Fractional Stable Matchings and their Applications

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Stable Marriage

1	3	1	4	2
2	1	3	4	2
3	4	3	1	2
4	3	2	1	4

1	4	3	1	2
2	3	4	1	2
3	3	2	1	4
4	4	2	3	1

Example 1: Stable Marriage: male and female preference lists

- Let μ be a matching of men to women
- Blocking pair: a pair (m, w) such that:

  man m prefers woman w to his current partner $\mu(m)$; AND

  woman w prefers man m to her current partner $\mu(w)$
- Stable Marriage: A marriage with no blocking pairs
- The matching $\{(1,1)(2,2)(3,3)(4,4)\}$ is unstable; (4,2) blocks.
- A stable marriage always exists!

Stable Marriage Polytope

$$x_{i,j} = \begin{cases} 1 & \text{if man } i \text{ is matched to woman } j \\ 0 & \text{otherwise} \end{cases}$$

Let Γ be the set of mutually acceptable pairs.

Stability constraint for pair (3,1):

$$x_{3,1} + x_{3,3} + x_{3,4} + x_{4,1} \geq 1.$$

Stable Marriage Polytope

$$\text{Max} \quad \sum_{(i,j)\in\Gamma} x_{i,j}$$

subject to:

$$\sum_{j:(m,j)\in\Gamma} x_{m,j} \leq 1, \quad \forall m \in M$$

$$\sum_{i:(i,w)\in\Gamma} x_{i,w} \leq 1, \quad \forall w \in W$$

$$x_{m,w} + \sum_{j>_m w} x_{m,j} + \sum_{i>_w m} x_{i,w} \geq 1, \quad \forall (m,w) \in \Gamma$$

$$x_{m,w} \geq 0, \quad \forall (m,w) \in \Gamma.$$

Stable Marriage Polytope: Dual

$$\text{Min} \quad \sum_{i \in M} \alpha_i + \sum_{j \in W} \beta_j - \sum_{(i,j) \in \Gamma} \gamma_{i,j}$$

subject to:

$$\alpha_m + \beta_w - \sum_{j <_m w} \gamma_{m,j} - \sum_{i <_w m} \gamma_{i,w} - \gamma_{m,w} \geq 1, \quad \forall (m, w) \in \Gamma$$

$$\alpha_m, \beta_w, \gamma_{m,w} \geq 0.$$

Stable Marriage Polytope

Roth, Rothblum and Vande Vate (1993):

- For any primal solution x

$$\gamma_{m,w} = x_{m,w}, \quad \alpha_m = \sum_j x_{m,j}, \quad \beta_w = \sum_i x_{i,w}$$

  is dual feasible.

- Objective values of the above primal and dual solutions are identical.
- x is primal optimal; (α, β, γ) is dual optimal.

By complementary slackness:

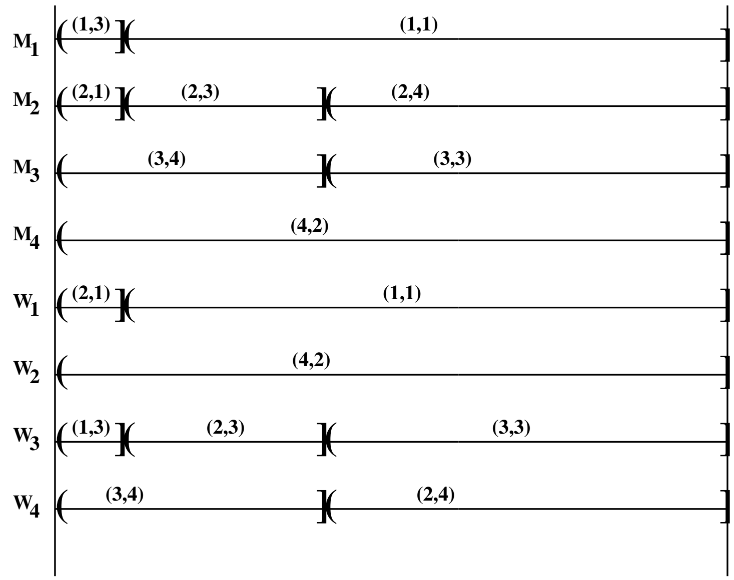
$$x_{m,w} > 0 \text{ only if } x_{m,w} + \sum_{i >_w m} x_{i,w} + \sum_{j >_m w} x_{m,j} = 1.$$

Geometry of Fractional Solutions

$$x_{1,1} = 9/10 \quad x_{2,1} = 1/10 \quad x_{3,3} = 6/10 \quad x_{4,2} = 1$$

$$x_{1,3} = 1/10 \quad x_{2,3} = 3/10 \quad x_{3,4} = 4/10$$

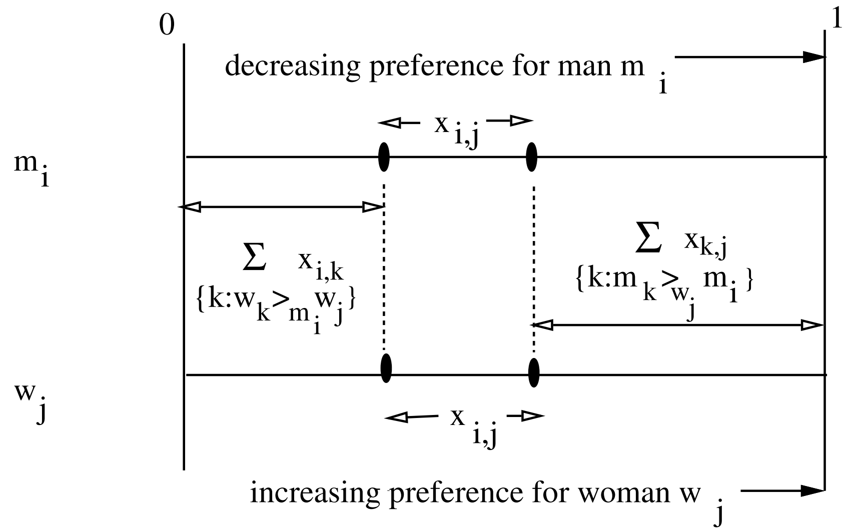
$$x_{2,4} = 6/10$$



- Pick a random number u in $(0, 1)$; say: u=0.3
- stable matching corresponding to $u = 0.3$ is:

– $\{(1,1), (2,3), (3,4), (4,2)\}$.

Geometry of Fractional Solutions



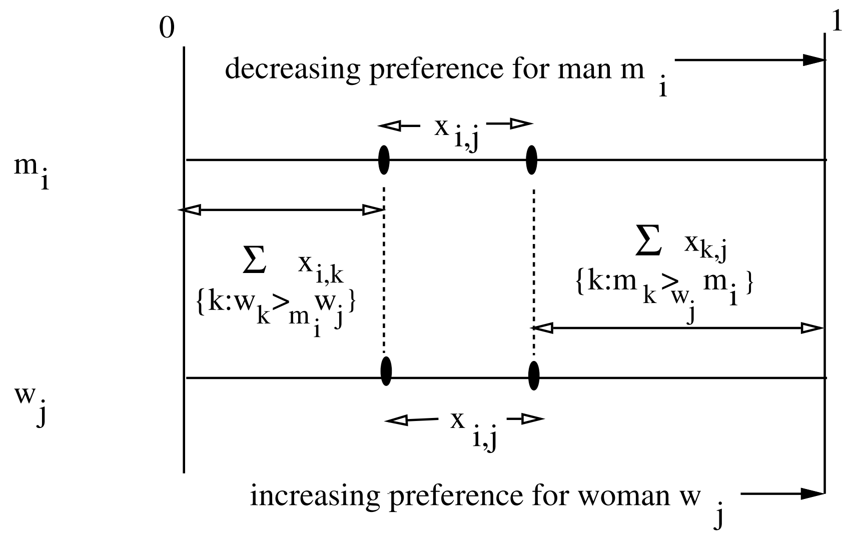
For any optimal primal solution $\mathbf{x}^*$:

Arrange each man's intervals in *decreasing* preference order to cover $[0, 1]$;

Arrange each woman's intervals in *increasing* preference order to cover $[0, 1]$.

By complementary slackness: the interval $x_{m,w}$ in m's arrangement and w's arrangement coincide!

Integrality Proof



- Generate u in $(0, 1]$ uniformly at random
- Set $x_{i,j} = 1$ if u falls into $x_{i,j}$'s sub-interval
- $E(x_{i,j}) = x^*_{i,j}$
- Hence, stable marriage polytope is integral!

Fair Stable Marriages

- Fractional marriages form a lattice. In particular, there is a best marriage for each side.

- Let $M_1, M_2, \ldots, M_r$ be r distinct stable marriage solutions. Each man m_i has r possible mates under these marriages. Assign him the woman whose rank is k among the r (possibly non-distinct) women. For each woman w_j, assign her to the man whose rank is $r + 1 - k$ among the r men she was assigned to under the matchings. This assignment gives rise to another stable marriage solution.

- Setting $k = r/2$ or $(r + 1)/2$ gives the median stable marriage

Fair Stable Marriages

- Open Problem: compute the median stable marriage efficiently.

All these results are proved by finding a suitable fractional solution, and picking an appropriate $u \in (0, 1)$ in the picture.

Stable Admissions

- Many-to-one stable marriage (college admissions)

  Two-sided market: universities and students

  University u has quota $q_u \geq 1$

  Student a has quota $q_a = 1$!

- Blocking pair: a pair (u, a) such that:

  student a prefers university u to her current university; AND

  university u prefers student a to *at least one* of its currently assigned students

- Stable Assignment: An assignment with no blocking pairs

Stable Admissions

1	1	2	3	4
2	4	3	2	1

1	2	1
2	2	1
3	2	1
4	1	2

Example 2: Stable Admissions; $q_1 = q_2 = 2$

- The “matching” $\{(1,1)(1,3)(2,2)(2,4)\}$ is unstable; (2,3) blocks.
- The “matching” $\{(1,1)(1,2)(2,3)(2,4)\}$ is stable

Stable Admissions reduces to stable marriage

1	1	2	3	4
2	4	3	2	1

1	2	1
2	2	1
3	2	1
4	1	2

Example 2: Stable Admissions; $q_1 = q_2 = 2$

is equivalent to

1	1	2	3	4
1'	1	2	3	4
2	4	3	2	1
2'	4	3	2	1

1	2	2'	1	1'
2	2	2'	1	1'
3	2	2'	1	1'
4	1	1'	2	2'

- One-to-one correspondence of stable solutions
- A stable assignment <u>always</u> exists!

Stable Admissions Polytope

Can we characterize the convex hull of all stable assignments in the stable admission problem using a set of linear inequalities?

Use the reduction to stable marriage + stable marriage polytope.

Not satisfactory:

- Does not capture indifference: $\{(1,2),(1',1)\}$ is blocked by $(1,1)$
- does not use the natural assignment variables.
- many-to-many?

Stable Admissions: A natural formulation

Max $\displaystyle\sum_{(u,a)\in\Gamma} x_{u,a}$

subject to:

$$\sum_{j:(u,j)\in\Gamma} x_{u,j} \le q_u, \quad \forall u \in U$$

$$\sum_{i:(i,a)\in\Gamma} x_{i,a} \le 1, \quad \forall a \in A$$

$$q_u x_{u,a} + \sum_{j >_u a} x_{u,j} + q_u \sum_{i >_a u} x_{i,a} \ge q_u, \quad \forall (u,a) \in \Gamma$$

$$x_{u,a} \ge 0, \quad \forall (u,a) \in \Gamma.$$

We get this LP if we use the equivalent stable marriage problem followed by aggregation:

$$x_{u,a} \;=\; x_{u_1,a} + x_{u_2,a} + \ldots + x_{u_{q_u},a}.$$

Stable Admissions: A natural formulation

Unfortunately, the integrality result does not hold.

Suppose University 1 is assigned $(1, 1/2, 1/2, 0)$; and University 2 is assigned $(0, 1/2, 1/2, 1)$

This is an extreme-point solution to the natural formulation.

Difficulty first observed by Balinski-Baiou (2000)

Stable Admissions: Balinski-Baiou formulation

$$S(u, a) \equiv \{(u, j) \in \Gamma \mid j \geq_u a\}.$$

$$T(u, a) \equiv \{(i, a) \in \Gamma \mid i \geq_a u\}.$$

For $a_1 <_u a_2 <_u \ldots <_u a_{q_u}$,

$$C(u; a_1, a_2, \ldots, a_{q_u}) \;\equiv\; S(u, a_1) \;\cup\; T(u, a_1) \;\cup \ldots \cup\; T(u, a_{q_u}).$$

An assignment μ is stable if and only if for each university u, every *comb* associated with u contains at least q_u pairs of μ.

Stable Admissions: Balinski-Baiou formulation

$$\text{Max} \quad \sum_{(u,a)\in\Gamma} x_{u,a}$$

subject to:

$$\sum_{j:(u,j)\in\Gamma} x_{u,j} \leq q_u, \quad \forall u \in U$$

$$\sum_{i:(i,a)\in\Gamma} x_{i,a} \leq 1, \quad \forall a \in A$$

$$\sum_{(i,j)\in C} x_{ij} \geq q_u, \quad \forall C \in \mathcal{C}_u, u \in U$$

$$x_{u,a} \geq 0, \quad \forall (u,a) \in \Gamma.$$

Stable Admissions: Balinski-Baiou formulation

1	1	2	3	4
2	4	3	2	1

1	2	1
2	2	1
3	2	1
4	1	2

Example 2: Stable Admissions; $q_1 = q_2 = 2$

$x_{1,.} = (1, 1/2, 1/2, 0)$ and $x_{2,.} = (0, 1/2, 1/2, 1)$ violates the comb inequality with teeth $(2, 3)$ and (2,4)

$$x_{2,3} + x_{2,4} + x_{1,4} \ \geq \ 2.$$

Stable Admissions: Balinski-Baiou formulation

- Balinski and Baiou show that the comb-inequality formulation is exact, and can be solved in polynomial-time.

- Proof is elementary, BUT not as elegant as in the marriage case.

  Question: Is there a "visual" proof?

Stable Admissions: A simpler proof

Given any fractional stable assignment $\mathbf{x}$:

- Think of university u as owning q_u bins, each with capacity 1. (Each bin represents a "seat.")
- Each student owns a bin with capacity 1
- Each $x_{ua} > 0$ is an item; needs to be packed into some bin owned by university u, and the bin owned by student a.

Stable Admissions: A simpler proof

- Each university packs its items in decreasing preference order.

- Phase t (for $t = 0, 1, \ldots$) of the packing procedure consists of

  (a) identifying the set, L_t, of bins with the maximum available space; and

  (b) assigning one item to each of the bins in L_t.

- The assignment of the items to the bins within a phase proceeds in a sequence of *steps*, indexed by $l = 1, 2, \ldots, |L_t|$.

- If bin $(i, u) \in L_t$ is considered in step l, university u's best remaining item is packed into it.

Stable Admissions

1	1	2	3	4	5	6	7
2	5	2					
3	6	7	3				
4	7	4					
5	2	1					

1	5	1	
2	2	5	1
3	3	1	
4	4	1	
5	1	2	
6	1	3	
7	1	3	4

Stable Admissions; $q_1 = 3$, $q_2 = q_3 = q_4 = q_5 = 1$

Fractional Solution:

$$x_{1,1} = 0.1, x_{1,3} = 0.6, x_{1,4} = 0.3, x_{1,5} = 0.9, x_{1,6} = 0.7, x_{1,7} = 0.4$$

$$x_{2,5} = 0.1, x_{2,2} = 0.9; x_{3,6} = 0.3, x_{3,7} = 0.3, x_{3,3} = 0.4$$

$$x_{4,7} = 0.3, x_{4,4} = 0.7; x_{5,2} = 0.1, x_{5,1} = 0.9$$

Stable Admissions

a_1	u_3	u_1	u_5	u_4	
a_2	u_1	u_3	u_4	u_2	u_5
a_3	u_4	u_5	u_3	u_1	u_2
a_4	u_3	u_4	u_1	u_5	
a_5	u_1	u_4	u_2		
a_6	u_4	u_3	u_2	u_1	u_5
a_7	u_2	u_5	u_1	u_3	
a_8	u_1	u_3	u_2	u_5	u_4
a_9	u_4	u_1	u_5		
a_{10}	u_3	u_1	u_5	u_2	u_4
a_{11}	u_5	u_4	u_1	u_3	u_2

Table 1: Preference lists for the students

capacity											
(4)	u_1	a_3	a_7	a_9	a_{11}	a_5	a_4	a_{10}	a_8	a_6	a_1	a_2
(1)	u_2	a_5	a_7	a_{10}	a_6	a_8	a_2	a_3	a_{11}			
(3)	u_3	a_{11}	a_6	a_8	a_3	a_2	a_4	a_7	a_1	a_{10}		
(2)	u_4	a_{10}	a_1	a_2	a_{11}	a_4	a_9	a_5	a_3	a_6	a_8	
(1)	u_5	a_2	a_4	a_{10}	a_7	a_6	a_1	a_8	a_3	a_{11}	a_9	

Table 2: Quotas and preference lists for the universities

Stable Admissions

Matching	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}
M_1	u_3	u_1	u_4	u_3	u_1	u_3	u_2	u_1	u_4	u_1	u_5
M_2	u_1	u_3	u_4	u_3	u_1	u_3	u_2	u_1	u_4	u_1	u_5
M_3	u_3	u_1	u_5	u_3	u_1	u_3	u_2	u_1	u_4	u_1	u_4
M_4	u_1	u_3	u_5	u_3	u_1	u_3	u_2	u_1	u_4	u_1	u_4
M_5	u_5	u_3	u_3	u_4	u_1	u_3	u_2	u_1	u_1	u_1	u_4
M_6	u_5	u_4	u_3	u_1	u_1	u_3	u_2	u_3	u_1	u_1	u_4
M_7	u_4	u_4	u_3	u_1	u_1	u_3	u_2	u_3	u_1	u_5	u_1

Table 3: List of all Stable Matchings

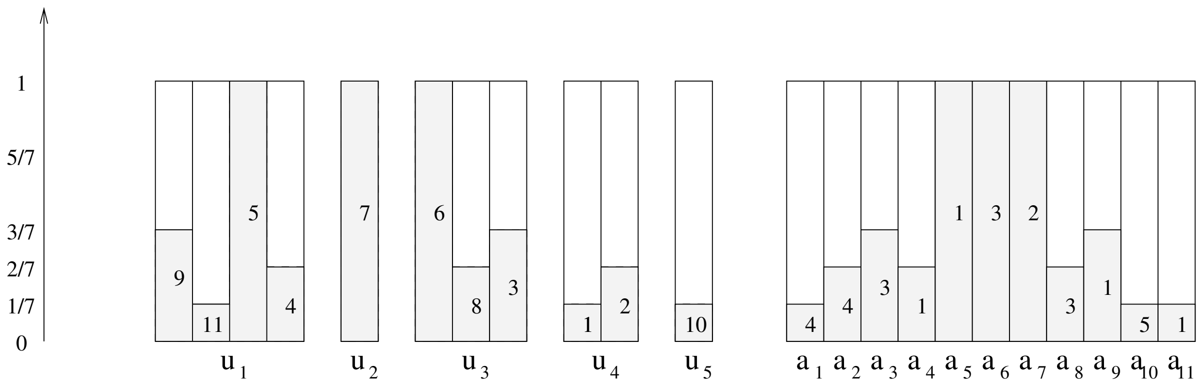


Figure 1: Bins at the end of Phase 1

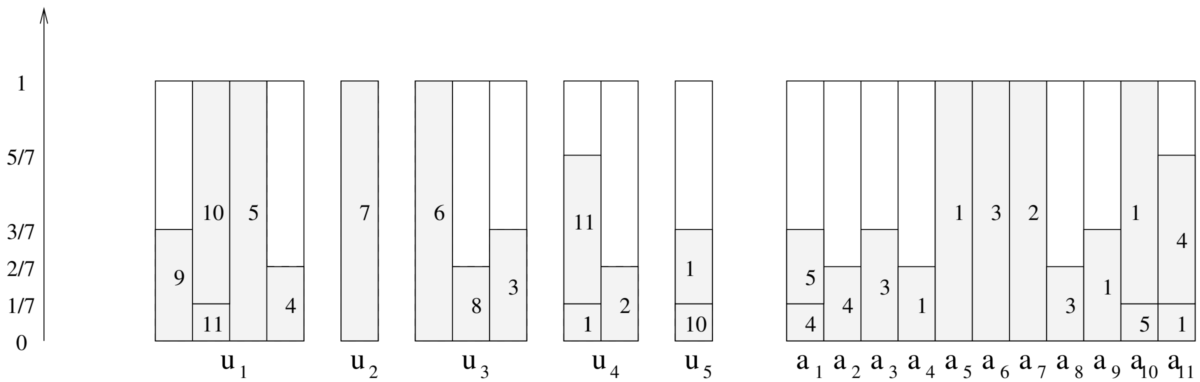


Figure 2: Bins at the end of Phase 2

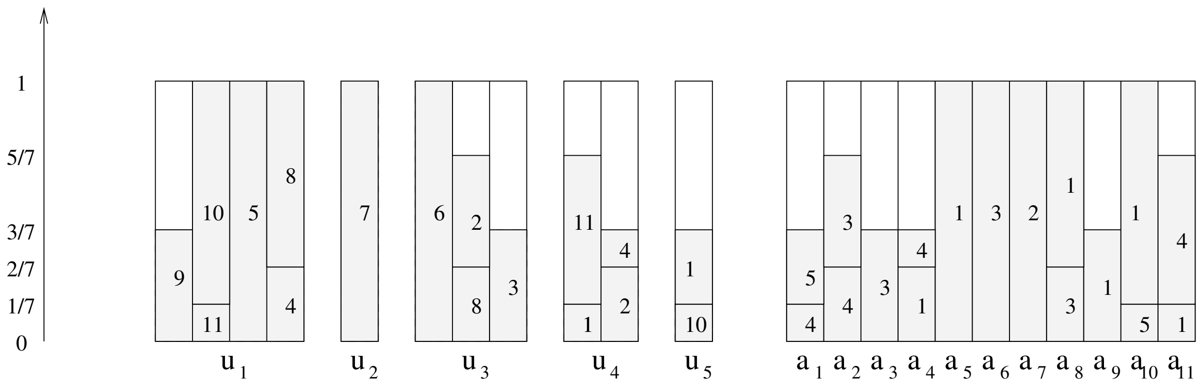


Figure 3: Bins at the end of Phase 3

Stable Admissions

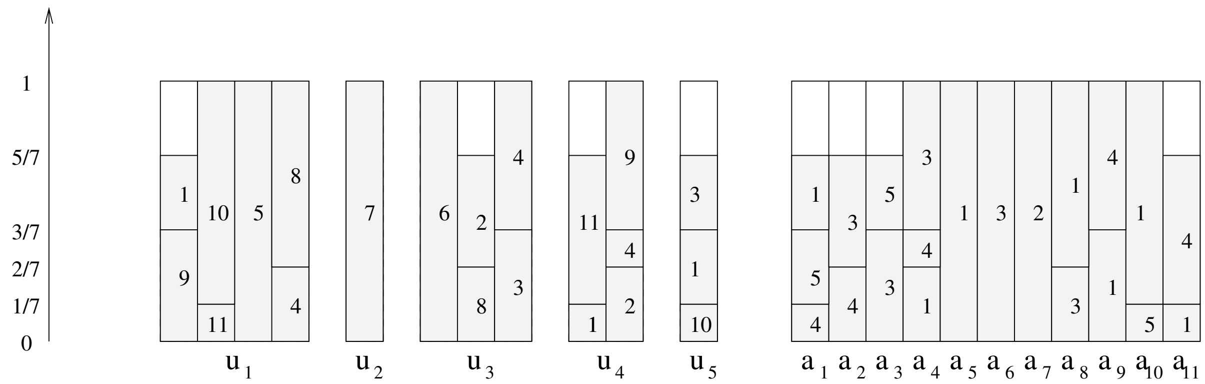


Figure 4: Bins at the end of Phase 4

Stable Admissions

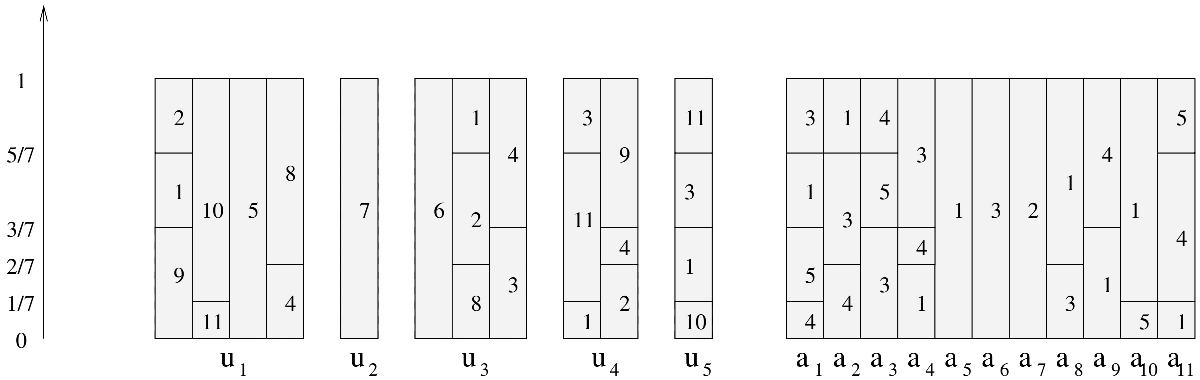


Figure 5: Bins at the end of Phase 5

Stable Admissions: A simpler proof

- $S_{i,u} \equiv$ sequence of students assigned to seat i of university u (at the end of any step)
- $a_{i,u} \equiv$ university u's least preferred student in $S_{i,u}$.

For any fractional stable assignment $\mathbf{x}$:

(a) For all (i, u),

$$x(S_{i,u} \setminus a_{i,u}) + x(T(u, a_{i,u})) \; = \; 1;$$

(b) If for a student a, item (v, a) is packed but (u, a) is not, then a prefers u to v.

(c) At the end of any phase, the $a_{i,u}$ are all distinct. In particular, for each $a \in A$, there is some (i, u) such that $a \equiv a_{i,u}$.

Proof: induction on the number of phases.

Stable Admissions: A simpler proof

- Each student packs her items in increasing preference order.
- The positions of item x_{ua} in u's bin and a's bin coincide!
- Pick any $r \in (0, 1)$: all "items" containing r define a *stable* matching.
- Integrality follows by choosing r uniformly at random!
- Polarity, Lattice structure follow.
- Independently assigning each student her kth best university gives a stable matching.

Another Marriage Connection

- No stable assignment in the convex combination can contain two students assigned to the same seat!

- Equivalence with the marriage problem, with members of $S_{i,u}$ competing for seat i.

Potential problem: $S_{i,u}$ depends on x?

Another Marriage Connection

- There is a partition that works for *all* stable assignments. (Independently discovered by Fleiner)

- Existence can be proved using "polarity" property.

- Partition can be constructed if we know the stable partners of each participant.

- Extends to the many-to-many version.

(In effect: we need to "solve" the problem before we can write down the equivalent marriage problem.)

Dual of the comb formulation

$$\text{Min} \quad \sum_{i \in U} q_i \alpha_i + \sum_{j \in A} \beta_j - \sum_{u \in U} \sum_{C \in \mathcal{C}_u} q_u \gamma_{u,C}$$

subject to:

$$\alpha_u + \beta_a - \sum_{i \in U} \sum_{C \in \mathcal{C}_i ; (u,a) \in C} \gamma_{i,C} \quad \geq \; 1, \quad \forall (u,a) \in \Gamma$$

$$\alpha_u, \beta_a, \gamma_{u,a} \geq 0.$$

Dual of the comb formulation

Given any $\mathbf{x}$ feasible for the comb formulation,

$$\alpha_u = \max\Big\{\sum_{j\in A} x_{uj} - q_u + 1, 0\Big\},$$

$$\beta_a = \sum_{i\in U} x_{ia},$$

and

$$\gamma_{u;a_1,a_2,\ldots,a_{q_u}} = \min_j\{x(T(u,a_j))\} - \max_j\{x(T(u,a_j)) - x_{u,j}\},$$

is an optimal solution to the dual and has the same objective value as the primal.

So $\mathbf{x}$ is primal optimal and (α, β, γ) is dual optimal!

Work in progress

- Computing fair stable assignments
- Lattice structure of fractional matchings
- Relationship to the Shapley-Shubik Assignment game (b-matching problem)
- Many-to-many matching: formulation analogous to the comb formulation
- Non-bipartite generalizations
- Indifference
  - Dichotomous preferences: matching (b-matching) problem
  - Strict, complete preferences: stable marriage (admissions) problem
  - (general) Indifference: NP-complete
- Lattice structure is the key!