# Anonymous Monotonic Social Welfare Functions

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**Abstract**

This paper presents two results about preference domain conditions that deepen our understanding of anonymous and monotonic Arrovian social welfare functions (ASWFs). We characterize the class of anonymous and monotonic ASWFs on domains without Condorcet triples. This extends and generalizes an earlier characterization (as Generalized Majority Rules) by Moulin [10] for single-peaked domains. We also describe a domain where anonymous and monotonic ASWFs exist only when there are an odd number of agents. This is a counter-example to a claim by Muller [11], who asserted that the existence of 3-person anonymous and monotonic ASWFs guaranteed the existence of $n$-person anonymous and monotonic ASWFs for any $n > 3$. Both results build upon the integer programming approach to the study of ASWFs introduced in Sethuraman, Teo, and Vohra [15].

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# 1 Introduction

Perhaps the most popular answer to the question of what is the most democratic way to aggregate preferences is the simple majority rule. If all voters agree that some alternative $x$ is preferred to another alternative $y$, then the majority rule will return this ranking (**unanimity**). An increase in the support for $x$ over $y$ can never disadvantage $x$ (**monotonicity**). The majority rule does not respect the labels given to alternatives (**neutrality**). It does not respect the names of voters; the preferences of a bandit receive as much consideration as that of a benedictine (**anonymous**).

However, as is well known, there are profiles of orderings on which the majority rule 'cycles'. In fact, as we know from Arrow [1], aggregation procedures that return a strict ordering satisfying much weaker conditions than majority rule, do not exist, unless one places restrictions on the class of preferences being considered. Nevertheless, one can still ask: on preference domains where coherent aggregation procedures exist, what form do they take? So as to continue the discussion it will be useful to introduce some notation.

Let $\mathcal{A}$ denote a finite set of (at least three) alternatives. Let $\Sigma$ denote the set of all *strict* preference orderings over $\mathcal{A}$. The set of admissible preference orderings for members of a society of $n$ agents will be a subset of $\Sigma$ and denoted $\Omega$. Let $\Omega^n$ be the set of all $n$-tuples of preferences from $\Omega$, called **profiles**. An element of $\Omega^n$ will typically be denoted as $\mathbf{P} = (\mathbf{p_1}, \mathbf{p_2}, \dots, \mathbf{p_n})$, where $\mathbf{p_i}$ is interpreted as the preference ordering of agent $i$. In the language of Le Breton and Weymark [7], we assume the **common preference domain** framework. An $n$-person Social Welfare Function is a function $\mathcal{F} : \Omega^n \to \Sigma$. Thus for any $\mathbf{P} \in \Omega^n$, $\mathcal{F}(\mathbf{P})$ is an ordering of the alternatives. We write $x\mathcal{F}(\mathbf{P})y$ if $x$ is ranked above $y$ under $\mathcal{F}(\mathbf{P})$. An $n$-person **Arrovian Social Welfare Function** (ASWF) on $\Omega$ is a function $\mathcal{F} : \Omega^n \to \Sigma$ that satisfies the following two conditions:

1. **Unanimity:** If for $\mathbf{P} \in \Omega^n$ and some $x, y \in \mathcal{A}$ we have $x\mathbf{p_i}y$ for all $i$ then $x\mathcal{F}(\mathbf{P})y$.

2. **Independence of Irrelevant Alternatives:** For any $x, y \in \mathcal{A}$ suppose $\exists \mathbf{P}, \mathbf{Q} \in \Omega^n$ such that $x\mathbf{p_i}y$ if and only if $x\mathbf{q_i}y$ for $i = 1, \dots, n$. Then $x\mathcal{F}(\mathbf{P})y$ if and only if $x\mathcal{F}(\mathbf{Q})y$.

The first axiom stipulates that if all agents prefer alternative $x$ to alternative $y$, then the social welfare function $\mathcal{F}$ must rank $x$ above $y$. The second axiom states that the ranking of $x$ and $y$ in $\mathcal{F}$ is not affected by how the agents rank the other alternatives.

An obvious ASWF is the *dictatorial rule*: rank the alternatives in the order of the preferences of a particular agent (the dictator). An ASWF is **dictatorial** if there is an $i$ such that $\mathcal{F}(\mathbf{P}) = \mathbf{p_i}$ for all $\mathbf{P} \in \Omega^n$. An ordered pair of alternatives $(x, y)$ is called **trivial** if $x\mathbf{p}y$ for all $\mathbf{p} \in \Omega$. In view of unanimity, any ASWF must have $x\mathcal{F}(\mathbf{P})y$ for all $\mathbf{P} \in \Omega^n$ whenever $(x, y)$ is a trivial pair. If $\Omega$ consists only of trivial pairs then distinguishing between dictatorial and non-dictatorial ASWFs becomes nonsensical, so we assume that $\Omega$ contains at least one non-trivial pair. The domain $\Omega$ is **Arrovian** if it admits a non-dictatorial ASWF. When $\Omega = \Sigma$, Arrow [1] proved that the only ASWFs are the dictatorial functions. This conclusion has inspired a huge literature devoted to identifying Arrovian domains (see, for example, the monograph [5]).

Not only will an Arrovian domain admit a non-dictatorial ASWF, it may admit many different kinds of them. Thus one may wish to impose additional restrictions to whittle down the number of choices. Maskin [8], for example, shows that if a domain admits an ASWF that is neutral and anonymous, then the majority rule is an ASWF on that domain. In some cases one can make a stronger claim: every ASWF satisfying anonymity and neutrality can be represented as a kind of majority rule (see for example Moulin [9]). Results of this nature have been used to explain or support the importance of the majority rule.

While there is little objection to the first two properties of the majority rule listed above, it is not clear the other two (neutrality and anonymity) are essential for a 'democratic' aggregation procedure. One can point to institutions that are said to exemplify democratic principles that use aggregation procedures that violate both of these properties. Within the General Assembly of the U.N., procedural matters are decided by a simple majority but substantive matters by a two-thirds majority; a violation of neutrality. In the Security Council, only five particular nations have a veto; a violation of anonymity.[1] For this reason it is interesting to obtain analogs of Maskin's [8] theorem for ASWFs that satisfy either neutrality or anonymity but not both.

In Sethuraman, Teo, and Vohra [15] we show that if a domain admits a neutral ASWF then either the majority rule or the born loser rule[2] is an ASWF on that domain. In this paper we investigate the case of ASWFs that satisfy anonymity but not neutrality. A precise description of our results appears in the next section.

## 2 The Results

The axioms defining an ASWF do not preclude a small number of agents from essentially dictating the final outcome. Moreover, additional support in favor of an alternative over another does not necessarily improve the relative ranking of the initial alternative. One can eliminate these two possibilities by requiring that the ASWFs be **anonymous** and **monotonic**.

- An ASWF $\mathcal{F}$ is **anonymous** if its ranking over pairs of alternatives remains unchanged when the labels of the agents are permuted. Observe that a dictatorial rule is not anonymous.

- An ASWF $\mathcal{F}$ is **monotonic** if for all $x, y \in \mathcal{A}$, and all subsets of agents $S$, if $x\mathbf{p_i}y$ for all agents $i \in S$ implies $x\mathcal{F}(\mathbf{P})y$, then $x\mathbf{p'_i}y$ for all agents $i \in S' \supset S$ implies $x\mathcal{F}(\mathbf{p}')y$. In other words, if $x$ is socially preferred to $y$, then if additional agents change their relative $x, y$ preferences in favor of $x$, $x$ remains socially preferred to $y$.

[1] Readers who look upon the U.N. with a jaundiced eye, need only only turn to the U.S. Houses of Congress for similar examples.

[2] This is a modification of the anti-dictator rule that satisfies unanimity.

It is well known (cf. [14]) that any domain that excludes a Condorcet triple admits an anonymous, monotonic ASWF (AMASWF). The majority rule[3], which is anonymous and monotonic, is an SWF on such domains. Is it the only such rule? Moulin [10, Theorem 11.6, pp. 303-305], generalizing his earlier work [9], provided an answer when the preference domain consists of *all* single-peaked preference orderings with respect to a given linear order. Moulin proved that every AMASWF on this domain can be represented as a **generalized majority rule** (GMR). A GMR $\mathcal{M}$ for $n$ agents is of the following form:

- Add $n-1$ dummy agents, each with a fixed preference ordering drawn from $\Sigma$.
- $x$ is ranked above $y$ under $\mathcal{M}$ if and only if the majority (of real and dummy agents) prefer $x$ to $y$.

Each instance of a GMR can be described algebraically as follows. Fix a profile $\mathbf{R} \in \Sigma^{n-1}$ and let $m(x, y)$ be the number of orderings in $\mathbf{R}$ where $x$ is ranked above $y$. Given any profile $\mathbf{P} \in \Omega^n$, GMR ranks $x$ above $y$ if the number of agents who rank $x$ above $y$ under $\mathbf{P}$ is at least $n - m(x, y)$.

Our first contribution is to generalize Moulin's theorem to preference domains that do not contain any Condorcet triples (of which the single peaked domain is a special case). Specifically, we show that if the domain contains a preference $\mathbf{q}$ and its inversion $\mathbf{q}^{-1}$ (obtained by inverting the ordering of the alternatives in $\mathbf{q}$), then every AMASWF on this domain is a GMR. Furthermore, we show that these requirements are necessary by exhibiting a domain violating these requirements and an AMASWF on that domain that cannot be expressed as a GMR.

Our second contribution is motivated by a desire to prove an analog of the theorem of Kalai and Muller [6] for AMASWFs. They showed that the existence of a non-dictatorial ASWF does not depend on the number of agents; in other words, a domain $\Omega$ admits an $n$-person non-dictatorial ASWF if and only if it admits a 2-person non-dictatorial ASWF. While this result reduces the problem of verifying whether a domain is Arrovian or not to examining 2-person SWFs, it does not characterize Arrovian domains.

In our case, does the existence of a 2-person (or $k$-person, for any fixed $k$) AMASWF guarantee the existence of such ASWFs for all $n \geq k$? Muller [11] observed that the existence of a 2-person AMASWF does not guarantee the existence of an $n$-person AMASWF for all $n > 2$. However, Muller [11, Theorem 2, pp. 617] claimed the following:

**Claim 1** *There exists an $n$-person AMASWF on $\Omega$ for every $n > 3$ iff $\Omega$ admits a 3-person AMASWF.*

The claim, if true, implies that the existence of an AMASWF depends only on the domain and not the number of agents.

[3]For an even number of agents, the majority rule is not well-defined since we require the output of the ASWF to be a *strict* preference order; we can overcome this by introducing a dummy agent with a fixed preference $\mathbf{p} \in \Sigma$, and applying the majority rule.

We show that the claim is false. In fact, we prove more: We show that there is no odd number $k \geq 3$ such that the existence of a $k$-person AMASWF will guarantee the existence of an $n$-person AMASWF for all $n > k$. Thus, the existence of an AMASWF must depend not only on the domain $\Omega$, but also on $n$, the number of agents.

On the positive side we show that the existence of 2 and 3-person AMASWFs on a domain $\Omega$ satisfying some additional conditions is sufficient to guarantee the existence of an $n$-person AMASWF for all $n \geq 3$.

All of these results are obtained using insights from an integer program that characterizes the class of $n$-person ASWFs. This integer program was first introduced in Sethuraman, Teo, and Vohra [15]. In the next section we describe this integer program as specialized to the case of AMASWFs. In the subsequent section we describe the generalization of Moulin's result. The penultimate section of the paper exhibits the counterexample to Muller's claim. We conclude with some open problems.

While we restrict attention to the case of strict preferences, we conjecture, by analogy with previous results in the literature, that the results here would hold when indifferences are permitted. In support of this conjecture, we explicitly show that one of our main results—the existence of an anonymous, monotonic social welfare function depends on both the domain $\Omega$ and the number of voters $n$—continues to hold, even if social indifference is permitted.

## 3 An Integer Programming Description

Denote the set of all ordered pairs of alternatives by $\mathcal{A}^2$. Let $E$ denote the set of all agents, and $S^c$ denote $E \setminus S$ for any $S \subseteq E$. The independence of irrelevant alternatives condition allows us to specify an ASWF in terms of which ordered pair of alternatives a given subset, $S$, of agents is decisive over.

**Definition 1** *For a given ASWF $\mathcal{F}$, a subset $S$ of agents is* **weakly decisive for $x$ over** *$y$ if whenever all agents in $S$ rank $x$ over $y$ and all agents in $S^c$ rank $y$ over $x$, the ASWF $\mathcal{F}$ ranks $x$ over $y$.*

Since this is the only notion of decisiveness used in the paper, we omit the qualifier 'weak' in what follows. For each non-trivial element $(x, y) \in \mathcal{A}^2$, we define a 0-1 variable as follows:

$$d_S(x,y) = \begin{cases} 1, & \text{if the subset } S \text{ of agents is decisive for } x \text{ over } y; \\ 0, & \text{otherwise.} \end{cases}$$

If $(x, y) \in \mathcal{A}^2$ is a trivial pair then by default we set $d_S(x, y) = 1$ for all $S \neq \emptyset$. Given an ASWF $\mathcal{F}$, we can determine the associated $d$ variables as follows: for each $S \subseteq E$, and each non-trivial pair $(x, y)$, pick a $\mathbf{P} \in \Omega^n$ in which agents in $S$ rank $x$ over $y$, and agents in $S^c$ rank $y$ over $x$; if $x\mathcal{F}(\mathbf{P})y$, set $d_S(x, y) = 1$, else set $d_S(x, y) = 0$. Anonymity requires that $d_S(x, y) = d_T(x, y)$ whenever $|S| = |T|$. For this reason we write $d_r(x, y)$ in place of $d_S(x, y)$ whenever $|S| = r$. By definition, an ASWF satisfies unanimity and independence of irrelevant alternatives; moreover, its outcome on any given profile is a transitive ordering of the alternatives. We now express these properties in terms of the associated $d$ variables. **Unanimity**: To ensure unanimity, for all $(x, y) \in \mathcal{A}^2$, we must have

$$d_n(x, y) = 1. \tag{1}$$

**Independence of Irrelevant Alternatives**: Consider a pair of alternatives $(x, y) \in \mathcal{A}^2$, a $\mathbf{P} \in \Omega^n$, and let $S$ be the set of agents who prefer $x$ to $y$ in $\mathbf{P}$. (Thus, each agent in $S^c$ prefers $y$ to $x$ in $\mathbf{P}$.) Suppose $x\mathcal{F}(\mathbf{P})y$. Let $\mathbf{Q}$ be any other profile such that all agents in $S$ rank $x$ over $y$ and all agents in $S^c$ rank $y$ over $x$. By the independence of irrelevant alternatives condition $x\mathcal{F}(\mathbf{Q})y$. Hence the set $S$ is decisive for $x$ over $y$. However, had $y\mathcal{F}(\mathbf{P})x$ a similar argument would imply that $S^c$ is decisive for $y$ over $x$. Thus, for all $1 \leq r \leq n$ and non-trivial $(x, y) \in \mathcal{A}^2$, we must have

$$d_r(x, y) + d_{n-r}(y, x) = 1. \tag{2}$$

A consequence of Eqs. (1) and (2) is that $d_0(x, y) = 0$ for all $(x, y) \in \mathcal{A}^2$. **Transitivity**: The next class of constraints ensure transitivity. Let $A$, $B$, $C$, $U$, $V$, and $W$ be (possibly empty) *disjoint* sets of agents whose union includes all agents. Denote their cardinalities by $|A|, |B|, |C|, |U|, |V|$ and $|W|$ respectively. For each such partition of the agents, and any triple $x, y, z$,

$$d_{|A|+|U|+|V|}(x, y) + d_{|B|+|U|+|W|}(y, z) + d_{|C|+|V|+|W|}(z, x) \leq 2, \tag{3}$$

where the sets satisfy the following conditions (see Figure 1):

$$\begin{aligned}
&|A| \neq 0 \text{ only if there exists } \mathbf{p} \in \Omega, x\mathbf{p}z\mathbf{p}y,\\
&|B| \neq 0 \text{ only if there exists } \mathbf{p} \in \Omega, y\mathbf{p}x\mathbf{p}z,\\
&|C| \neq 0 \text{ only if there exists } \mathbf{p} \in \Omega, z\mathbf{p}y\mathbf{p}x,\\
&|U| \neq 0 \text{ only if there exists } \mathbf{p} \in \Omega, x\mathbf{p}y\mathbf{p}z,\\
&|V| \neq 0 \text{ only if there exists } \mathbf{p} \in \Omega, z\mathbf{p}x\mathbf{p}y,\\
&|W| \neq 0 \text{ only if there exists } \mathbf{p} \in \Omega, y\mathbf{p}z\mathbf{p}x.
\end{aligned}$$

These constraints ensure that on any profile $\mathbf{P} \in \Omega^n$, the ASWF $\mathcal{F}$ does not produce a ranking that "cycles." A consequence of (2) and (3) that is useful:

$$d_{|A|+|U|+|V|}(x, y) + d_{|B|+|U|+|W|}(y, z) + d_{|C|+|V|+|W|}(z, x) \geq 1.$$

To deduce it, interchange the roles of $z$ and $x$ in (3). Then the roles of $A$ and $V$ (resp. $B$ and $W$, $C$ and $U$) can be interchanged to obtain the new inequality:

$$d_{|A|+|C|+|V|}(z, y) + d_{|B|+|C|+|W|}(y, x) + d_{|U|+|A|+|B|}(x, z) \leq 2.$$

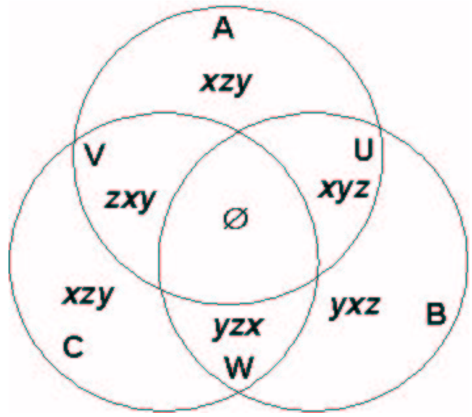


Figure 1: The sets and the associated orderings

Using (2), we obtain

$$d_{|B|+|U|+|W|}(y,z) + d_{|A|+|U|+|V|}(x,y) + d_{|C|+|V|+|W|}(z,x) \geq 1.$$

<u>**Monotonicity**</u>:

$$d_r(x,y) \leq d_s(x,y) \quad \forall r \leq s, \ \forall (x,y) \in \mathcal{A}^2. \tag{4}$$

Constraints (1-4) are both necessary and sufficient to characterize all $n$-person AMASWFs. One consequence of (3) which we invoke repeatedly in the rest of the paper is the following: Suppose there are $\mathbf{p}, \mathbf{q} \in \Omega$ and three alternatives $x, y$ and $z$ such that $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$. Then

$$d_r(x,y) \leq d_r(x,z), \tag{5}$$

$$d_r(z,x) \leq d_r(y,x). \tag{6}$$

The first inequality follows from using a profile $\mathbf{P}$ in which $r$ agents rank $x$ over $y$ over $z$ and $|E| - r$ agents rank $y$ over $z$ over $x$. If $d_r(x,y) = 1$ then $x\mathcal{F}(\mathbf{P})y$. By unanimity, $y\mathcal{F}(\mathbf{P})z$. By transitivity, $x\mathcal{F}(\mathbf{P})z$. Hence $d_r(x,z) = 1$; the second inequality can be proved similarly. Another consequence, which follows from transitivity, is: Suppose we know only that there is a $\mathbf{p} \in \Omega$ with $x\mathbf{p}y\mathbf{p}z$. Then

$$d_r(x,y) + d_r(y,z) \leq 1 + d_r(x,z), \tag{7}$$

$$d_r(z,y) + d_r(y,x) \geq d_r(z,x). \tag{8}$$

We omit the easy proof.

## 4 Domains without Condorcet Triples

Throughout this section, we assume that $\Omega$ is a domain without Condorcet triples. In this case, it is well-known that the majority rule, with a tie-breaking preference order in the case of an even number of agents, is an AMASWF on $\Omega$. Here we are interested in *characterizing* AMASWFs on such domains. We show that all such functions are essentially "majority rules." For subsequent discussion, we need to introduce some terms. A linear extension of a partially-ordered set (poset) $P$ is a linear (total) order $L$ with $a <_P b$ implying $a <_L b$ for all $a, b$ in the poset. Every poset $P$ can be obtained as the intersection of its linear extensions (cf. Dushnik and Miller (1941)). The minimum number of extensions defining $P$ is called the *dimension* of $P$. Let $\mathcal{F}$ be an AMASWF on $\Omega$, and $d$ be the associated decisiveness variables. (For convenience, we often refer to an SWF $\mathcal{F}$ in terms of its associated decisiveness variables $d$.) Let $D^r[d] = \{(x,y) \in \mathcal{A}^2 : d_r(x,y) = 1\}$. When there is no ambiguity we suppress the dependence on $d$ and write $D^r$ instead of $D^r[d]$. We can think of each $D^r[d]$ as being the arc set of a directed graph. We start with a key lemma on the structure of decisive sets of any AMASWF on $\Omega$.

**Lemma 1** *If $\Omega$ is a domain without any Condorcet triples and $d$ an anonymous, monotonic ASWF, then $D^r$ is transitive for any $r \leq \lfloor n/2 \rfloor$. Moreover, for any pair of alternatives $x, y$, $D^r$ does not contain both $(x,y)$ and $(y,x)$.*

**Proof.** Fix an $r < n/2$, and suppose $(x,y), (y,z) \in D^r$. We show that $(x,z) \in D^r$, i.e. $D^r$ is transitive. We may assume that at least one preference order in $\Omega$ ranks $z$ above $x$, otherwise there is nothing to prove. If there is a preference with $x$ above $y$ above $z$ in $\Omega$, then we can invoke (7) to argue that $(x,z) \in D^r$. Suppose otherwise. Consider the preference domain $\Omega$ restricted to the three alternatives $x$, $y$, and $z$; we abuse notation and refer to this restricted preference domain by $\Omega$. Let $\mathbf{p}$ and $\mathbf{q}$ be preference orders such that $z\mathbf{p}x\mathbf{p}y$ and $y\mathbf{q}z\mathbf{q}x$. We complete the proof by considering all possible cases:

- $\mathbf{p}, \mathbf{q} \in \Omega$: Consider the profile in which $r$ agents rank $p$ and $(n-r)$ agents rank $q$. The given SWF $\mathcal{F}$ produces a cycle; so this case cannot occur.

- $\mathbf{p}, \mathbf{q} \notin \Omega$: There must be some preference order in which $x$ is ranked above $y$, because $d_r(x,y) = 1$. Similarly, there must be some preference order in which $y$ is ranked above $z$. By assumption, $\Omega$ does not contain the preference order $xyz$. So $\Omega$ must contain the preference orders $xzy$ and $yxz$. Note that both of these rank $x$ above $z$. If $\Omega$ contains also $zyx$, then these preference orders form a Condorcet cycle. So this case cannot occur.

- $\mathbf{p} \in \Omega, \mathbf{q} \notin \Omega$: We know that $\Omega$ does not contain the orders $\{xyz, yzx\}$. But $\Omega$ must contain a preference order with $y$ ranked above $z$, otherwise $(z,y)$ will be a trivial pair; so $\Omega$ must contain the preference order $yxz$. Consider the profile with $r$ agents ranking $yxz$ and $(n-r)$ agents ranking $zxy$. The given SWF $\mathcal{F}$ ranks $x$ above $y$ and $y$ above $z$ in this profile, hence $x$ is ranked above $z$. This profile has exactly $r$ agents ranking $x$ above $z$. Hence $d_r(x,z) = 1$. The case with $\mathbf{q} \in \Omega$ but not $\mathbf{p}$ can be treated in the same way.

Thus, we have proved that $D^r$ is transitive. Suppose $(x, y) \in D^r$; then by (2), $(y, x) \notin D^{n-r}$, which, by monotonicity, implies $(y, x) \notin D^r$. Thus, $D^r$ cannot contain both $(x, y)$ and $(y, x)$ for a pair of alternatives $x, y$. ■

The question of characterizing all social welfare functions satisfying certain properties on a given preference domain is a natural one. In particular, Moulin [10, Theorem 11.6, pp. 303-305] provided an elegant characterization for single-peaked domains, which are a subclass of the domains without Condorcet triples. Specifically, he showed that every AMASWF on a (complete) single-peaked domain is a *generalized majority rule*. Our main contribution is to formulate an appropriate generalization of Moulin's theorem to preference domains that do not contain any Condorcet triples. We first show that if the domain $\Omega$ contains a preference ordering $\mathbf{q}$ and its inversion $\mathbf{q}^{-1}$ (obtained by inverting the ordering of the alternatives in $q$), then every AMASWF on this domain is a GMR.

**Theorem 1** *Let $\Omega$ be a domain with no Condorcet triples that contains an ordering $\mathbf{q}$ and its inversion $\mathbf{q}^{-1}$, for some $\mathbf{q} \in \Sigma$. Then any $n$-person AMASWF on $\Omega$ can be represented as a generalized majority rule.*

**Proof**. Without loss of generality, suppose $x_1 \mathbf{q} x_2 \mathbf{q} \ldots \mathbf{q} x_N$, where $N$ is the total number of alternatives. Note that $x_N \mathbf{q}^{-1} x_{N-1} \mathbf{q}^{-1} \ldots \mathbf{q}^{-1} x_1$. Fix an AMASWF $d$ on $\Omega$; Lemma 1 implies $D^r[d]$ is a partial order for all $r \leq \lfloor n/2 \rfloor$. Let $G_r$ be the (undirected) graph of the partial order corresponding to $D^r$, and let $G_r^c$ denote its complement. Our goal is to show that both $G_r$ and $G_r^c$ have transitive orientations. That $G_r$ has a transitive orientation is clear (orient the edge $\{x_i, x_j\}$ of $G_r$ from $x_i$ to $x_j$ if and only if $(x_i, x_j) \in D^r$). For the graph $G_r^c$, construct a new directed graph $D(G_r^c)$ by orienting the edge $\{x_i, x_j\}$ of $G_r^c$ from $x_i$ to $x_j$ if and only if $i < j$. We show that $D(G_r^c)$ is transitive. Consider the directed arcs $(x_i, x_j)$ and $(x_j, x_k)$ in $D(G_r^c)$. By definition,

- $i < j < k$; and
- $(x_i, x_j) \notin D^r$, $(x_j, x_k) \notin D^r$, $(x_j, x_i) \notin D^r$, and $(x_k, x_j) \notin D^r$.

By (8), and the existence of $\mathbf{q}$ and $\mathbf{q}^{-1}$, we have

$$d_r(x_i, x_k) = 0, \quad d_r(x_k, x_i) = 0.$$

Hence $(x_i, x_k) \in D(G_r^c)$, which implies that the graph $D(G_r^c)$ is transitive. Since the graph $G_r$ and its complement $G_r^c$ have transitive orientations, it follows from Dushnik and Miller [4] that the partial order corresponding to $D^r$ has dimension 2. We show this explicitly by constructing two linear extensions of $D^r$ whose intersection determines $D^r$. Consider the two (directed) graphs $T_1^r \equiv D^r \cup D(G_r^c))$, and $T_2^r \equiv D^r \cup D^{-1}(G_r^c)$, where $D^{-1}(G_r^c)$ is the transitive directed graph obtained by inverting the orientation of all the arcs in $D(G_r^c)$. Since $D^r$ and $D(G_r^c)$ are transitive graphs, and their union forms a complete directed graph, $T_1^r$ is a complete transitive graph. Let $\mathbf{p_1^r}$ be the (linear) ordering induced by $T_1^r$. Similarly, let

$\mathbf{p_2^r}$ be the (linear) ordering induced by $T_2^r$. Note that $\mathbf{p_1^r}$ and $\mathbf{p_2^r}$ agree on the ordering of the arcs in $D^r$, and disagree on all the arcs not in $D^r$. To get a GMR, we use $n-1$ dummy agents. For odd $n$, we use the profiles

$$\{\mathbf{p_1^r}, \mathbf{p_2^r}\}_{r=1}^{(n-1)/2}.$$

For even $n$, we use the profiles

$$\{\mathbf{p_1^r}, \mathbf{p_2^r}\}_{r=1}^{n/2-1},$$

which specifies the preferences of all but one of the dummy agents; the preference of the last dummy agent is simply the transitive order induced by $D^{n/2}$. Note that $D^{n/2}$ must be a complete transitive graph: transitivity follows from Lemma 1 and completeness follows from an application of Eq. (2). We show that the GMR with the specified preferences of the dummy agents is equivalent to the AMASWF $d$ we started with. Consider any pair of alternatives $x$, $y$, and suppose $r$ is the smallest number such that $d_r(x,y) = 1$. It is enough to show that $r$ (or more) real agents rank $x$ above $y$ if and only if a majority of the real and dummy agents rank $x$ above $y$. Since there are $(n-1)$ dummy agents, this is equivalent to showing that *exactly* $(n-r)$ dummy agents rank $x$ above $y$.

- $n$ **is odd**: If $r < n/2$, then, for $k = 1, 2, \ldots, r-1$, exactly one dummy agent ranks $x$ above $y$, and for $k = r, r+1, \ldots, (n-1)/2$, both dummy agents rank $x$ above $y$. Thus, the number of dummy agents ranking $x$ above $y$ is

$$(r-1) + 2 \times \left(\frac{n-1}{2} - r + 1\right) \;=\; n - r,$$

  as required. If $r > n/2$, then, let $r' = n - r + 1$; it is clear that $r'$ is the smallest number such that $d_{r'}(y,x) = 1$. If $r' < n/2$, by the argument just made, exactly $(n-r') = r-1$ dummy agents rank $y$ above $x$; since there are totally $(n-1)$ dummy agents, exactly $(n-r)$ dummy agents rank $x$ above $y$, as required. The only remaining possibility is for both $r, r' > n/2$. This is possible if and only if $r = r' = (n+1)/2$; this simply means the voting rule is really a simple majority rule on the pair $\{x, y\}$. In this case, exactly half the dummy agents rank $x$ above $y$; but $(n-r) = (n-1)/2$, which is exactly half the number of dummy agents; so the number of dummy agents ranking $x$ above $y$ is exactly $(n-r)$.

- $n$ **is even**: If $r \leq n/2$, then, for $k = 1, 2, \ldots, r-1$, exactly one dummy agent ranks $x$ above $y$, and for $k = r, r+1, \ldots, n/2-1$, both dummy agents rank $x$ above $y$; finally, the lone dummy agent with a preference order consistent with $D^{n/2}$ ranks $x$ above $y$. Thus, the number of dummy agents ranking $x$ above $y$ is $(r-1) + 2 \times (n/2 - r) + 1 \;=\; n - r$, as required. If $r > n/2$, then, let $r' = n - r + 1$; it is clear that $r' \leq n/2$, and it is the smallest number such that $d_{r'}(y,x) = 1$. By the argument just made, exactly $(n - r') = r - 1$ dummy agents rank $y$ above $x$; since there are totally $(n-1)$ dummy agents, exactly $(n-r)$ dummy agents rank $x$ above $y$, as required.

■

The remainder of this section is devoted to two issues. First, we discuss examples of "natural" domains that meet the conditions of Theorem 1. Second, we discuss the effect of relaxing the various assumptions of the Theorem.

**Domains meeting the assumptions of Theorem 1.** The set of all single-peaked preferences or single-dipped preferences with respect to a linear order $\mathbf{q}$ satisfy the assumptions of Theorem 1: they contain no Condorcet triples, and they contain both $\mathbf{q}$ and $\mathbf{q^{-1}}$. We describe next domains that naturally generalize these properties. Suppose the alternatives can be represented by pairs of real numbers

$$\{(a_1, b_1), (a_2, b_2), \ldots, (a_N, b_N)\},$$

with $a_1 < a_2 \ldots < a_N$ and $b_1 > b_2 > \ldots > b_N$. Each agent is represented by a non-negative pair of real numbers $(\alpha_i, \beta_i)$. Agent $i$ has a "utility" of $\alpha_i a_j + \beta_i b_j$ for alternative $j$, and the agents' preferences are determined by their utilities. Hence agent $i$ prefers alternative $j$ to $k$ if

$$\alpha_i a_j + \beta_i b_j > \alpha_i a_k + \beta_i b_k.$$

Since we assume all preferences are strict, we break ties using a pre-determined priority ordering. See Figure 2 for an illustration. The above contains single peaked and single dipped domains as special cases. Consider the piecewise linear graph obtained by connecting the alternatives using straight-line segments. If this graph is convex, the domain is single-dipped; and if this graph is concave, the domain is single-peaked. It is clear that this preference domain has no Condorcet triples, and contains a ranking $\mathbf{q}$ and its inversion $\mathbf{q^{-1}}$. By theorem 1, the only AMASWFs in this domain are GMRs. Another example satisfying the domain condition is the class of "order-restricted" preferences, first considered by Rothstein [13]. This is a restriction on *profiles*, not individual preferences; moreover, it is defined with respect to an ordering of the individuals, not alternatives. (We adapt the standard definition to the case of strict preferences.) A profile $\mathbf{P} \in \Omega^n$ is order-restricted if and only if there is an ordering $\sigma$ of the $n$ agents such that, for all distinct pair of alternatives $(x, y)$, the first $j(x, y)$ agents strictly prefer $x$ to $y$, and the remaining agents strictly prefer $y$ to $x$. Note that the ordering, $\sigma$, of the agents cannot depend on the pair of alternatives $(x, y)$, but the function $j(\cdot)$ can. It is well-known that the class of order-restricted preferences and the class of single-peaked preferences intersect each other, but neither one of them contains the other, see [2]. Importantly a profile that is order restricted does not contain a Condorcet triple. One domain that produces profiles that are order restricted can be described as follows. Let $t$, a real number, be a type and the utility that an agent with type $t$ assigns to alternative $x$ is $u(x|t)$. Now alternative $x$ is ranked above $y$ by a agent with type $t$ if $u(x|t) > u(y|t)$. Thus each type induces a preference ordering over the alternatives. We can thus associate a preference domain with a set of types. If $u(x|t) - u(y|t)$ is increasing in $t$, then every profile

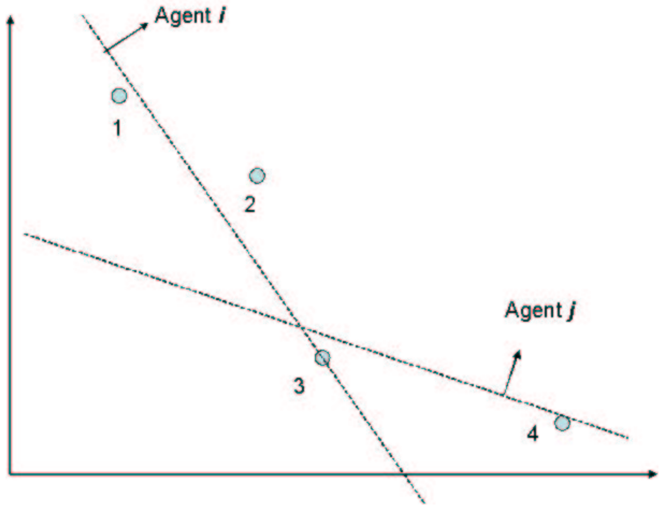


Figure 2: Agent $i$ prefers alternative 4 to 2 to 3 to 1; Agent $j$ prefers alternative 1 to 2 to 4 to 3

of types induces a profile of orderings that is order restricted. If the domain of types is 'rich' enough, the domain will contain an ordering $\mathbf{q}$ and its inverse. One such example is in Chapter 4, section 6 of [2].

**Relaxing the assumptions of Theorem 1.** The requirement that $\Omega$ contains at least one ordering and its inversion is crucial as the next example shows.

**Example 1.** Let $\mathcal{A} = \{a, b, c, d, e, f\}$ and

$$\Omega' = \{abcdef, dbcafe, eacbfd, fbaced\}.$$

Notice that $\Omega'$ does not contain any Condorcet triples or the inversion of any of its orderings. Consider the 3-person ASWF, defined by:

$$D^1 = \{(a, e), (a, f), (b, d), (b, f), (c, d), (c, f)\},$$

$$D^2 = \mathcal{A}^2 \setminus \{(e, a), (f, a), (d, b), (f, b), (d, c), (f, c)\}.$$

See Figure 3 for an illustration of $D^1$ and $D^2$. The rule implied by $D^1$ and $D^2$ is clearly anonymous and monotonic. We show that it is an SWF. If not, the rule should cycle on some three person profile on some triple of alternatives. We rule out this possibility by considering every triple of alternatives.

- Consider the triple $\{a, b, c\}$. Observe that these three items are ordered according to majority rule. Since $\Omega'$ has no Condorcet triples, the rule will not 'cycle' on this triple. A similar argument applies to the triple $\{d, e, f\}$.

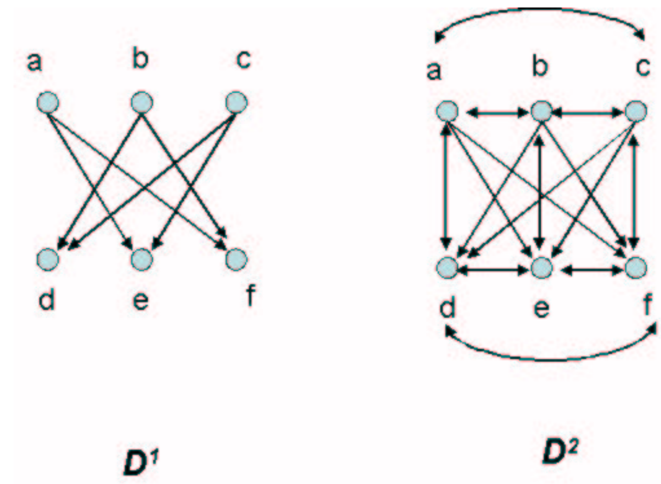


Figure 3: The sets $D^1$ and $D^2$ represented as directed graphs

- Consider the triple $\{a, e, f\}$. For the rule to cycle on this triple, we need all the 3 agents to rank $e$ above $a$, or all 3 persons to rank $f$ above $a$. However, since there is only one ordering in $\Omega'$ which ranks $e$ above $a$ (and only one which ranks $f$ above $a$), we conclude that all 3 persons must submit the same preference orderings. Since they are unanimous, the rule will not cycle. The same argument applies to the triples $\{b, d, f\}$, $\{c, d, e\}$, $\{b, c, d\}$, $\{a, c, e\}$ and $\{a, b, f\}$.

- Consider the triple $\{a, b, d\}$. For the rule to cycle on this triple, we only need to consider the case where $b$ is ranked above $d$, $d$ above $a$ and $a$ above $b$ under our rule. The other cycle (in opposite orientation) has already been ruled out by the previous case, since it needs all 3 agents to rank $d$ above $b$. In this case, we need at least two agents to rank $d$ above $a$, and at least 2 agents to rank $a$ above $b$. However, since there is no $\mathbf{p}$ in $\Omega'$ with $d\mathbf{p}a\mathbf{p}b$, and there are only three agents, we do not have enough agents to enforce this cycle. This argument applies to all the remaining triples not considered so far.

Notice that the poset corresponding to $D^1$ is an instance of a 3-crown, which is the smallest poset with dimension 3 (see [16]). Thus the 3-crown poset cannot be expressed as an intersection of two linear orders that extend it. We conclude that the SWF cannot be represented by a GMR. ■

**Remark.** We pause to comment on the relationship of Theorem 1 to Moulin's characterization [10] and an earlier extension of that result due to the authors [15, Theorem 12]. Moulin's original result considered the domain consisting of *all* single-peaked preference orderings with respect to a given linear order $\mathbf{q}$. This domain has no Condorcet triples, and contains both $\mathbf{q}$ and $\mathbf{q}^{-1}$, and so, Theorem 1 could be viewed as a direct generalization of this result. Our earlier result [15, Theorem 12] extended Moulin's characterization to domains consisting of *some* (not necessarily all) single-peaked preference orderings with respect to a given linear order $\mathbf{q}$. Finally, we note that Theorem 1 and our earlier result [15, Theorem 12] are not comparable. The domain $\{abc, cba, acb\}$ satisfies the hypothesis of Theorem 1, but cannot be a part of any single-peaked domain. The domain $\{abc, bca\}$ is a (subdomain of a) single-peaked domain, but violates the hypothesis of Theorem 1.

Suppose $\Omega$ is free of Condorcet triples, but also does not contain a pair $\mathbf{q}, \mathbf{q^{-1}}$. Example 1 shows that in this case there are AMASWFs that are not representable as generalized majority rules. From Lemma 1, however, we see that every AMASWF on such a domain could be viewed as a *block majority rule*. For $n$ odd, there will be $(n-1)/2$ *blocks* of dummy agents, with each block containing *at least* two dummy agents. For each pair of alternatives $x, y$, whenever *all* of the dummy agents within a block rank $x$ above $y$, that block contributes 2 votes for $x$ over $y$, otherwise it contributes 0. Every real agent ranking $x$ above $y$ contributes 1 for $x$ over $y$. Let $m(x, y)$ be the number of votes for $x$ over $y$. Then, $x$ is ranked above $y$ socially iff $m(x, y) > m(y, x)$. Clearly, the GMR is obtained as a special case when each block has exactly two dummy agents. In this case, by definition, if both dummy agents in a block rank $x$ above $y$, the block contributes 2 to $m(x, y)$, otherwise it contributes 1 each to $m(x, y)$ and $m(y, x)$, which can be treated as a zero contribution to both $m(x, y)$ and $m(y, x)$. We conclude by noting that the assumption regarding the absence of Condorcet triples cannot be weakened. Consider the following example due to Muller [11].

**Example 2.** Let $\mathcal{A} = \{x, y, z, w\}$ and $\Omega = \{yzxw, zxyw, xwyz, ywzx, xyzw, zwxy\}$. Consider the following 5-person ASWF, defined by $D^1 = \emptyset$,

$$D^2 = D^3 = \{(x, w), (x, z), (y, w), (y, x), (z, y), (z, w)\},$$

and $D^4 = \mathcal{A}^2$. It is straightforward (but tedious) to verify that this is an AMASWF. However, $D^2$ has a cycle involving $\{x, y, z\}$, and so this rule cannot be represented as a block majority rule.

# 5 General Domains and a Counterexample

In this section we show that, in general, the existence of an AMASWF depends not just on the domain condition, but also on the number of agents involved. To this end, we first construct a counterexample to Claim 1. We identify a domain $\Omega$ that admits a 3-person AMASWF, but does not admit a 2-person AMASWF. As any $2k$-person AMASWF can be used to obtain a 2-person solution, $\Omega$ cannot admit an AMASWF for any even number of agents; thus, this domain serves as a counterexample to Claim 1. Before proceeding further with the example, it is instructive to examine why Muller's proof of Claim 1 is incorrect. Muller's proof of Claim 1 relied on the following lemma [11, Lemma 1], re-written in our terminology.

**Lemma 2** *Suppose $\Omega$ permits construction of a monotonic SWF. Let $S$ be a non-empty subset of the set of all pairs of alternatives. Suppose $A_1, A_2, A_3$ is a partition of the agents such that each $A_i$ is decisive over $S$, then $S$ is a transitive sink. In particular, $S$ contains exactly half the pairs.*

We have not defined what a "sink" is, but this is not important for our purposes. Muller's proof of Claim 1 fails because Lemma 2 is false (see also Example 3 described below). The flaw in Muller's [11, pp. 619] proof of Lemma 2 is his (false) claim that if a coalition is decisive over $S$, then its complement is decisive over the inverse complement

$$S^* = \{(a,b)|a,b \in \mathcal{A}, (b,a) \notin S\}.$$

This is true if $S$ consists of *all* pairs of alternatives over which a coalition is decisive, but not otherwise. Muller's application of this claim, however, violates this restriction. A correct restatement is "if $S$ is the set of all pairs of alternatives over which a coalition is decisive, then $S^*$ is the set of all pairs of alternatives over which the complementary coalition is decisive." The former statement fixes $S$ regardless of the coalition; the latter statement chooses the "right" $S$ for each coalition. The counter example is as follows:

**Example 3.** Let $\mathcal{A} = \{a,b,c,d,e,f\}$ and

$$\overline{\Omega} = \{\text{cbeadf}, \text{ceabfd}, \text{fabecd}, \text{dcbaef}, \text{aecbfd}, \text{adbcef}, \text{cbdaef}, \text{afbced}, \text{abfced}\}.$$

The possible rankings of every triple of alternatives in $\overline{\Omega}$ are listed in Table 1. Observe that $\overline{\Omega}$ has no trivial pairs. Given an ASWF in terms of the $d$ variables (of the integer program), let $D^r = \{(x,y) \in \mathcal{A}^2 : d_r(x,y) = 1\}$.

## 5.1 Two agents

We first show that $\overline{\Omega}$ does not admit a 2-person AMASWF. We begin with a simple lemma, whose proof is immediate from equation 2.

**Lemma 3** *For all $x, y$, $d_1(x,y) + d_1(y,x) = 1$. In particular, $D^1$ contains exactly half the ordered pairs of alternatives.*

**Lemma 4** *Suppose $\Omega$ contains all possible orders of a triple $x, y, z$ of alternatives except the order $xyz$. Then,*

*(a)* $zx \in D^1$,

*(b)* $d_1(y,z) = d_1(y,x)$,

*(c)* $d_1(x,y) = d_1(z,y)$.

| Category | Alternatives | Possible Ranking | Properties |
|---|---|---|---|
| C1 | a, b, c | acb, cba, cab, abc | no Condorcet triple |
| | d, e, f | fed, edf, def, efd | no Condorcet triple |
| C2 | a, b, d | bad, abd, dba, adb, bda | no dab |
| | a, b, e | bea, eab, abe, bae, aeb | no eba |
| | a, c, d | cad, acd, dca, adc, cda | no dac |
| | a, c, f | caf, fac, acf, cfa, afc | no fca |
| | b, c, e | cbe, ceb, bec, ecb, bce | no ebc |
| | b, c, f | cbf, fbc, bcf, cfb, bfc | no fcb |
| C3 | a, b, f | baf, afb, fab, abf | f above a only when f above b<br>f above b only when a above b |
| | a, c, e | cae, aec, ace, cea | e above a only when c above a<br>e above c only when a above c |
| | a, e, f | eaf, afe, fae, aef, | e above a only when e above f<br>f above a only when f above e |
| | b, c, d | dcb, cbd, dbc, bcd | d above b only when d above c<br>d above c only when d above b |
| | b, d, f | fbd, bdf, dbf, bfd | d above b only when d above f<br>f above b only when f above d |
| | c, d, e | ecd, cde, dce, ced | d above c only when d above e<br>e above c only when e above d |
| C4 | a, d, e | ead, ade, dae, aed | e above d only when a above d<br>d above e only when a above e |
| | a, d, f | fad, adf, daf, afd | d above a only when d above f<br>d above f only when a above f |
| | b, d, e | ebd, bde, dbe, bed | d above e only when b above e<br>e above d only when b above d |
| | b, e, f | ebf, bfe, fbe, bef | e above b only when e above d<br>e above f only when b above f |
| | c, d, f | fcd, cdf, dcf, cfd | d above f only when c above f<br>f above d only when c above d |
| | c, e, f | fec, ecf, fce, cef | e above f only when c above f<br>c above f only when e above f |

Table 1: Triplets associated with $\overline{\Omega}$

**Proof**: From the 5 orderings $yzx, zxy, zyx, yxz, xzy$ and (5-6) we have the following implications:

- $d_1(y,z) \leq d_1(y,x)$, $d_1(x,y) \leq d_1(z,y)$;
- $d_1(z,y) \leq d_1(z,x)$, $d_1(x,z) \leq d_1(y,z)$;
- $d_1(y,x) \leq d_1(y,z)$, $d_1(z,y) \leq d_1(x,y)$;
- $d_1(x,z) \leq d_1(x,y)$, $d_1(y,x) \leq d_1(z,x)$.

Hence $d_1(y,z) = d_1(y,x), d_1(x,y) = d_1(z,y)$. If $d_1(x,z) = 1$, then $d_1(x,y) = d_1(z,y) = d_1(z,x) = 1$. This is a contradiction. Hence we have $d_1(x,z) = 0$, i.e. $d_1(z,x) = 1$. ■

**Lemma 5** *The domain $\overline{\Omega}$ does not admit a 2-person anonymous and monotonic ASWF.*

**Proof**: Suppose there exists a 2-person AMASWF on $\overline{\Omega}$. Then $D^1$ is well-defined. From Table 1, we see that all possible orderings of the set $\{a,b,d\}$ are present except for the order $dab$. Applying Lemma 4, we conclude that

$$d_1(b,d) = 1, \;\; d_1(a,b) = d_1(a,d), \;\; d_1(d,a) = d_1(b,a).$$

Applying Lemma 4 to the sets $\{a,b,e\}$, $\{a,c,d\}$, $\{a,c,f\}$, $\{b,c,e\}$, and $\{b,c,f\}$ respectively, we deduce

- $d_1(b,d) = 1$, $d_1(a,b) = d_1(a,d)$, $d_1(d,a) = d_1(b,a)$;
- $d_1(a,e) = 1, d_1(b,a) = d_1(b,e), d_1(e,b) = d_1(a,b)$;
- $d_1(c,d) = 1, d_1(a,c) = d_1(a,d), d_1(d,a) = d_1(c,a)$;
- $d_1(a,f) = 1, d_1(c,a) = d_1(c,f), d_1(f,c) = d_1(a,c)$;
- $d_1(c,e) = 1, d_1(b,c) = d_1(b,e), d_1(e,b) = d_1(c,b)$; and
- $d_1(b,f) = 1, d_1(c,b) = d_1(c,f), d_1(f,c) = d_1(b,c)$.

Now, observe that from these relationships, we must have

$$d_1(d,a) = d_1(b,a) = d_1(b,e) = d_1(b,c) = d_1(f,c) = d_1(a,c) = d_1(a,d),$$

which is a contradiction, since $d_1(d,a) + d_1(a,d) = 1$. ■

**Remarks.** While we require the ASWF to be anonymous and monotonic, we *do not* require it to be neutral. Clearly, a 2-person ASWF cannot be anonymous and neutral if social indifference is not permitted. The absence of neutrality gives the ASWF the freedom to resolve "ties" based on the labels of the alternatives. Lemma 5 essentially says that there is no "consistent" way to resolve ties in the domain $\overline{\Omega}$. It may be tempting to conclude that Lemma 5 holds because social indifference is not permitted. This is not true, as we show next. (Assuming strict individual preferences only strengthens our negative result.)

**Social Indifference.** We briefly argue that Lemma 5 remains valid even if social indifference is allowed. Note that $\overline{\Omega}$ has only strict preferences, so a 2-person AMASWF $f$ can be fully described by specifying what it does when the agents disagree on the ranking of a pair of alternatives $x$ and $y$. (We have used both unanimity and IIA.) We encode the AMASWF $f$ using the decisiveness variables $d(x,y) \in \{0, 1/2, 1\}$. Whenever the agents disagree on their $x, y$ ranking, $d(x,y) = 0$ indicates that $f$ ranks $y$ above $x$, $d(x,y) = 1$ indicates that $f$ ranks $x$ above $y$, and $d(x,y) = 1/2$ indicates that $f$ is indifferent between $x$ and $y$. From this definition, it is clear that $d(x,y) + d(y,x) = 1$. Lemma 5 will follow if we can prove Lemma 4, which is based on the inequalities (5-6), so we consider these next. Suppose there are $\mathbf{p}, \mathbf{q} \in \Omega$ and three alternatives $x, y$ and $z$ such that $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$. Then

$$d_r(x,y) \le d_r(x,z),$$
$$d_r(z,x) \le d_r(y,x).$$

The first inequality follows from using a profile $\mathbf{P}$ in which $r$ agents rank $x$ over $y$ over $z$ and $|E| - r$ agents rank $y$ over $z$ over $x$. If $d_r(x,y) = 1/2$ or $d_r(x,y) = 1$ then $f$ is indifferent between $x$ and $y$ or prefers $x$ to $y$; by unanimity, $f$ prefers $y$ to $z$. Since both the indifference and preference relations are transitive, $f$ must prefer $x$ to $z$, i.e., $d_r(x,z) = 1$. The second inequality can be proved similarly. Parts (b) and (c) of Lemma 5 are now immediate. To see that part (a) is also true, note that $d_1(x,z) = 1$ or $d_1(x,z) = 1/2$ implies $d_1(x,y) = d_1(z,y) = d_1(z,x) = 1$, which is a contradiction. Thus, $d_1(x,z) = 0$, i.e. $d_1(z,x) = 1$.

## 5.2 Three agents

We now prove that $\overline{\Omega}$ must admit an AMASWF when there are three agents.

**Theorem 2** *The domain $\overline{\Omega}$ admits a 3-person anonymous and monotonic ASWF.*

**Proof**: It suffices to specify $D^1$ and $D^2$. Let

$$D^1 = \{(b,d), (b,f), (a,e), (a,f), (c,d), (c,e)\},$$

$$D^2 = \mathcal{A}^2 \setminus \{(d,b), (f,b), (e,a), (f,a), (d,c), (e,c)\}.$$

(See Figure 3.) The definition of $D^1$ and $D^2$ allows us to construct an ASWF $\mathcal{F}$ to rank the alternatives, given any 3-person profile from $\Omega$. By construction, the $d$ variables associated with $\mathcal{F}$ satisfy (1, 2, 4). It remains to show that they satisfy (3). We do this on a case-by-case basis. In Table 1 we have divided the triples into categories. We select one triple from each category and show that (3) holds. An identical argument applies to all of the other triples in the same category.

(C1) Consider the triple {a,b,c}. In (3) set $x = a, y = b, z = c$. Note that the domain $\overline{\Omega}$ does not contain any Condorcet triples associated with the 3 alternatives. Given the orderings of $a, b, c$ in $\overline{\Omega}$ it follows that $B = W = \emptyset$. Since there are only three agents, it is not possible for

$$|A| + |U| + |V| \geq 2, |U| \geq 2, \text{ and } |C| + |V| \geq 2$$

at the same time. Hence

$$d_{|A|+|U|+|V|}(a,b) + d_{|U|}(b,c) + d_{|C|+|V|}(c,a) \leq 2.$$

Similarly, it is not possible for

$$|C| \geq 2, |A| + |V| + |C| \geq 2, \text{ and } |A| + |U| \geq 2$$

at the same time. Hence

$$d_{|C|}(b,a) + d_{|A|+|V|+|C|}(c,b) + d_{|A|+|U|}(a,c) \leq 2.$$

(C2) Consider the triple {a,b,d}. In (3) set $x = a, y = b, z = d$. Given the orderings of $a, b, d$ in $\overline{\Omega}$ it follows that $V = \emptyset$. Since there are only three agents, it is not possible for

$$|A| + |U| \geq 2, |B| + |U| + |W| \geq 1, \text{ and } |C| + |W| \geq 2$$

at the same time. Hence

$$d_{|A|+|U|}(a,b) + d_{|B|+|U|+|W|}(b,d) + d_{|C|+|W|}(d,a) \leq 2.$$

Similarly, it is not possible for

$$|C| + |W| + |B| \geq 2, |A| + |C| \geq 3, \text{ and } |A| + |B| + |U| \geq 2$$

at the same time. Hence

$$d_{|C|+|W|+|B|}(b,a) + d_{|A|+|C|}(d,b) + d_{|A|+|B|+|U|}(a,d) \leq 2.$$

(C3) Consider the triple $\{a, e, f\}$. In (3) set $x = a, y = e, z = f$. Given the orderings of $a, e, f$ in $\overline{\Omega}$ it follows that $C = W = \emptyset$. Since there are only three agents, it is not possible for

$$|A| + |U| + |V| \geq 1, |B| + |U| \geq 2, \text{ and } |V| \geq 3$$

at the same time. Hence

$$d_{|A|+|U|+|V|}(a,e) + d_{|B|+|U|}(e,f) + d_{|V|}(f,a) \leq 2.$$

Similarly, it is not possible for

$$|B| \geq 3, |A| + |V| \geq 2, \text{ and } |A| + |B| + |U| \geq 1$$

at the same time. Hence

$$d_{|B|}(e,a) + d_{|A|+|V|}(f,e) + d_{|A|+|B|+|U|}(a,f) \leq 2.$$

(C4) Consider the triple {a,d, e}. In (3) set $x = a, y = d, z = e$. Given the orderings of $a, d, e$ in $\overline{\Omega}$ it follows that $C = W = \emptyset$. Since there are only three agents, it is not possible for

$$|A| + |U| + |V| \geq 2, |B| + |U| \geq 2, \text{ and } |V| \geq 3$$

at the same time. Hence

$$d_{|A|+|U|+|V|}(a, d) + d_{|B|+|U|}(d, e) + d_{|V|}(e, a) \leq 2.$$

Similarly, it is not possible for

$$|B| \geq 2, |A| + |V| \geq 2, \text{ and } |A| + |B| + |U| \geq 1$$

at the same time. Hence

$$d_{|B|}(d, a) + d_{|A|+|V|}(e, d) + d_{|A|+|B|+|U|}(a, e) \leq 2.$$

By an exhaustive argument, we conclude that the $\mathcal{F}$ constructed using $D^1$ and $D^2$ is an AMASWF. ∎

Any $2n$-person AMASWF can be used to construct a 2-person solution: consider the restricted set of profiles in which agents $2, 3, \ldots, n$ have the same preferences as agent 1, and agents $n + 2, n + 2, \ldots, 2n$ have the same preferences as agent $n + 1$. Therefore, the domain $\overline{\Omega}$ constitutes a counterexample to the claim made by Muller [11].

### 5.3 Discussion

Example 3 highlights a key difficulty in the study of AMASWFs: the existence of such rules depends *both* on the domain and on the number of agents. (This is true even if social indifference is permitted.) It is natural to ask if there exists a number $\hat{n}$ such that the existence of an $\hat{n}$-person AMASWF will guarantee the existence of an AMASWF for all $n > \hat{n}$. Muller [11] showed that $\hat{n} > 2$, and our example shows that $\hat{n} > 3$. We next argue that such an $\hat{n}$ does not exist for $\hat{n}$ odd.

**Theorem 3** *Let $\Omega$ be a domain that admits a 3-person anonymous and monotonic ASWF $d'$. If the set $D^1[d']$ is transitive, then $\Omega$ admits a $2k + 1$-person anonymous, monotonic, ASWF for all $k > 1$.*

**Remark.** Theorem 3 generalizes the following well-known result: If the majority rule (i.e.,$D^1 = \emptyset$) is a social welfare function on a domain for 3 agents, then the majority rule is a social welfare function on that domain for any odd number of agents.

**Proof**: Let $d'$ be the given 3-person AMASWF on $\Omega$. We construct a $2k + 1$ rule, $d$ on $\Omega$ as follows: $d_r = d'_1$ for $r \leq k$ and $d_r = d'_2$ for $r \geq k + 1$. The rule $d$ is clearly anonymous and monotonic. We prove that it is an ASWF by showing that if $d$ violates (3) then $d'$ must also violate (3), contradicting the fact that $d'$ is an ASWF on $\Omega$. Suppose then a violation of (3) on the triple $x, y, z$. Let $A$, $B$, $C$, $U$, $V$, and $W$ (possibly empty) be the associated partition of the $2k+1$ agents. Violation of (3) implies that

$$d_{|A|+|U|+|V|}(x,y) + d_{|B|+|U|+|W|}(y,z) + d_{|C|+|V|+|W|}(z,x) \;=\; 3,$$

and therefore

$$d_{|A|+|U|+|V|}(x,y) = d_{|B|+|U|+|W|}(y,z) = d_{|C|+|V|+|W|}(z,x) \;=\; 1.$$

Case 1: All of $|A|+|U|+|V|$, $|B|+|U|+|W|$ and $|C|+|V|+|W|$ are at least $k+1$. We conclude from the definition of $d$ that $d'_2(x,y) = d'_2(y,z) = d'_2(z,x) = 1$. Next, from the hypothesis of the case each pair of sets from the collection $\{A \cup U \cup V,\ B \cup U \cup W,\ C \cup V \cup W\}$ has a non-empty intersection, and so $U, V, W \neq \emptyset$; this implies we can find the orderings $xyz$, $yzx$ and $zxy$ in $\Omega$. Now consider the 3-person rule on the preference profile $(xyz, yzx, zxy)$. Observe that $2 \geq d'_2(x,y) + d'_2(y,z) + d'_2(z,x) = 3$, a contradiction.

Case 2: Exactly two of $|A|+|U|+|V|$, $|B|+|U|+|W|$ and $|C|+|V|+|W|$ are at least $k+1$. Without loss of generality, suppose that $|A|+|U|+|V|, |B|+|U|+|W| \geq k+1$ and $|C|+|V|+|W| \leq k$. We conclude from the definition of $d$ that $d'_2(x,y) = d'_2(y,z) = 1$ and $d'_1(z,x) = 1$. Since $\{A \cup U \cup V\} \cap \{B \cup U \cup W\} \neq \emptyset$ it follows that $U \neq \emptyset$. Thus the ordering $xyz$, is present in $\Omega$. Since $\{C \cup V \cup W\} \neq \emptyset$, the ordering $xz$ is present in $\Omega$. Therefore $\Omega$ contains an ordering of $\{x, y, z\}$ (different from $xyz$) in which $z$ is ranked above $x$. It is clear that $d'$ violates (3) on any preference profile in which two agents rank $xyz$ and the third has this other ranking ($z$ above $x$).

Case 3: Exactly one of $|A|+|U|+|V|$, $|B|+|U|+|W|$ and $|C|+|V|+|W|$ is at least $k+1$. Without loss of generality suppose that $|B|+|U|+|W|, |C|+|V|+|W| \;\leq\; k$. Now $d_{|A|+|U|+|V|}(x,y) = 1$ implies $d'_2(x,y) = 1$. Similarly $d'_1(y,z) = d'_1(z,x) = 1$. However transitivity of $D^1[d']$ implies $d'_1(y,x) = 1$, contradicting the fact that $d'_2(x,y) = 1$.

Case 4: None of $|A|+|U|+|V|$, $|B|+|U|+|W|$ and $|C|+|V|+|W|$ is at least $k+1$. In this case, violation of (3) by $d$ implies that $d'_1(x,y) = d'_1(y,z) = d'_1(z,x) = 1$. First suppose that $A$, $B$, and $C$ are all non-empty; in this case, $\Omega$ must contain the orderings $xzy, yxz, zyx$. The given $d'$ violates (3) on the 3-agent profile $(xzy, yxz, zyx)$, a contradiction. So, we may suppose at least one of $A$, $B$ and $C$ is empty. Without loss of generality, suppose $C = \emptyset$. Then,

$$2k+1 \;=\; |A|+|U|+|V|+|W|+|B| \;\leq\; k+|W|+|B| \;\leq\; 2k,$$

a contradiction.

∎

Note that Example 3 discussed earlier satisfies the hypothesis of Theorem 3. In particular, the domain $\overline{\Omega}$ defined there admits an AMASWF whenever the number of agents is odd, and does not admit such a solution whenever the number of agents is even. The next result provides an easy sufficient condition in terms of a 2-person solution. This result is essentially due to Muller [11], but we state and prove it here for the sake of completeness.

**Theorem 4** *Let $\Omega$ be a domain that admits a 2-person anonymous and monotonic ASWF $d'$. If $D^1[d']$ is transitive, then $\Omega$ admits an anonymous, monotonic, ASWF for all $n > 2$.*

**Proof**: Let $d'$ be the 2-person AMASWF on $\Omega$. We define an $n$-person rule $d$ such that $d_j(x, y) = d'_1(x, y)$ for all $x$ and $y$ and $j \leq n - 1$. This rule is clearly anonymous and monotonic; we next show that it is an ASWF. Note that the set $D^1[d']$ is transitive and contains exactly half the pairs. Suppose $d$ violates (3) on the triple $\{x, y, z\}$ and let $A, B, C, U, V, W$ be the associated partition. Violation of (3) implies that $d_{|A|+|U|+V|}(x, y) = d_{|B|+|U|+|W|}(y, z) = d_{|C|+|V|+|W|}(z, x) = 1$. Suppose that $|A| + |U| + |V|$, $|B| + |U| + |W|$ and $|C| + |V| + |W|$ are all at most $n - 1$. Then $d'_1(x, y) = d'_1(y, z) = d'_1(z, x) = 1$ which violates the transitivity assumption. Hence at least one of $|A| + |U| + |V|$, $|B| + |U| + |W|$ or $|C| + |V| + |W|$ is $n$. In fact, at most one of them can have value $n$, for otherwise, all the agents must have the same ranking, yielding a unanimous preference profile in which a violation of (3) cannot occur. Suppose $|A| + |U| + |V| = n$. Since $|B| + |U| + |W|$ and $|C| + |V| + |W|$ have value at most $n - 1$ it follows that $d'_1(y, z) = d'_1(z, x) = 1$. By transitivity, $d'_1(y, x) = 1$ and $d'_1(x, y) = 0$. Suppose both $U$ and $V$ are non-empty. Then, the given rule $d'$ cycles on the 2-person profile $(zxy, xyz)$, a contradiction. However, if either of $U$ or $V$ is empty, a violation of (3) cannot occur. ∎

**Remark.** The social welfare function constructed in the proof of Theorem 4 may be viewed as a *status-quo* rule: The transitive ordering, $\sigma$, consistent with $D^1$ may be regarded as the status-quo, which can be overturned only by unanimity. Such a $D^1$ is termed a *transitive sink* in Muller [11]. In a recent paper, Dasgupta and Maskin [3, pg. 30] call such a rule a *unanimity rule with order of preference* $\sigma$. In fact, our Theorem 4 is closely related to Dasgupta and Maskin [3, Theorem 12]. In their work, Dasgupta and Maskin show that any domain admitting an ASWF satisfying anonymity and *tie-breaking consistency* also admits a *status-quo* rule as an ASWF. (Such rules are clearly anonymous and monotonic.) When all preferences are strict, their *tie-breaking consistency* condition is equivalent to requiring a *transitive* $D^1$. Dasgupta and Maskin do not assume monotonicity; it is clear that our proof of Theorem 4 does not use monotonicity either. Monotonicity has no bite because all 2 person ASWFs are monotonic anyway.

Consider an arbitrary domain $\Omega$. If $\Omega$ does not admit a 2-person AMASWF, it cannot admit a $2k$-person AMASWF for any $k > 1$. This observation, along with Theorem 4, helps us identify the "difficult" domains: those $\Omega$ for which *every* 2-person AMASWF $d$ has an *intransitive* $D^1[d]$. Muller [11] gives an example of such a domain; in his example, the domain admits a unique 2-person anonymous ASWF (all 2-person ASWFs are monotonic because of unanimity) for which $D^1$ happens to be intransitive. Muller's example does not admit a 3-person solution (hence cannot admit a 6-person solution), but admits a 4-person solution. Furthermore, he (implicitly) claimed that if $\Omega$ admits an AMASWF for all $n \geq 3$, then it must necessarily admit a 2-person solution $d$ with a transitive $D^1[d]$; his proof of this result relied on Claim 1, which, unfortunately, is not true. The result (stated next as Muller's conjecture), however, may well be true; moreover, if true, it provides a complete characterization of domains that admit an AMASWF, regardless of the number of agents.

**Conjecture 1** *(Muller) If $\Omega$ admits an anonymous, monotonic, ASWF for all $n \geq 3$, then it must necessarily admit a 2-person solution $d$ with a transitive $D^1[d]$.*

As far as we know, Conjecture 1 is still unresolved; note that its converse is stated and proved as Theorem 4. Conjecture 1 is useful because it characterizes domains that admit an AMASWF regardless of the number of agents. We conclude this section by proving the following related result.

**Theorem 5** *Suppose $\Omega$ admits an anonymous, monotonic, 3-person ASWF $d'$, with $D^1[d']$ transitive. Suppose also that $\Omega$ admits a 2-person solution $\hat{d}$ with a (possibly intransitive) $D^1[\hat{d}]$ that contains $D^1[d']$. Then $\Omega$ admits an anonymous, monotonic ASWF for all $n > 3$.*

**Proof.** For odd $n$, an AMASWF exists by Theorem 3, so we assume $n = 2k$, for $k > 1$. We know that $D^1[\hat{d}]$ is transitive, and that $(x, y) \in D^2[d']$ if and only if $(y, x) \notin D^1[d']$. Also, we know that $D^1[d'] \subseteq D^1[\hat{d}]$. Consider the $2k$-person rule $d$ with the decisive sets

$$D^1[d] = \ldots = D^{k-1}[d] = D_1[d'], \quad D^k = D^1[\hat{d}], \quad D^{k+1}[d] = \ldots = D^{2k-1}[d] = D^2[d'].$$

The rule $d$ is clearly anonymous; to show that it is monotonic, we must show that $D^1[\hat{d}] \subseteq D^2[d']$. Consider any $(x, y) \in D^1[\hat{d}]$. Then, $(y, x) \notin D^1[\hat{d}]$, which implies $(y, x) \notin D^1[\hat{d}]$, which implies $(x, y) \in D^2[d']$. We next show that $d$ is transitive. The proof is similar to the proof of Theorem 3. Referring to that proof we see that it covers all cases except when at least one of $|A| + |U| + |V|$, $|B| + |U| + |W|$ or $|C| + |V| + |W|$ is equal to $k$. Without loss of generality, we assume throughout that $|A| + |U| + |V| = k$.

Case 1: Both $|B| + |U| + |W|, |C| + |V| + |W| \geq k + 1$ In this case $U, V, W \neq \emptyset$, so, $\Omega$ must contain the Condorcet triple $xyz, zxy, yzx$. As in the proof of Case 1 of Theorem 3, the given rule $d'$ will violate (3) on this triple.

Case 2: Exactly one of $|B| + |U| + |W|, |C| + |V| + |W| \geq k + 1$ and the other is at most $k - 1$ Without loss of generality, let $|B| + |U| + |W| \leq k - 1$. Observe first that $A \cup U \cup V$ and $C \cup V \cup W$ have a non-empty intersection. Thus $V \neq \emptyset$. Now $B \cup U \cup W$ must have non-empty intersection with either $A \cup U \cup V$ or $C \cup V \cup W$. Suppose the first, a similar argument applies in the second case. Then $U \neq \emptyset$. Since $U$ and $V$ are non-empty, we know that $\Omega$ contains the orderings $xyz$ and $zxy$. Violation of (3) by $d$ implies that $d'_1(y,z) = 1$ and $d'_2(z,x) = 1$. Consider now a 3-person profile where two agents have the ordering $zxy$ and the other $xyz$. It is easy to see that $d'$ cycles on this profile, a contradiction.

Case 3: Exactly one of $|B| + |U| + |W|, |C| + |V| + |W| \geq k+1$ and the other is exactly $k$ Without loss of generality, suppose $|B|+|U|+|W| = k, |C|+|V|+|W| \geq k+1$. In this case $A \cup U \cup V$ and $B \cup U \cup W$ have a non-empty intersection with $C \cup V \cup W$. Thus $V$ and $W$ are non-empty, i.e., $\Omega$ contains the orderings $zxy$ and $yzx$. Violation of (3) by $d$ implies that $\hat{d}_1(x,y) = \hat{d}_1(y,z) = 1$. Apply $\hat{d}$ to the two-agent profile consisting of $zxy$ and $yzx$ and a cycle results, a contradiction.

Case 4: $|B| + |U| + |W| = k, |C| + |V| + |W| = k$ In this case one of the three sets $A \cup U \cup V$, $B \cup U \cup W$, $C \cup V \cup W$ has a non-empty intersection with the other two. Without loss of generality, suppose that it is $B \cup U \cup W$. Then $U$ and $W$ are non-empty, and the argument follows Case 3.

Case 5: $|B| + |U| + |W|, |C| + |V| + |W| \leq k-1$ Violation of (3) by $d$ implies that $\hat{d}_1(x,y) = d'_1(y,z) = d'_1(z,x) = 1$. Transitivity of $D^1[d']$ implies $d'_1(y,x) = 1$. Since $D^1[d'] \subseteq D^1[\hat{d}]$ it follows that $\hat{d}_1(x,y) = 0$ a contradiction.

■

## 6 Conclusion

We show in this paper that the existence of an anonymous, monotonic ASWF function depends on both the domain and on the number of the agents. We also generalize Moulin's characterization of anonymous, monotonic ASWFs on the domain of single-peaked preferences. In addition to the obvious open problems such as characterizing domains admitting AMASWFs and characterizing all AMASWFs of such domains, our investigation suggests three problems:

- What are the conditions on $\Omega$ such that $dim(P(D^r)) \leq 2$ for all $r \leq (n-1)/2$? i.e. is there a more general condition on the domain that will ensure that all anonymous monotonic social welfare functions are GMR?

- The examples in this paper were constructed using the crown graph, which is known to be the smallest poset with dimension 3. Using posets with higher dimension, say $k$, can one construct a domain such that $k$-person AMASWF exists but not $l$-person AMASWF, for any $l < k$?

- What is the connection between the directed graphs $D^r$ obtained from the social welfare function and the domain $\Omega$? In general, given $D^r$, is there a polynomial time algorithm to construct the corresponding $\Omega$, or prove that no domain corresponding to $D^r$ exists? Note that finding a linear order given conditions on triples ordering is already in general NP-Complete (cf. Opatrny [12]).

We leave these issues for future research.

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