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A Polynomial-time Algorithm for the Bistable Roommates Problem

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In a recent paper, Weems introduced the bistable matching problem, and asked if a polynomial-time algorithm exists to decide the feasibility of the bistable roommates problem. We resolve this question in the affirmative using linear programming. In addition, we show that several (old and new) results for the bistable marriage and roommates problem become transparent using the polyhedral approach. This technique has been used recently by the authors to address classical stable matching problems.

Key Words: stable matching; linear programming; rounding; bistability.

1. INTRODUCTION

Preliminaries. The stable marriage problem and its generalization, the stable roommates problem, are well-known matching problems that have been investigated extensively. In the marriage problem, each of n men and n women has a list of all the n members of the opposite sex in decreasing preference order. A stable matching is a set of n man–woman pairs such that no man and woman strictly prefer each other to their assigned spouses. In the stable roommates problem, each of $n = 2k$ persons has a list of the remaining $2k-1$ persons in decreasing preference order. A stable matching is a set of k pairs such that no two persons strictly prefer each other to their assigned roommates.

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In their pioneering paper, Gale and Shapley [1] designed an elegant algorithm that determines a stable matching for *any* instance of the stable marriage problem. Furthermore, they showed that stable matchings may not exist for some instances of the stable roommates problem. The structure of the marriage problem has since been studied by computer scientists and economists; see for instance the books by Knuth [5], Gusfield and Irving [3], Roth and Sotomayor [7], and the references therein. For the roommates problem, Irving [4] designed an algorithm to construct a feasible solution, whereas one exists.

Bistability. Associated with any instance of a stable matching (marriage or roommates) problem is a new instance obtained by reversing the order of each preference list. In a recent paper, Weems [12] introduced the concept of *bistable* matchings, which are matchings that are stable with respect to both the original and reversed preference lists. For the bistable marriage problem, Weems described an elegant way of adapting the original Gale–Shapley algorithm to find a bistable matching, if one exists, and showed that several structural results known for the stable marriage problem hold for the bistable version as well. However, Weems left open analogous questions for the *bistable roommates* problem, which is the subject of the present paper. Our main contribution is to observe that the linear programming (LP) approach can be used effectively to find a feasible bistable solution for the roommates problem, *if one exists*, and to prove the non-existence of a feasible solution otherwise. We also show that several known results for the bistable marriage problem become transparent using the LP approach.

LP for Stable Matchings. Vande Vate [11] initiated the study of the marriage problem using a mathematical programming approach and obtained a complete characterization of the convex hull of the stable marriage solutions. This polyhedral description was later extended and simplified by Rothblum [8], Roth *et al.* [6], and Teo and Sethuraman [9, 10]. One of the goals of this paper is to show that the LP-based approach appears to have some inherent advantages compared to the combinatorial approach using rotational posets. In fact, Weems [12] extended the rotational posets approach to the bistable roommates problem, but found that it leads to 3-SAT clauses, which are not easily seen to be solvable in polynomial time. In contrast, the LP-based rounding approach of Teo and Sethuraman [9] can be extended relatively easily to the bistable problem.

The rest of this paper is organized as follows. In Section 2 we revisit the LP formulation of the bistable marriage problem and provide genuinely simple proofs of known results. In addition, we prove a strong structural property of the set of bistable marriage solutions and introduce the concept of a median bistable marriage. In Section 3 we consider the bistable roommates problem and formulate it as an integer program; we also show that the underlying LP relaxation is not strong enough to decide feasibility. In Section 4, we strengthen the LP relaxation using additional inequalities and show that the strengthened LP has the power to distinguish feasible instances from infeasible ones.

2. BISTABLE MARRIAGES REVISITED

Consider an instance of the stable marriage problem and recall the definition of a *blocking pair*. A pair (m, w) is a blocking pair for a matching M if

- m and w are not matched to each other in M; and
- *both* m and w prefer each other to their assigned partners in M.

A stable marriage is one that does not have any blocking pair. Stated differently, a matching M is stable if for every man–woman pair (m, w)

- m and w are assigned to each other in M; or
- *At least* one of m, w has a better partner in M.

Notice that this definition allows for *both* m and w to have better partners in M. In such a case, however, (m, w) would be a blocking pair for the reversed instance of the problem (referred to as a *reverse blocking pair*)! This naturally leads to the following definition for bistability: a matching M is *bistable* if for every man–woman pair (m, w)

- m and w are assigned to each other in M; or
- *exactly* one of m, w has a better partner in M.

It is an easy exercise to verify that this definition is equivalent to the definition of Weems [12].

We now proceed to formulate the bistable matching problem as an integer programming problem. For women w_1 and w_2, we write $w_1 >_m w_2$ if man m prefers w_1 to w_2. Let

$$x_{i,j} = \begin{cases} 1 & \text{if } m_i \text{ is matched to } w_j \\ 0 & \text{otherwise.} \end{cases}$$

Any incidence vector that corresponds to a (bi)stable matching is called a (bi)stable marriage solution.

Consider the following formulation of the bistable marriage problem:

$$(I_{BM}) \qquad \sum_j x_{i,j} = 1 \qquad \forall i, \tag{1}$$

$$\sum_i x_{i,j} = 1 \qquad \forall j, \tag{2}$$

$$x_{i,j} + \sum_{k:\, w_k <_{m_i} w_j} x_{i,k} + \sum_{k:\, m_k <_{w_j} m_i} x_{k,j} = 1 \qquad \forall i, j, \tag{3}$$

$$x_{i,j} \in \{0, 1\} \qquad \forall i, j. \tag{4}$$

Equation (3) is clearly valid, otherwise, we must have either

$$\sum_{k:\, m_k <_{w_j} m_i} x_{k,j} = 1 \qquad \text{and} \qquad \sum_{k:\, w_k <_{m_i} w_j} x_{i,k} = 1,$$

or

$$\sum_{k:\, m_k <_{w_j} m_i} x_{k,j} = 0 \qquad \text{and} \qquad \sum_{k:\, w_k <_{m_i} w_j} x_{i,k} = 0.$$

In the former case, m_i and w_j are both matched to less favorable mates in the matching than each other, and hence form a blocking pair; in the latter case, m_i and w_j are both matched to more favorable mates in the matching than each other, and hence form a reverse blocking pair.

Let P_{BM} be the linear programming relaxation of I_{BM} obtained by removing the integrality restrictions on the x_{ij} variables. This formulation P_{BM} suggests a very simple geometry for each of its fractional points $\{x_{ij}\}$: Since $\sum_j x_{i,j} = 1$, for each man m_i, we can construct n intervals of the type $(a, b]$ (left-open, right-closed) of length $x_{i,j}$—one for each woman w_j (some of these intervals can be empty). We can arrange these n intervals in any *non-overlapping* order to cover the line segment $(0, 1]$. Likewise, we can do the same for the women. Consider the following table of $2n$ rows:

- for each man m_i, $x_{i,j}$, $j = 1, \ldots, n$ are arranged in *decreasing* preference of m_i to cover the interval $(0, 1]$; and
- for each woman w_j, $x_{i,j}$, $i = 1, \ldots, n$ are arranged in *increasing* preference of w_j to cover the interval $(0, 1]$.

Note that constraint (3) implies an interesting property of the arrangement: The subintervals spanned by $x_{i,j}$ in the arrangement corresponding to m_i and w_j coincide! (See Fig. 1.)

Using this geometry, and the proof technique discussed by Teo and Sethuraman [9], it is easy to see that the polytope is integral. We present the argument in full here for the sake of completeness. (We note that these methods have been used to prove the integrality of stable marriage polytope by exploiting the above geometry,

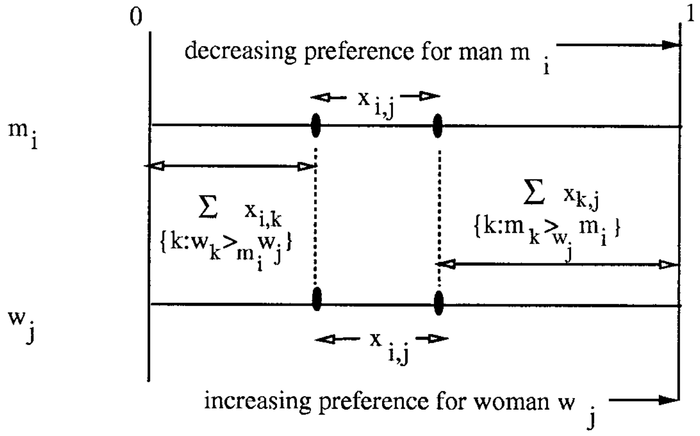

FIG. 1. Geometry of the layout of x_{ij}.

which is a consequence of linear programming duality; in contrast, the bistability case is much simpler: the geometry is an immediate consequence of the formulation.)

THEOREM 2.1. *The polytope* (P_{BM}) *is the convex hull of the bistable marriage solutions.*

Proof. Let x be any feasible solution in P_{BM}. We abuse terminology and use $x_{i,j}$ to also refer to the disjoint subintervals in the rows corresponding to man m_i and woman w_j in the arrangement of Fig. 1.

We show that x can be written as a convex combination of bistable matchings, using the geometric structure of the fractional feasible solutions. We generate a random number U uniformly in $(0, 1]$ and construct a matching in the following way: Match m_i to w_k if $x_{i,k} > 0$ and in the row of m_i in Fig. 1, U lies in the subinterval spanned by $x_{i,k}$; similarly, match w_j to m_l if in the row of w_j in Fig. 1, U lies in the subinterval spanned by $x_{l,j}$.

From the formulation of the bistable roommates problem, and by the construction of the arrangement in Fig. 1, m_i is matched to w_j if and only if w_j is matched to m_i. Furthermore, no two men can be matched to the same woman, and no two women can be matched to the same man, which makes the assignment under consideration a perfect matching. Any woman (say w_k) who is preferred by m_i to his mate w_j in this assignment (i.e., the subinterval $x_{i,k}$ is to the left of $x_{i,j}$ in the row of m_i in Fig. 1) is assigned a mate whom she strictly prefers to m_i, since in the row of w_k in Fig. 1, the random number U lies strictly to the right of the subinterval $x_{i,k}$. Similarly, any man (say m_l) who is preferred by w_j to her mate m_i in this assignment (i.e., the subinterval $x_{l,j}$ is to the right of $x_{i,j}$ in the row of w_j) is assigned a mate whom he strictly prefers to w_j, since in the row of m_l, the random number U lies strictly to the left of the subinterval $x_{l,j}$. Thus, if m and w are not paired in the matching, exactly one of m, w gets a better partner, which makes the matching bistable. Let X_U be the assignment obtained. In other words, $X_U(i, j) = 1$ if and only if m_i and w_j are matched to each other under the above assignment. Then

$$E(X_U(i, j)) = P(U \text{ lies in the subinterval spanned by } x_{i,j}) = x_{i,j}.$$

Thus $x(i, j) = \int_0^1 X_u(i, j)\, du$ and x can be written as a convex combination of X_u as u varies over the interval $(0, 1]$. Clearly, there are at most $O(n^2)$ distinct assignments arising from X_u as u varies. ∎

The rounding approach can also be used to derive the following surprising property of the bistable marriage solutions (see Teo and Sethuraman [9]):

THEOREM 2.2. *Let* $X_1, X_2, \ldots, X_l$ *be* l *distinct bistable marriage solutions. Each man* m_i *has* l *possible mates under these matchings. Assign him the woman whose rank is* k *among the* l *(possibly non-distinct) women. For each woman* w_j*, assign her to the man whom she ranked* $l+1-k$ *among the* l *men she was assigned to under the matchings. This assignment gives rise to another, not necessarily distinct, bistable marriage solution.*

In addition to establishing the lattice structure of bistable matchings, Theorem 2.2 generalizes several analogous results for stable matchings. For instance, when $l = 3$ and $k = 2$, Theorem 2.2 reduces to the median property for bistable solutions due to Weems [12]. When l corresponds to the total number of distinct stable marriage solutions in the problem, and $k = 1$, we obtain the men-optimal bistable matching. When the total number of solutions, l, is odd, the case $k = \frac{l+1}{2}$ is of special interest—there is a bistable marriage solution in which each person is assigned to a partner who is the "median" partner among his/her possible mates!

3. FRACTIONAL BISTABLE ROOMMATES POLYTOPE

We now turn to the bistable roommates problem. Determining, in polynomial time, if a given instance of the bistable roommates problem has a solution was left open by Weems [12]. In this section, we show that a certain LP formulation can be used to resolve the feasibility of a given bistable roommates instance in polynomial time. In other words, we show that a bistable matching exists for a given instance of the bistable roommates problem if and only if the associated LP formulation has a feasible solution.

3.1. LP for Bistable Roommates

Consider the following straightforward formulation of the bistable roommates problem:

$$(I_{BM}) \qquad \sum_j x_{i,j} = 1 \qquad \forall i, \tag{5}$$

$$x_{i,j} + \sum_{l:l<_j i} x_{l,j} + \sum_{l:l<_i j} x_{i,l} = 1, \qquad \forall i, j. \tag{6}$$

$$x_{i,j} \in \{0, 1\} \qquad \forall i, j. \tag{7}$$

Constraints (6) must be valid, otherwise, either (i) i has a roommate inferior to j, and j has a roommate inferior to i; or (ii) i has a roommate superior to j, and j has a roommate superior to i. In the former case, (i, j) is a blocking pair, and in the latter case, (i, j) is a reverse blocking pair. We call constraints (6) the *paired* inequalities. (In the roommates case, the variable $x_{i,j}$ models the decision whether person i is matched to person j.)

Let F_{BM} be the linear programming relaxation of the formulation I_{BM} in which the integrality restrictions on the variables x_{ij} are dropped. In contrast to the bistable marriage problem, however, there are infeasible instances of the bistable roommates problem for which F_{BM} is non-empty (see the 6 node example below), and so F_{BM} cannot be used to address the feasibility question for the bistable roommates problem.

EXAMPLE 1. Consider the bistable roommates problem with 6 nodes and the following preference lists:

$$P(1): 2, 3, 4, 5, 6$$
$$P(2): 3, 4, 6, 5, 1$$
$$P(3): 4, 5, 6, 1, 2$$
$$P(4): 5, 6, 1, 2, 3$$
$$P(5): 6, 1, 2, 3, 4$$
$$P(6): 1, 2, 3, 4, 5.$$

It can be verified easily that:

(a) The vector x with $x_{i,j} = \frac{1}{2}$ for $\{i, j\}$ in the set $\{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 1\}\}$ and $x_{i,j} = 0$ otherwise is a feasible solution in F_{BM};

(b) The matching $\{(1, 4), (2, 6), (3, 5)\}$ is the only stable matching to Example 1;

(c) Example 1 does not have a bistable matching.

This shows that we cannot use the feasibility of F_{SM} to determine the feasibility of the bistable roommates problem. We next propose a new class of valid inequalities to strengthen the formulation so as to resolve the existence of a bistable roommates solution efficiently.

3.2. On a New Class of Valid Inequalities

Let $x_{u,v} = 1$ if (u, v) is an edge in the matching, 0 otherwise. Consider distinct nodes i, j, k such that j prefers k to i. For the matching to be stable, the following must be valid,

$$S(i, j, k) \equiv \frac{1}{2}\left(\sum_{l: l \leqslant_j i} x_{lj} + \sum_{l: l \leqslant_k j} x_{lk}\right) \leqslant \frac{1}{2}$$

since the above is dominated by the paired inequalities (6) in F_{BM}.

As in the case of stable roommates (see Teo and Sethuraman [9]), the above inequality can be extended to an odd cycle version: Suppose $i_0, i_1, \ldots, i_C$ (C even) are such that i_k prefers i_{k+1} to i_{k-1}, where the indices are taken modulo $(C+1)$. Then by adding up the above inequality, we have

$$\sum_{k=0}^{C} S(i_{k-1}, i_k, i_{k+1}) \leqslant \frac{C+1}{2}.$$

Note that the nodes in the cycle *need not* be distinct. The coefficients of all the variables appearing in the LHS axe integral. Hence by rounding down the right hand side, we have the following *odd cyclic preference* inequalities:

$$\sum_{k=0}^{C} S(i_{k-1}, i_k, i_{k+1}) \leqslant \left\lfloor \frac{C+1}{2} \right\rfloor.$$

Thus we obtain an improved formulation:

$$(P_{SR}) \qquad \sum_j x_{i,j} = 1, \qquad \forall i, \tag{8}$$

$$x_{i,j} + \sum_{l:\, l <_j i} x_{l,j} + \sum_{l:\, l <_i j} x_{i,l} = 1 \qquad \forall i, j, \tag{9}$$

$$\sum_{k=0}^{C} S(i_{k-1}, i_k, i_{k+1}) \leqslant \left\lfloor \frac{C+1}{2} \right\rfloor, \qquad i_{k-1} <_{i_k} i_{k+1},\ k = 0, 1, \ldots, C. \tag{10}$$

We next argue that although P_{SR} may contain exponentially many constraints, its feasibility can be resolved in polynomial-time because the associated separation problem can be solved efficiently.

Separation Routine

Separation over (F_{SM}) (and hence (9)) is trivial since there are only $O(n^2)$ of these constraints. We define a new directed graph $G' = (V', A')$ with

$$V' = \{(i, j) \colon i \in V, j \in V\},$$
$$A' = \{((i, j), (j, k)) \colon i <_j k\}.$$

For each arc $((i, j), (j, k))$ in A', define a weight

$$c(i, j, k) = 1 - \sum_{l:\, l \leqslant_j i} x_{lj} - \sum_{l:\, l \leqslant_k j} x_{lk}.$$

Note that $c(i, j, k) \geqslant 0$ by (9). Furthermore, an odd directed cycle $\mathscr{C} = (i_0, i_1, \ldots, i_C)$ (in that order) in G' gives rise to an odd cyclic preference inequality, with cost

$$c(\mathscr{C}) = |\mathscr{C}| - 2 \sum_{0 \leqslant k \leqslant C} \ \sum_{l:\, l \leqslant_{i_k} i_{k-1}} x_{l, i_k}.$$

Thus $c(\mathscr{C}) \geqslant 1$ if and only if $\sum_{0 \leqslant k \leqslant C} \sum_{l:\, l \leqslant_{i_k} i_{k-1}} x_{l, i_k} \leqslant (|\mathscr{C}| - 1)/2$, which is (10).

It is well known that finding a shortest odd cycle in the directed graph G' with non-negative weight function can be solved easily by solving $O(|V'|)$ shortest directed path problems in an associated bipartite graph (see Grötschel *et al.* [2]).

3.3. Feasibility of the Bistable Roommates Problem

In this section, we prove the following result:

THEOREM 3.1. *Formulation (P_{SR}) is feasible if and only if the corresponding bistable roommates problem is feasible.*

The proof of this result essentially follows the proof for the stable roommates problem, discussed by Teo and Sethuraman [9], except that we now have to deal with bistability (rather than stability). We refer the readers to the above paper for

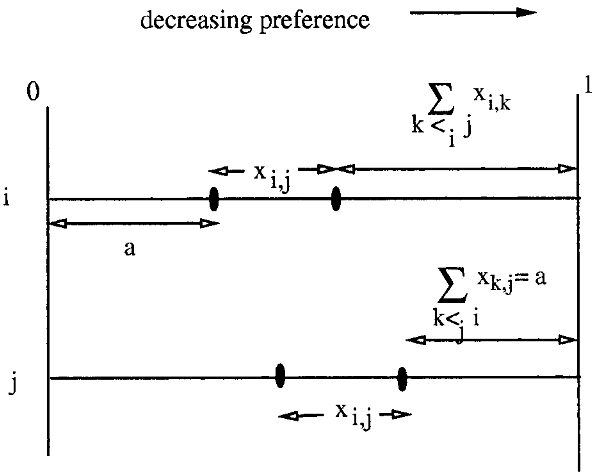

FIG. 2. Geometry of the layout of x_{ij}.

the details, but will just focus on the intuition behind the proof in this paper. It exploits the geometry of the fractional solutions (see Fig. 2). We need the following steps:

Step 1. For each person i, we arrange the n subintervals $x_{i,j}$ as j varies in decreasing preference of i. See Fig. 2. If $x_{i,k}$ is the subinterval that covers the point $\frac{1}{2}$, then we assign k to i. In this way, the assignment gives rise to a union of disjoint cycles and a partial matching.

Step 2. Using the odd cyclic preference inequalities, we show that the length of each cycle arising in this way must be even. By choosing alternate edges of the even cycles in a *careful* manner, we obtain the roommates assignment.

If there is any odd-cycle obtained in Step 1, then we must have the following situation (say, with an odd cycle of length 3, with i assigned to k, k assigned to j, and j assigned to i) as shown in Fig. 3.

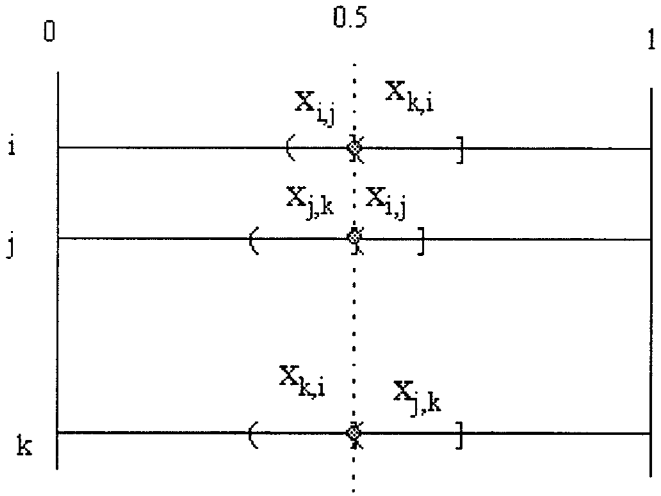

FIG. 3. Odd cycle instances.

The above example, for instance, would violate the odd cycle constraint $S(i, j, k)+S(j, k, i)+S(k, i, j) \leqslant 1$. This ensures that the solution obtained in Step 1 is simply a union of even cycles plus a partial matching, from which we can construct a bistable matching.

For instance, if we have a cycle of length 4 from the rounding procedure, say as in Fig. 4, where the assignments are $i \leftarrow j$, $j \leftarrow k$, $k \leftarrow l$, $l \leftarrow i$. We could round $x_{i,j}$, $x_{k,l}$ to 1 and $x_{j,k}$, $x_{l,i}$ to 0. However, this solution can fail to be bistable if $x_{ik}=0$ and $\sum_{j: j <_i k} x_{ij}=1/2$. Note that the position of the intervals $x_{i,k}$ and $x_{k,i}$ are as shown in Fig. 4. This means that i prefers j to k to l, and k prefers l to i to j. The pair (i, k) violates the bistability condition. Fortunately, the odd cyclic preference constraints rule out these possibilities: in this case, the inequality $S(i, k, l)+S(k, l, i)+S(l, i, k) \leqslant 1$ will be violated.

To complete the proof, we need to show that it is possible to choose alternate edges in the (even) cycles obtained in Step 1 to form a bistable solution. However, the edges must be chosen carefully to ensure bistability.

The edge (i, k) in the previous example plays a significant role in the way we select the edges in the bistable solution: it has the distinct property that

$$x_{ik}=0 \qquad \text{and} \qquad \sum_{j: j <_i k} x_{ij}=1/2.$$

We call such edges *obstruction* edges. It is clear that these are the only edges for which the bistability property may fail if we arbitrarily select alternate edges from the even cycles to form the assignment. By definition, if (i, j) is an obstruction edge, then the node i (and also j) must be in the vertex set of some even cycle C in $\mathscr{M}$. It cannot be the end node of any partial matching. If for an obstruction edge (i, k), i and k lie on a common cycle (as in the example just discussed) in the solution obtained in Step 1, we can easily derive a violated odd cycle inequality, as illustrated earlier. Thus, we need only examine the situation when i and k lie on disjoint cycles.

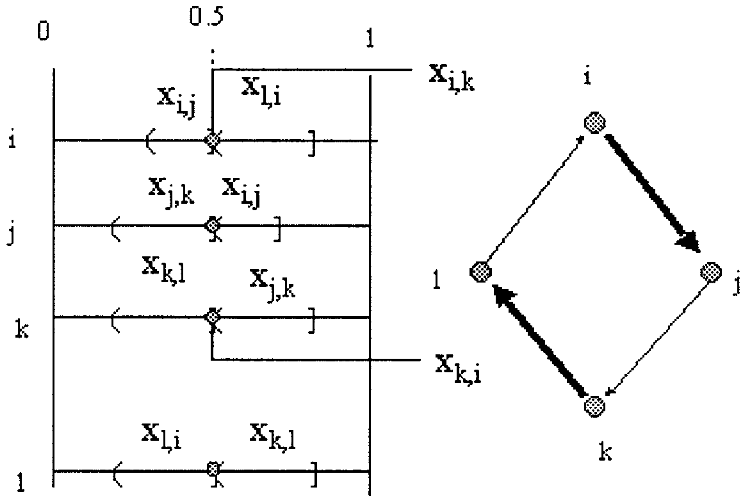

FIG. 4. Cycle of length 4.

Let $\mathscr{M}$ be the set containing all the obstruction edges (undirected) and the (oriented) edges obtained in Step 1 of the algorithm. If node j is assigned to node i in Step 1, then the orientation of the edge is from i to j. Let $G^* = (V, E^*)$ denote the graph defined on V with edge set in $\mathscr{M}$.

CLAIM. *G^* is bipartite.*

Proof. Suppose G^* contains an odd cycle $\mathscr{D}$, consisting of edges from $\mathscr{M}$. Fix an orientation for $\mathscr{D}$. Denote the directed odd cycle by $\bar{\mathscr{D}}$. Some of the edges in $\bar{\mathscr{D}}$ will receive a different orientation in $\bar{\mathscr{D}}$ and $\mathscr{M}$. These are the edges obtained in Step 1 of the algorithm.

Let $\{e_1, e_2, e_3, ..., e_k\}$ be a *maximal* subset of edges that forms a directed path in $\bar{D}$ and each e_i has received a different orientation from $\mathscr{M}$. Let C be the cycle that contains this path in $\mathscr{M}$. Replace the set $\{e_1, e_2, e_3, ..., e_k\}$ by $E(C) \setminus \{e_1, e_2, e_3, ..., e_k\}$, where $E(C)$ denotes the edge set of the cycle C. Since C is an even cycle, the parity the two sets are identical.

In this way, we maintain the parity of the number of nodes in the directed cycle, while ensuring that the orientation of the edges in $\bar{D}$ agree with that in $\mathscr{M}$.

On the other hand, we can also ensure that every arc that corresponds to an obstruction edge is isolated in the directed cycle $\bar{\mathscr{D}}$. In other words, no two obstruction edges are adjacent to one another in $\bar{\mathscr{D}}$. If necessary, we will tag an even cycle C from $\mathscr{M}$ to the nodes of $\bar{\mathscr{D}}$ to ensure this property.

In this way, we ensure that for every triplet (i, j, k) of consecutive nodes in $\bar{\mathscr{D}}$, $i <_j k$ and $S(i, j, k) \geqslant \frac{1}{2}$. However, this gives rise to an odd cyclic preference inequality that violates (10). ∎

Hence G^* is bipartite. Let A, B be the two partite sets of G^*. This splits the nodes of $\mathscr{M}$ into two parts each of equal size. Furthermore, each node in B is assigned to some node in A in Step 1. The assignment obtained in this way will be bistable since there are no obstruction edges for two nodes in the same set A and B, respectively. This concludes the proof of Theorem 3.1. ∎

Finally, we note the following interesting structural properties of the stable roommates solutions (the proofs are straightforward and are left as exercises for the reader).

THEOREM 3.2. *Let $X_1, X_2, ..., X_l$ be l distinct stable roommates solutions and assume l is odd. Each person has l possible roommates under these matchings. To each person, assign the person whose rank is $(l+1)/2$ among the l (possibly non-distinct) roommates. This assignment gives rise to another stable roommates solution.*

THEOREM 3.3. *Let $X_1, X_2, ..., X_l$ be l distinct stable roommates solutions and assume l is even. Each person has l possible roommates under these matchings. Then there is a stable roommates solution in which each person is assigned to a person whose rank is $\frac{l}{2}$ or $\frac{l}{2}+1$ among the l possible roommates.*

4. CONCLUSIONS

In this paper, we showed that the bistable roommates problem proposed by Weems [12] is solvable in polynomial time via linear programming. The techniques used are borrowed from Teo and Sethuraman [9], which exploits the geometry of the fractional solutions. Many of the structural results for stable matching (and its generalization, such as the bistable marriage/roommates problem) are straightforward consequences of this geometry.

The bistability problem also underscores an important advantage the LP based approach has over the classical combinatorial approach. The latter approach, for the bistable roommates problem, leads to 3SAT instances for which no efficient algorithm is known so far, whereas the former approach seems to be easily generalizable to many other variants of the roommates problem.

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