrix} 800 \\ 1200 \\ 2000 \end{bmatrix} + 300 - 2766.67 = 66.66 \tag{4}$$

h. If the contribution of chairs were to decrease to 2.5, what would be the optimal production mix and what contribution would this production plan give?

Current basis will still be optimal given that 2.5 is still in the given range for current basis to still be optimal. Moreover given that the RHS didint change, optimal production mix is the same and contribution is $[2.5, 5] \cdot \begin{bmatrix} 700 \\ 133.3333 \end{bmatrix} = 2416.66$.

2. Formulating the problem we have:

$$\begin{aligned} &\max 46 \cdot x_1 + 57 \cdot x_2 + 60 \cdot x_3 \\ &10 \cdot x_1 + 6 \cdot x_2 + 12 \cdot x_3 \leq 62 \\ &4 \cdot x_1 + 10 \cdot x_2 + 8 \cdot x_3 \leq 60 \\ &x_1, x2, x3 \geq 0 \end{aligned}$$

In canonical form we have:

$$\begin{aligned} &\max \quad 46 \cdot x_1 + 57 \cdot x_2 + 60 \cdot x_3 \\ &10 \cdot x_1 + 6 \cdot x_2 + 12 \cdot x_3 + x_4 = 62 \\ &4 \cdot x_1 + 10 \cdot x_2 + 8 \cdot x_3 + x_5 = 60 \\ &x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

a. The optimal solution is $x = \begin{bmatrix} 3.421 \\ 4.632 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

b. Yes the solution is unique because all the nonbasic variables have reduced cost strictly negative.

c. For source A: $[46 - 8.19, 46 + 48.9620]$
Given that x_1 is a basic variable we need to check how it will affect the reduced cost of the nonbasic variables:

$$\delta \cdot \left(B^{-1} \cdot A_3\right)_1 \geq \bar{c}_3 \Rightarrow \delta \geq \frac{-7.7559}{0.947} = -8.19 \tag{5}$$

$$\delta \cdot \left(B^{-1} \cdot A_4\right)_1 \geq \bar{c}_4 \Rightarrow \delta \geq \frac{-3.053}{0.132} = -23.1288 \tag{6}$$

$$\delta \cdot \left(B^{-1} \cdot A_5\right)_1 \geq \bar{c}_5 \Rightarrow \delta \leq \frac{3.868}{0.079} = 48.9620 \tag{7}$$

For source B: $[57 - 18.4226, 57 + 58.7115]$

$$\delta \cdot \left(B^{-1} \cdot A_3\right)_2 \geq \bar{c}_3 \Rightarrow \delta \geq \frac{-7.7559}{0.421} = -18.4226 \tag{8}$$

$$\delta \cdot \left(B^{-1} \cdot A_4\right)_2 \geq \bar{c}_4 \Rightarrow \delta \leq \frac{3.053}{0.052} = 58.7115 \tag{9}$$

$$\delta \cdot \left(B^{-1} \cdot A_4\right)_2 \geq \bar{c}_5 \Rightarrow \delta \geq \frac{-3.868}{0.132} = -29.3030 \tag{10}$$

For source C:
Given that x_3 is not a basic variable, we can increase it cost by 7.759 without changing current optimal solution, and we can reduced as much as we want.

d. The shadow prices are $p_1 = 3.053$ and $p_2 = 3.868$ which are the reduced cost of slack variables. They represent the change in the objective value if we increase the RHS of the respective constraint by 1.

e. If we change the limitation on the sale of the case of nuts then:

$$\delta_{b_1} \geq \tfrac{-3.421}{0.132} = -25.9167$$
$$\delta_{b_1} \leq \tfrac{-4.632}{-0.052} = 89.07$$

If we change the limitation on the sale of the case of nuts then:

$$\delta_{b_2} \leq \tfrac{-3.421}{-0.079} = 43.3038$$
$$\delta_{b_2} \geq \tfrac{-4.632}{0.132} = -35.0909$$

3. a.

$$\begin{aligned}
&\max \quad z = 2 \cdot x_1 + (x_2^+ - x_2^-) + 3 \cdot x_3 + (x_4^+ - x_4^-) \\
&\text{s.t} \\
&x_1 + (x_2^+ - x_2^-) + x_3 + (x_4^+ - x_4^-) + x_5 = 5 \\
&2 \cdot x_1 - (x_2^+ - x_2^-) + 3 \cdot x_3 = -4 \\
&x_1 - x_3 + (x_4^+ - x_4^-) - x_6 = 1 \\
&x_1, x_2^+, x_2^-, x_3, x_4^+, x_4^-, x_5, x_6 \geq 0
\end{aligned}$$

b. The dual of the given problem is:

$$\begin{aligned}
&\min 5 \cdot p_1 - 4 \cdot p_2 + p_3 \\
&\text{s.t} \\
&p_1 + 2 \cdot p_2 + p_3 \geq 2 \\
&p_1 - p_2 = 1 \\
&p_1 + 3 \cdot p_2 - p_3 \geq 3 \\
&p_1 + p_3 = 1 \\
&p_1 \geq 0, p_2 \text{ is free}, p_3 \leq 0
\end{aligned}$$

The dual of the problem found in part a is given by:

$$\begin{aligned}
&\min 5 \cdot y_1 - 4 \cdot y_2 + y_3 \\
&\text{s.t} \\
&y_1 + 2 \cdot y_2 + y_3 \geq 2 \\
&y_1 - y_2 \geq 1 \\
&-y_1 + y_2 \geq -1 \\
&y_1 + 3 \cdot y_2 - y_3 \geq 3 \\
&y_1 + y_3 \geq 1 \\
&-y_1 - y_3 \geq -1 \\
&y_1 \geq 0 \\
&-y3 \geq 0 \\
&y_1, y_2, y_3 \text{ are free}
\end{aligned}$$

Multiplying constraint 3 of the dual problem of the problem found in part a,together with equation 2, lead us to the second constraint of the dual of the given problem.Multiplying constraint 6 of the dual problem of the problem found in part a, we have that together with equation 5, lead us to the fourth constraint of the dual of the given problem. Constraint 7 and 8 are equivalent to constraints non negativity constraints in the dual of the given problem, and finally y_2 is free as p_2 in the dual problem of the given problem.

4. a. The optimal solution to the dual problem is given by the negative of the reduced cost of the slack variables which are:

$$\begin{bmatrix} \frac{44}{15} \\ \frac{4}{15} \end{bmatrix}$$

b. From the final tableau we have that the shadow prices are $\bar{y}_1 = \frac{-44}{15}$ and $\bar{y}_2 = \frac{-4}{15}$.

$$\begin{aligned}
\bar{c}_5 = c_5 - \textstyle\sum_{i=1}^{2} a_{i5} \cdot n\bar{y}_i = 0 - \left(-\frac{44}{15}\right) = \frac{44}{15} \\
\bar{c}_5 = c_5 - \textstyle\sum_{i=1}^{2} a_{i5} \cdot n\bar{y}_i = 0 - \left(-\frac{4}{15}\right) = \frac{4}{15}
\end{aligned}$$

c. Basic variables in the primal are x_1 and x_4 then:

$$\begin{aligned}
&x_1 \cdot \left(p^T \cdot A_1\right) - c_1) = 0 \\
&x_4 \cdot \left(p^T \cdot A_4\right) - c_4) = 0 \\
&p_1 \cdot (b_1 - a_1 \cdot x) = 0 \\
&p_2 \cdot (b_2 - a_2 \cdot x) = 0
\end{aligned}$$

In which A_i is the i_{th} column of A in the initial tableau, and a_i is the i_{th} row in the intial tableau.

The dual problem is given by:

$$\begin{aligned}
&\max \quad 3 \cdot p_1 + 2 \cdot p_2 \\
&\text{s.t} \\
&3 \cdot p_1 + p_2 \geq -4 \\
&p_1 - p_2 \geq 3 \\
&2 \cdot p_1 \geq 3 \\
&p_1 + p_2 \geq 2 \\
&-p_2 \geq -8 \\
&p_2 \leq 0, p1 \text{ is free}
\end{aligned}$$

Graphical solution of the dual:

Solving graphically the dual we have that the optimal dual solution is:

$$p^* = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Then by complementarity of slackness we have that $x_1^*, x_2^*, x_3^*, x_5^*$ are zero the replacing which let us with the unique solution and $x_4^* = 3$. Clearly the optimal solution for dual and primal have the same value, 6.