Recitation #9

1 The simplex method for uncapacitated network flow problems

1. A typical iteration starts with a basic feasible solution f associated with a tree T. (What is a tree? A tree is a graph with no cycles)

2. To compute the dual vector p, solve the system of equation:

$$\begin{array}{l} p_i - p_j = c_{ij} \forall (i,j) \in T \\ p_n = 0 \end{array} \tag{1}$$

   By proceeding from the root towards the leaves.

3. Compute the reduced costs $\bar{c}_{ij} = c_{ij} - (p_i - p_j)$ of all arcs $(i,j) \notin T$. If they are all nonnegative, the current basic feasible solution is optimal and the algorithm terminates; else, choose some (i,j) with $\bar{c}_{ij} < 0$ to be brought into the basis.

4. The entering arc (i,j) and the arcs in T form a unique cycle. If all arcs in the cycle are oriented the same way as (i,j) then the optimal cost is $-\infty$ and the algorithm terminates.

5. Let B be the set of arcs in the cycle that are oriented in the opposite direction from (i,j). Let $\theta^* = \min_{(k,l) \in B} f_{kl}$, and push θ^* units of flow around the cycle. A new flow vector is determined:

$$\hat{f}_{kl} = \begin{cases} f_{kl} + \theta & \text{if } (k,l) \in F \\ f_{kl} - \theta & \text{if } (k,l) \in B \\ f_{kl} & \text{Otherwise} \end{cases} \tag{2}$$

   Remove from the basis one of the old basic variables whose new value is equal to zero.

2 The simplex method for capacitated network flow problems

1. A typical iteration starts with a basic feasible solution f associated with a tree T and a partition of remaining arcs into two sets D, U, such that $f_{ij} = d_{ij}$ for $(i,j) \in D$, and $f_{ij} = u_{ij}$ for $(i,j) \in U$.

2. To compute the dual vector p, solve the system of equation:

$$\begin{array}{l} p_i - p_j = c_{ij} \forall (i,j) \in T \\ p_n = 0 \end{array} \tag{3}$$

   By proceeding from the root towards the leaves.

3. Compute the reduced costs $\bar{c}_{ij} = c_{ij} - (p_i - p_j)$ of all arcs $(i,j) \notin T$. If $\bar{c}_{ij} \geq 0$ for all $(i,j) \in D$, and $\bar{c}_{ij} \leq 0$ for all $(i,j) \in U$, the current basic feasible solution is optimal and the algorithm terminates.

4. Let (i,j) be an arc such that $\bar{c}_{ij} < 0$ and $(i,j) \in D$, or such that $\bar{c}_{ij} > 0$ and $(i,j) \in U$. This arc (i,j) together with the tree T forms a unique cycle. Choose the orientation of the cycle as follows. If $(i,j) \in D$, then (i,j) should be a forward arc. If $(i,j) \in U$, then (i,j) should be a backward arc.

5. Let F and B be the forward and backward arcs respectively, in the cycle. Determine θ^*:

$$\theta^* = \min\{\min_{(k,l)\in B}\{f_{kl} - d_{kl}\}, \min_{(k,l)\in F}\{u_{kl} - f_{kl}\}\} \tag{4}$$

A new flow vector is determined:

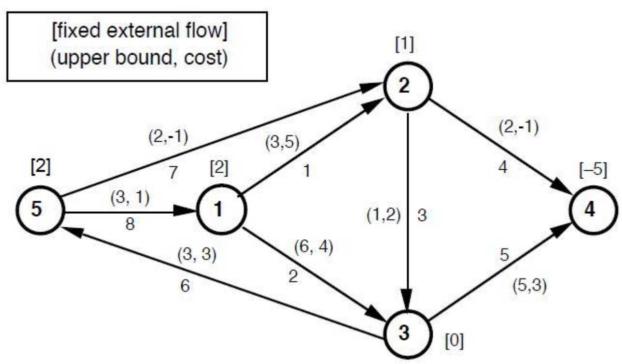
$$\hat{f}_{kl} = \begin{cases} f_{kl} + \theta & \text{if } (k,l) \in F \\ f_{kl} - \theta & \text{if } (k,l) \in B \\ f_{kl} & \text{Otherwise} \end{cases} \tag{5}$$

Finally update the set T, D, U.

3 Example

Consider the following network flow diagram.

Figure 1: The network



Using this diagram, determine the primal and dual basic solutions when

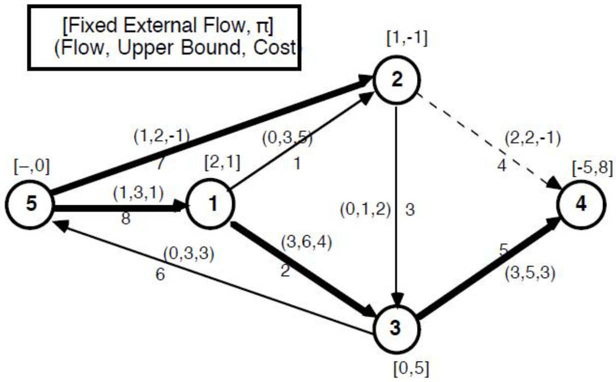
a. $n_B = \{8, 7, 2, 5\}$, $n_0 = \{1, 3, 6\}$, $n_1 = \{4\}$.

b. $n_B = \{8, 1, 2, 4\}$, $n_0 = \{6, 7\}$, $n_1 = \{3, 5\}$.

c. $n_B = \{8, 1, 2, 4\}$, $n_0 = \{6, 7\}$, $n_1 = \{3, 5\}$.

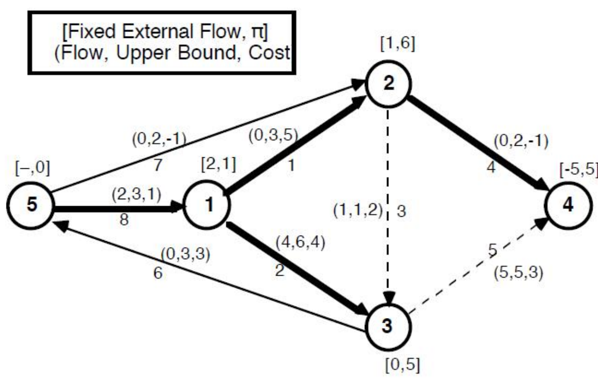
Classify each as feasible, infeasible or no solution. Classify feasible solution as optimal or non optimal. In each case, indicate the reason for your classification.

Figure 2: part a.



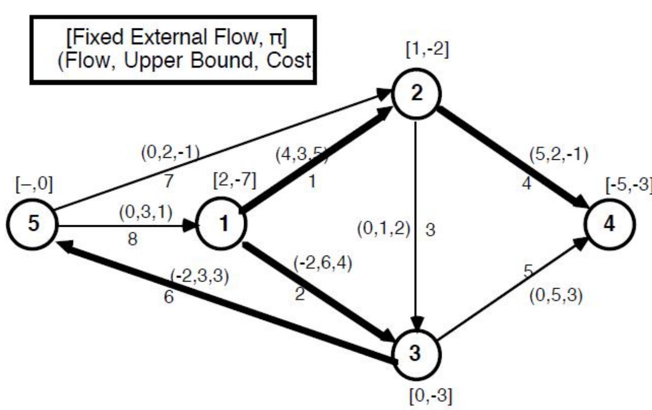
Marginal costs: $d_1 = 7$, $d_3 = -4$, $d_4 = -10$, $d_6 = 8$. The solution is feasible, but not optimal. Arc 3 is a candidate to enter the basis.

Figure 3: part b.



Marginal costs: $d_3 = 3$, $d_5 = 3$, $d_6 = 8$, $d_7 = -7$. The solution is feasible, but not optimal. Arcs 3, 5 and 7 are candidates to enter the basis.

Figure 4: part c.



The solution is not feasible because some flows are above the upper bound and some flows are negative.