A note on the average rank of rank-optimal assignments

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Abstract

Results on the random assignment problem due to Aldous [2] are used to show that the average rank of rank-optimal assignments is bounded by a constant that is independent of the market size. This result was first shown by Nikzad [4] using a different approach.

Keywords: matching, random assignment, Pareto efficient assignments, rank distribution

1 Discussion

We consider an assignment problem with n agents and n objects. Each agent must be assigned an object and each object can be assigned to exactly one agent. We consider the problem of finding a minimum-cost assignment of agents to objects under the following two cost matrices:

(a) The cost matrix $\mathbb{X} = [x_{ij}]$ with i.i.d. entries, each uniformly distributed in $[0, 1]$.

(b) The cost matrix $\mathbb{Z} = [z_{ij}]$, with $(z_{i1}, z_{i2}, \ldots, z_{in})$ being a uniform random permutation of the vector $(\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n})$

Letting $\mathbb{C}$ denote any of these cost matrices, the object of interest in each case is the cost of an optimal assignment given by

$$\mathcal{A}(n, \mathbb{C}) \;=\; \min_{\pi} \sum_{i} c_{i,\pi(i)}.$$

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The random assignment problem with cost matrix $\mathbb{X}$ has been investigated thoroughly in the literature. In particular, the mean of this random variable, $\mathbb{E}[\mathcal{A}(n, \mathbb{X})]$, has been analyzed extensively and has been shown to converge to $\pi^2/6$ as $n \to \infty$. These results have straightforward implications for the same problem when the cost matrix is $\mathbb{Z}$.

We let i index the agents and j index the objects. Given a matrix $\mathbb{X}$ one can construct a corresponding $\mathbb{Z}$ by letting $z_{ij} = \frac{k}{n}$ if and only if x_{ij} is the kth smallest entry in $\{x_{i1}, x_{i2}, \dots, x_{in}\}$. One can pass from $\mathbb{Z}$ to $\mathbb{X}$ in the same way as well: for each agent i, generate n i.i.d. random variables $\{u_1, u_2, \dots, u_n\}$, each uniformly distributed in $[0, 1]$; and rearrange these so that the kth smallest entry in this collection is set to be x_{ij} if and only if $z_{ij} = k/n$.

Aldous [2] (see also Aldous [1]) proved the following remarkable results about the optimal assignments for the cost matrix $\mathbb{X}$:

Theorem 1

(a) $\lim_{n\to\infty} \mathbb{E}[\mathcal{A}(n, \mathbb{X})] = \frac{\pi^2}{6}$

(b) Let $q_n(k)$ be the probability that (a fixed) agent i is assigned to an object with the kth smallest value in an optimal assignment. Then $\lim_{n\to\infty} q_n(k) = 2^{-k}$ for each $k = 1, 2, \dots, n$.

Theorem 1 immediately implies the following bounds for $\mathbb{E}[\mathcal{A}(n, \mathbb{Z})]$.

Theorem 2 $\lim_{n\to\infty} \mathbb{E}[\mathcal{A}(n, \mathbb{Z})] \in [\frac{\pi^2}{6}, 2]$.

Proof. Let π_n^* be an optimal matching for the cost matrix $\mathbb{X}$ and let σ_n^* be an optimal matching for the cost matrix $\mathbb{Z}$ when there are n agents. (That is agent i is assigned object $\pi_n^*(i)$ in the first matching and $\sigma_n^*(i)$ in the second matching.) By the correspondence between $\mathbb{X}$ and $\mathbb{Z}$, the expected cost incurred by agent i in the matching π_n^* under the cost matrix $\mathbb{Z}$ is given by $\sum_{k=1}^{n} \frac{k}{n} \cdot q_n(k)$. This implies the total expected cost of the matching π_n^* under the cost matrix $\mathbb{Z}$ is simply $\sum_{k=1}^{n} k q_n(k)$, as there are n agents. Using Theorem 1 (b), we have

$$\lim_{n\to\infty} \mathbb{E}[\mathcal{A}(n, \mathbb{Z})] \;\le\; \lim_{n\to\infty} \sum_{k=1}^{n} k \cdot 2^{-k} \;=\; 2,$$

which establishes the upper bound.

For the lower bound, we again appeal to the correspondence between the $\mathbb{X}$ and the $\mathbb{Z}$ matrices. Pick any permutation μ of the objects. Conditional on the ordinal preferences of the agents over all the objects, indicated by the $\mathbb{Z}$ matrix, observe that the $x_{i,\mu(i)}$ is distributed as the kth order statistic of n Uniform $[0,1]$ random variables if $z_{i,\mu(i)} = k/n$, for all permutations μ. Therefore, for all $1 \ \leq i, j, k, \ \leq n$,

$$E[z_{ij} - x_{ij} \mid \mathbb{Z}, z_{ij} = k/n] \ = \ \frac{k}{n} - \frac{k}{n+1} \ = \ \frac{k}{n(n+1)} \ > \ 0. \tag{1}$$

By Theorem 1 (a),

$$\lim_{n\to\infty} \mathbb{E}\left[\sum_{i=1}^{n} x_{i,\sigma_n^*(i)}\right] \ \geq \ \frac{\pi^2}{6}.$$

Moreover,

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{E}\left[\sum_{i=1}^{n} x_{i,\sigma_n^*(i)}\right] \ &\leq \ \lim_{n\to\infty} \mathbb{E}\left[\sum_{i=1}^{n} x_{i,\sigma_n^*(i)}\right] + \lim_{n\to\infty} \mathbb{E}\left[\sum_{i=1}^{n} z_{i,\sigma_n^*(i)} - x_{i,\sigma_n^*(i)}\right] \\
&= \ \lim_{n\to\infty} \mathbb{E}\left[\sum_{i=1}^{n} z_{i,\sigma_n^*(i)}\right] \\
&= \ \lim_{n\to\infty} \mathbb{E}[\mathcal{A}(n,\mathbb{Z})]
\end{aligned}$$

where the first inequality follows from the non-negativity of the second term established in Eq. (1). ■

Nikzad [4] considered this problem and proved a bound of $7\frac{3}{4}$ and observed that simulations suggested a bound that is below 2. Making a direct connection to the random assignment literature, which has generally dealt with i.i.d. and continuous distributions, was left as an open problem in his work. The main purpose of this note is to make such a connection and to observe that doing so makes the proofs simpler and the bounds sharper. We end with a couple of additional remarks.

1. Simulations suggest that the value of $\lim_{n\to\infty} \mathbb{E}[\mathcal{A}(n,\mathbb{Z})]$ is approximately 1.83, roughly midway between the upper and lower bounds observed here. Finding the exact value appears to be challenging. The exact values for small n are shown in Table 1.

2. The literature on the standard random assignment model suggests a path forward for the case of finite n as well. Consider a cost matrix $\mathbb{Y} = [y_{ij}]$ with i.i.d. entries, each

n	1	2	3	4	5
$\mathbb{E}[\mathcal{A}(n,\mathbb{Z})]$	1	$\frac{5}{4}(1.25)$	$\frac{49}{36}$ (1.361)	$\frac{13259}{9216}$ (1.439)	$\frac{310084531}{207360000}$ (1.495)

Table 1: Exact Values of $\mathbb{E}[\mathcal{A}(n,\mathbb{Z})]$ for small n

equally likely to be $\{\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, \frac{n}{n} = 1\}$. This is a discrete model, but amenable to the elegant primal-dual analysis of Karp [3] (see also Steele [5, Chapter 4]). A simple calculation shows that $\mathbb{E}[\mathcal{A}(n,\mathbb{Y})] < 3$ for all n. A bound on $\mathbb{E}[\mathcal{A}(n,\mathbb{Z})]$ follows by observing that $\mathcal{A}(n,\mathbb{Y})$ is stochastically larger than $\mathcal{A}(n,\mathbb{Z})$ in the sense of first-order stochastic dominance. The bound of 3 on $\mathbb{E}[\mathcal{A}(n,\mathbb{Y})]$ is clearly not sharp. For large n, it is intuitive that $\mathbb{E}[\mathcal{A}(n,\mathbb{Y})]$ is larger than $\mathbb{E}[\mathcal{A}(n,\mathbb{X})]$ by approximately $1/2$, and this can be verified by simulation, as well as by an argument along the lines of Theorem 2. We omit these details because the bounds obtained are necessarily weaker than the ones in Theorem 2.

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