Integer Programming and Arrovian Social Welfare Functions

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Abstract. We formulate the problem of deciding which preference domains admit a non-dictatorial Arrovian Social Welfare Function as one of verifying the feasibility of an integer linear program. Many of the known results about the presence or absence of Arrovian social welfare functions, impossibility theorems in social choice theory, and properties of majority rule etc., can be derived in a simple and unified way from this integer program. We characterize those preference domains that admit a non-dictatorial, neutral Arrovian social welfare Function and give a polyhedral characterization of Arrovian social welfare functions on single-peaked domains.

1 Introduction

The Old Testament likens the generations of men to the leaves of a tree. It is a simile that applies as aptly to the literature inspired by Arrow's impossibility theorem [2]. Much of it is devoted to classifying those preference domains that admit or exclude the existence of a non-dictatorial Arrovian social welfare function (ASWF)[1]. We add another leaf to that tree. Specifically, we formulate the problem of deciding which preference domains admit a non-dictatorial Arrovian social welfare function as one of verifying the feasibility of an integer linear program. Many of the known results about the presence or absence of Arrovian social welfare functions, impossibility theorems in social choice theory, properties of the majority rule etc., can be derived in a simple and unified way from this integer program. The integer program also leads to some interesting new results such as (a) a characterization of preference domains that admit a non-dictatorial, neutral Arrovian social welfare function; and (b) a polyhedral characterization of Arrovian social welfare Functions on single-peaked domains.

[1] An ASWF is a social welfare function that satisfies the axioms of the Impossibility theorem.

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Let $\mathcal{A}$ denote the set of alternatives (at least three). Let Σ denote the set of all transitive, antisymmetric and total binary relations on $\mathcal{A}$. An element of Σ is a preference ordering. The set of admissible preference orderings for members of a society of n-agents (voters) will be a subset of Σ and denoted Ω. Let Ω^n be the set of all n-tuples of preferences from Ω, called *profiles*. An element of Ω^n will typically be denoted as $\mathbf{P} = (\mathbf{p_1}, \mathbf{p_2}, \ldots, \mathbf{p_n})$, where $\mathbf{p_i}$ is interpreted as the preference ordering of agent i. (In the language of Le Breton and Weymark [7], we assume the "common preference domain" framework; this assumption can be relaxed, see Sect. 2.) An n-person social welfare function is a function $f : \Omega^n \to \Sigma$. Thus for any $\mathbf{P} \in \Omega^n$, $f(\mathbf{P})$ is an ordering of the alternatives. We write $xf(\mathbf{P})y$ if x is ranked above y under $f(\mathbf{P})$. An n-person *Arrovian social welfare function* (ASWF) on Ω is a function $f : \Omega^n \to \Sigma$ that satisfies the following two conditions:

1. **Unanimity:** If for $\mathbf{P} \in \Omega^n$ and some $x, y \in \mathcal{A}$ we have $x\mathbf{p_i}y$ for all i then $xf(\mathbf{P})y$.
2. **Independence of Irrelevant Alternatives:** For any $x, y \in \mathcal{A}$ suppose $\exists \mathbf{P}, \mathbf{Q} \in \Omega^n$ such that $x\mathbf{p_i}y$ if an only if $x\mathbf{q_i}y$ for $i = 1, \ldots, n$. Then $xf(\mathbf{P})y$ if an only if $xf(\mathbf{Q})y$.

The first axiom stipulates that if all voters prefer alternative x to alternative y, then the social welfare function f must rank x above y. The second axiom states that the ranking of x and y in f is not affected by how the voters rank the other alternatives. An obvious social welfare function that satisfies the two conditions is the *dictatorial rule*: rank the alternatives in the order of the preferences of a particular voter (the dictator). Formally, an ASWF is *dictatorial* if there is an i such that $f(\mathbf{P}) = \mathbf{p_i}$ for all $\mathbf{P} \in \Omega^n$. An ordered pair $x, y \in \mathcal{A}$ is called *trivial* if $x\mathbf{p}y$ for all $\mathbf{p} \in \Omega$. In view of unanimity, any ASWF must have $xf(\mathbf{P})y$ for all $\mathbf{P} \in \Omega^n$ whenever x, y is a trivial pair. If Ω consists only of trivial pairs then distinguishing between dictatorial and non-dictatorial ASWF's becomes nonsensical, so we assume that Ω contains at least one non-trivial pair. The domain Ω is *Arrovian* if it admits a non-dictatorial ASWF.

The main contributions of this paper are summarized below.

- We provide an integer linear programming formulation of the problem of finding an n-person ASWF. For each Ω we construct a set of linear inequalities with the property that every feasible 0-1 solution corresponds to an n-person ASWF.
- When restricted to the class of neutral ASWF's the integer program yields a simple and easily checkable characterization of domains that admit neutral, non-dictatorial ASWF's. This result contains as a special case the results of Sen [14] and Maskin [8] about the robustness of the majority rule.
- For the case when Ω is single-peaked, we show that the polytope defined by the set of linear inequalities is *integral*: the vertices of the polytope correspond to ASWF's and every ASWF corresponds to a vertex of the polytope. This gives the first characterization of ASWF's on this domain. The same proof technique yields a characterization of the generalized majority rule on single peaked domains, originally due to Moulin [10].

- We show that the computational complexity of deciding whether a domain is Arrovian depends critically on the way the domain is described. We propose a graph-theoretical method to identify stronger linear inequalities for ASWF's. For cases with a small number of alternatives (3 or 4), our approach is able to characterize the polytope of all ASWF's. Thus for *any* Ω and any set of alternatives size at most 4 we characterize the polyhedral structure of all ASWF's.

2 The Integer Program

Denote the set of all ordered pairs of alternatives by $\mathcal{A}^2$. Let E denote the set of all agents, and S^c denote $E \setminus S$ for all $S \subseteq E$.

To construct an n-person ASWF we exploit the independence of irrelevant alternatives condition. This allows us to specify an ASWF in terms of which ordered pair of alternatives a particular subset, S, of agents is decisive over.

Definition 1. *For a given ASWF f, a subset S of agents is* weakly decisive for x over y *if whenever all agents in S rank x over y and all agents in S^c rank y over x, the ASWF f ranks x over y.*

Since this is the only notion of decisiveness used in the paper, we omit the qualifier 'weak' in what follows.

For each non-trivial element $(x, y) \in \mathcal{A}^2$, we define a 0-1 variable as follows:

$$d_S(x,y) = \begin{cases} 1, \text{ if the subset } S \text{ of agents is decisive for } x \text{ over } y; \\ 0, \text{ otherwise.} \end{cases}$$

If $(x, y) \in \mathcal{A}^2$ is a trivial pair then by default we set $d_S(x, y) = 1$ for all $S \neq \emptyset$.

Given an ASWF f, we can determine the associated d variables as follows: for each $S \subseteq E$, and each non-trivial pair (x, y), pick a $\mathbf{P} \in \Omega^n$ in which agents in S rank x over y, and agents in S^c rank y over x; if $xf(\mathbf{P})y$, set $d_S(x, y) = 1$, else set $d_S(x, y) = 0$.

In the rest of this section, we identify some conditions satisfied by the d variables associated with an ASWF f.

Unanimity: To ensure unanimity, for all $(x, y) \in \mathcal{A}^2$, we must have

$$d_E(x, y) = 1. \tag{1}$$

Independence of Irrelevant Alternatives: Consider a pair of alternatives $(x, y) \in \mathcal{A}^2$, a $\mathbf{P} \in \Omega^n$, and let S be the set of agents that prefer x to y in $\mathbf{P}$. (Thus, each agent in S^c prefers y to x in $\mathbf{P}$.) Suppose $xf(\mathbf{P})y$. Let $\mathbf{Q}$ be any other profile such that all agents in S rank x over y and all agents in S^c rank y over x. By the independence of irrelevant alternatives condition $xf(\mathbf{Q})y$. Hence the set S is decisive for x over y. However, had $yf(\mathbf{P})x$ a similar argument would imply that S^c is decisive for y over x. Thus, for all S and $(x, y) \in \mathcal{A}^2$, we must have

$$d_S(x, y) + d_{S^c}(y, x) = 1. \tag{2}$$

A consequence of Eqs. (1) and (2) is that $d_{\emptyset}(x,y) = 0$ for all $(x,y) \in \mathcal{A}^2$.

Transitivity: To motivate the next class of constraints, it is useful to consider the majority rule. If the number n of agents is odd, the majority rule can be described using the following variables:

$$d_S(x,y) = \begin{cases} 1, & \text{if} \quad |S| > n/2, \\ 0, & \text{otherwise.} \end{cases}$$

These variables satisfy both (1) and (2). However, if Ω admits a *Condorcet* triple (e.g., $\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3} \in \Omega$ with $x\mathbf{p_1}y\mathbf{p_1}z$, $y\mathbf{p_2}z\mathbf{p_2}x$, and $z\mathbf{p_3}x\mathbf{p_3}y$), then such a rule does not always produce an *ordering* of the alternatives for each preference profile. Our next constraint (*cycle elimination*) is designed to exclude this and similar possibilities.

Let A, B, C, U, V, and W be (possibly empty) *disjoint* sets of agents whose union includes all agents. For each such partition of the agents, and any triple x, y, z,

$$d_{A \cup U \cup V}(x,y) + d_{B \cup U \cup W}(y,z) + d_{C \cup V \cup W}(z,x) \leq 2, \tag{3}$$

where the sets satisfy the following conditions (hereafter referred to as conditions (*)):

$$\begin{aligned}
A \neq \emptyset &\text{ only if there exists } \mathbf{p} \in \Omega, x\mathbf{p}z\mathbf{p}y,\\
B \neq \emptyset &\text{ only if there exists } \mathbf{p} \in \Omega, y\mathbf{p}x\mathbf{p}z,\\
C \neq \emptyset &\text{ only if there exists } \mathbf{p} \in \Omega, z\mathbf{p}y\mathbf{p}x,\\
U \neq \emptyset &\text{ only if there exists } \mathbf{p} \in \Omega, x\mathbf{p}y\mathbf{p}z,\\
V \neq \emptyset &\text{ only if there exists } \mathbf{p} \in \Omega, z\mathbf{p}x\mathbf{p}y,\\
W \neq \emptyset &\text{ only if there exists } \mathbf{p} \in \Omega, y\mathbf{p}z\mathbf{p}x.
\end{aligned}$$

The constraint ensures that on any profile $\mathbf{P} \in \Omega^n$, the ASWF f does not produce a ranking that "cycles".

Theorem 1. *Every feasible integer solution to (1)-(3) corresponds to an ASWF and vice-versa.*

Proof. Given an ASWF, it is easy to see that the corresponding d vector satisfies (1)-(3). Now pick any feasible solution to (1)-(3) and call it d. To prove that d gives rise to an ASWF, we show that for every profile of preferences from Ω, d generates an ordering of the alternatives. Unanimity and Independence of Irrelevant Alternatives follow automatically from the way the d_S variables are used to construct the ordering.

Suppose d does not produce an ordering of the alternatives. Then, for some profile $\mathbf{P} \in \Omega^n$, there are three alternatives x, y and z such that d ranks x over y, y over z and z over x. For this to happen there must be three non-empty sets H, I, and J such that

$$d_H(x,y) = 1, \; d_I(y,z) = 1, \; d_J(z,x) = 1,$$

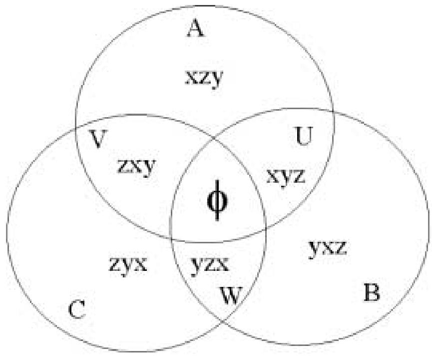

Fig. 1. The sets and the associated orderings

and for the profile $\mathbf{P}$, agent i ranks x over y (resp. y over z, z over x) if and only if i is in H (resp. I, J). Note that $H \cup I \cup J$ is the set of all agents, and $H \cap I \cap J = \emptyset$.

Let

$$A \leftarrow H \setminus (I \cup J), B \leftarrow I \setminus (H \cup J), C \leftarrow J \setminus (H \cup I),$$
$$U \leftarrow H \cap I, V \leftarrow H \cap J, W \leftarrow I \cap J.$$

Now A (resp. B, C, U, V, W) can only be non-empty if there exists $\mathbf{p}$ in Ω with $x\mathbf{p}z\mathbf{p}y$ (resp. $y\mathbf{p}x\mathbf{p}z$, $z\mathbf{p}y\mathbf{p}x$, $x\mathbf{p}y\mathbf{p}z$, $z\mathbf{p}x\mathbf{p}y$, $y\mathbf{p}z\mathbf{p}x$).

In this case constraint (3) is violated since

$$d_{A\cup U\cup V}(x,y)+d_{B\cup U\cup W}(y,z)+d_{C\cup V\cup W}(z,x) = d_H(x,y)+d_I(y,z)+d_J(z,x) = 3.$$

□

For the case $n = 2$, constraint (3) can be simplified as follows: (i) if for some $\mathbf{p}, \mathbf{q} \in \Omega$ and $x, y, z \in \mathcal{A}$, we have $x\mathbf{p}y\mathbf{p}z$ and $y\mathbf{q}z\mathbf{q}x$, then

$$d_S(x,y) \leq d_S(x,z), \quad (4)$$
$$d_S(z,x) \leq d_S(y,x); \quad (5)$$

and (ii) if for some $\mathbf{p} \in \Omega$ and $x, y, z \in \mathcal{A}$, we have $x\mathbf{p}y\mathbf{p}z$, then

$$d_S(x,y) + d_S(y,z) \leq 1 + d_S(x,z), \quad (6)$$
$$d_S(z,y) + d_S(y,x) \geq d_S(z,x). \quad (7)$$

These inequalities, discovered earlier by Kalai and Muller [5], are called *decisiveness implications*. Thus, Constraints (3) generalize the decisiveness implication conditions to $n \geq 3$. We will sometimes refer to (1)-(3) as IP.

General Domains. The IP characterization obtained above can be generalized to the case in which the domain of preferences for each voter is non-identical. In general, let D be the domain of profiles over alternatives. In this case, for each set S, the d_S variables need not be well-defined for each pair of alternatives x, y, if there is no profile in which all agents in S (resp. S^c) rank x over y (resp. y over x). d_S is thus only defined for (x, y) if such profiles exist. Note that $d_S(x, y)$ is well-defined if and only if $d_{S^c}(y, x)$ is well-defined. With this proviso inequalities (1) and (2) remains valid. We only need to modify (3) to the following:

Let A, B, C, U, V, and W be (possibly empty) *disjoint* sets of agents whose union includes all agents. For each such partition of the agents, and any triple x, y, z,

$$d_{A \cup U \cup V}(x, y) + d_{B \cup U \cup W}(y, z) + d_{C \cup V \cup W}(z, x) \leq 2, \tag{8}$$

where the sets satisfy the following conditions (hereafter referred to as condition (**)):

$$\begin{aligned} A \neq \emptyset &\text{ only if there exists } \mathbf{p_i}, i \in A, \text{ with } x\mathbf{p_i}z\mathbf{p_i}y, \\ B \neq \emptyset &\text{ only if there exists } \mathbf{p_i}, i \in B, \text{ with } y\mathbf{p_i}x\mathbf{p_i}z, \\ C \neq \emptyset &\text{ only if there exists } \mathbf{p_i}, i \in C, \text{ with } z\mathbf{p_i}y\mathbf{p_i}x, \\ U \neq \emptyset &\text{ only if there exists } \mathbf{p_i}, i \in U, \text{ with } x\mathbf{p_i}y\mathbf{p_i}z, \\ V \neq \emptyset &\text{ only if there exists } \mathbf{p_i}, i \in V, \text{ with } z\mathbf{p_i}x\mathbf{p_i}y, \\ W \neq \emptyset &\text{ only if there exists } \mathbf{p_i}, i \in W, \text{ with } y\mathbf{p_i}z\mathbf{p_i}x. \\ &\text{and } (\mathbf{p_1}, \ldots, \mathbf{p_n}) \in D. \end{aligned}$$

The following theorem is immediate from our discussion. We omit the proof.

Theorem 2. *Every feasible integer solution to (1), (2) and (8) corresponds to an ASWF on domain D and vice-versa.*

This yields a new characterization of non-dictatorial profile domains D, and can be used to obtain a simple proof of a result due to Fishburn and Kelly [4] on super non-Arrovian domains; we state this result without proof.

A domain D is called *super non-Arrovian* if it is non-Arrovian and every domain D' containing D is also non-Arrovian. Furthermore, if d_S is well defined for every pair of alternatives x, y and every S, we say that the domain D satisfies the *near-free doubles* condition.

Theorem 3 (Fishburn and Kelly [4]). *A domain D is super-non-Arrovian if and only if it is non-Arrovian and satisfies the near-free doubles condition.*

3 Applications

Arrow's Theorem. Our first use of IP is to provide a simple proof of Arrow's theorem.

Theorem 4 (Arrow's Impossibility theorem). *When $\Omega = \Sigma$, the 0-1 solutions to the IP correspond to dictatorial rules.*

Proof: When $\Omega = \Sigma$, we know from constraints (4-5) and the existence of all possible triples that $d_S(x,y) = d_S(y,z) = d_S(z,u)$ for all alternatives x, y, z, u. We will thus write d_S in place of $d_S(x,y)$ in the rest of the proof.

We show first that $d_S = 1 \Rightarrow d_T = 1$ for all $S \subset T$. Suppose not. Let T be the set containing S with $d_T = 0$. Constraint (2) implies $d_{T^c} = 1$. Choose $A = T \setminus S$, $U = T^c$ and $V = S$ in (3). Then, $d_{A \cup U \cup V} = d_E = 1$, $d_{B \cup U \cup W} = d_{T^c} = 1$ and $d_{C \cup V \cup W} = d_S = 1$, which contradicts (3).

The same argument implies that $d_T = 0 \Rightarrow d_S = 0$ whenever $S \subset T$. Note also that if $d_S = d_T = 1$, then $S \cap T \neq \emptyset$, otherwise the assignment $A = (S \cup T)^c, U = S, V = T$ will violate the cycle elimination constraint. Furthermore, $d_{S \cap T} = 1$, otherwise the assignment $A = (S \cup T)^c, U = T \setminus S, V = S \setminus T, W = S \cap T$ will violate the cycle elimination constraint. Hence there exists a minimal set S^* with $d_{S^*} = 1$ such that all T with $d_T = 1$ contains S^*. We show that $|S^*| = 1$. If not there will be $j \in S$ with $d_j = 0$, which by (2) implies $d_{E \setminus \{j\}} = 1$. Since $d_{S^*} = 1$ and $d_{E \setminus \{j\}} = 1$, $d_{E \setminus \{j\} \cap S^*} \equiv d_{S^* \setminus \{j\}} = 1$, contradicting the minimality of S^*. □

Born Loser rule. For subsequent applications we introduce the *born loser* rule. For each j, we define the *born loser* rule with respect to j (denoted by B_j) in the following way: (i) set $d_E^{B_j}(x,y) = 1$ for every $x, y \in \mathcal{A}^2$; (ii) set $d_\emptyset^{B_j}(x,y) = 0$ for every $x, y \in \mathcal{A}^2$; and (iii) for every non-trivial pair (x,y), and for any $S \neq \emptyset, E$, $d_S^{B_j}(x,y) = 0$ if $S \ni j$, $d_S^{B_j}(x,y) = 1$ otherwise.

Theorem 5. *For any j and $n > 2$, the born loser rule B_j is a non-dictatorial n-person ASWF if and only if for all x, y, z, there do not exist $\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}$ in Ω with*

$$x\mathbf{p_1}z\mathbf{p_1}y,\ x\mathbf{p_2}y\mathbf{p_2}z,\ z\mathbf{p_3}x\mathbf{p_3}y,$$

Proof. It is clear that by definition, d^{B_j} satisfies (1, 2). To see that it satisfies (3), observe that in every partition of the agents, one of the sets obtained must contain j. Say $j \in A \cup U \cup V$. If $d_{A \cup U \cup V}^{B_j}(x,y) = 0$, then (3) is clearly valid. So we may assume that $d_{A \cup U \cup V}^{B_j}(x,y) = 1$. This happens only when $A \cup U \cup V = E$ (or if (x,y) is trivial, which in turns imply that all the other sets are empty). We may assume $U, V \neq \emptyset$ and $j \in A$, otherwise (3) is clearly valid. But according to condition (*), this implies existence of $\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}$ in Ω with

$$x\mathbf{p_1}z\mathbf{p_1}y,\ x\mathbf{p_2}y\mathbf{p_2}z,\ z\mathbf{p_3}x\mathbf{p_3}y,$$

which is a contradiction.

So, d^{B_j} satisfies (1-3) and hence corresponds to an ASWF. When $n > 2$, B_j is clearly non-dictatorial. □

Anonymous and Neutral Rules. Two additional conditions that are sometimes imposed on an ASWF are anonymity and neutrality. An ASWF is called *anonymous* if its ranking over pairs of alternatives remains unchanged when the labels of the agents are permuted. Hence $d_S(x,y) = d_T(x,y)$ for all $(x,y) \in \mathcal{A}^2$

whenever $|S| = |T|$. In particular a dictatorial rule is not anonymous. An ASWF is called *neutral* if its ranking over any pair of alternatives depends only on the pattern of agents' preferences over that pair, not on the alternatives' labels. Neutrality implies that $d_S(x, y) = d_S(a, b)$ for any $(x, y), (a, b) \in \mathcal{A}^2$. Thus the value of $d_S(\cdot, \cdot)$ is determined by S alone. When anonymity and neutrality are combined, $d_S(\cdot, \cdot)$ is determined by $|S|$ alone. In such a case, we write d_S as d_r where $r = |S|$. If n is even, it is not possible for an anonymous ASWF to be neutral because Eq. (2) cannot be satisfied for $|S| = n/2$. The IP (1)-(3) can be used to derive a number of old and new results regarding anonymous and neutral ASWF's in a unified way; we state these results next.

Recall that Ω admits a *Condorcet* triple if there are x, y and $z \in \mathcal{A}$ and $\mathbf{p_1}$, $\mathbf{p_2}$ and $\mathbf{p_3} \in \Omega$ such that $x\mathbf{p_1}y\mathbf{p_1}z$, $y\mathbf{p_2}z\mathbf{p_2}x$, and $z\mathbf{p_3}x\mathbf{p_3}y$.

The following results are well known and follow directly from the IP characterization:

Theorem 6. *(Sen [14]) For an odd number of agents, the majority rule is an ASWF on Ω if and only if Ω does not contain a Condorcet triple.*

Theorem 7. *(Maskin [8]) Suppose there are at least 3 agents. If Ω admits an anonymous, neutral ASWF, then Ω has no Condorcet triples.*

Theorem 8. *(Maskin [8]) Suppose that g is anonymous, neutral, satisfies unanimity and independence of irrelevant alternatives, and is not the majority rule. Then there exists a domain Ω on which g is not an ASWF but the majority rule is.*

The next result, which is new, shows that checking whether Ω admits a neutral, non-dictatorial ASWF reduces to checking whether the majority rule or the born loser rule is an ASWF on that domain. Notice that no parity assumption on the number of voters is needed.

Theorem 9. *For $n \geq 3$, a domain Ω admits a neutral, non-dictatorial ASWF if an only if the majority rule or the born loser rule is an ASWF on Ω.*

Proof. If either the majority rule or the born loser rule is an ASWF on Ω, Ω clearly admits a neutral, non-dictatorial ASWF. Suppose then Ω admits a neutral, non-dictatorial ASWF, but neither the majority rule nor the born loser rule is an ASWF on Ω. Since the majority rule is not an ASWF, Ω admits a Condorcet triple $\{a, b, c\}$. Since the born loser rule is not an ASWF on Ω, by corollary 1 there exist $\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}$ in Ω and $x, y, z \in \mathcal{A}$ with

$$x\mathbf{p_1}z\mathbf{p_1}y, \ x\mathbf{p_2}y\mathbf{p_2}z, \ z\mathbf{p_3}x\mathbf{p_3}y.$$

We will need the existence of these orderings to construct a partition of the agents that satisfies the cycle elimination constraints. The proof will mimic the proof of Arrow's theorem (Theorem 4) given earlier.

Neutrality implies that $d_S(x, y) = d_S(y, z) = d_S(z, u)$ for all alternatives x, y, z, u. We will thus write d_S in place of $d_S(x, y)$ in the rest of the proof.

First, $d_S = 1 \Rightarrow d_T = 1$ for all $S \subset T$. Suppose not. Let T be the set containing S with $d_T = 0$. Constraint (2) implies $d_{T^c} = 1$. Choose $A = T \setminus S$, $U = T^c$ and $V = S$ in (3). We can do this because of $\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}$. Then, $d_{A \cup U \cup V} = d_E = 1$, $d_{B \cup U \cup W} = d_{T^c} = 1$ and $d_{C \cup V \cup W} = d_S = 1$, which contradicts (3).

The same argument implies that $d_T = 0 \Rightarrow d_S = 0$ whenever $S \subset T$. Note also that if $d_S = d_T = 1$, then $S \cap T \neq \emptyset$, otherwise the assignment $A = (S \cup T)^c, U = S, V = T$ will violate the cycle elimination constraint.

Next we show that $d_{S \cap T} = 1$. Suppose not. Consider the assignment $U = E \setminus S$, $V = S \setminus T$ and $W = S \cap T$. We can choose such a partition because $\{a, b, c\}$ form a Condorcet triple. For this specification, $d_{A \cup U \cup V} = d_{E \setminus \{S \cap T\}} = 1$. Since $T \subset B \cup U \cup W$, $d_{B \cup U \cup W} = 1$ and $d_{C \cup V \cup W} = d_S = 1$, which contradicts (3).

Hence there exists a minimal set S^* such that $d_{S^*} = 1$ and all T with $d_T = 1$ contains S^*. We show that $|S^*| = 1$. If not there will be $j \in S$ with $d_j = 0$, and hence $d_{E \setminus \{j\} \cap S^*} = 1$, contradicting the minimality of S^*. □

A simple consequence of this result is the following theorem due to Kalai and Muller [5]. The proof is new.

Theorem 10. *A non-dictatorial solution to (1, 2, 4 - 7) exists for the case $n = 2$ agents if and only if a non-dictatorial solution to (1-3) exists for any n.*

Proof. Given a 2 person non-dictatorial AWSF, we can build an ASWF for the n-person case by focusing only on the preferences submitted by the first two voters and ranking the alternatives using the 2-person ASWF. This is clearly a non-dictatorial ASWF for the n-person case. Hence we only need to give a proof of the converse.

Let d^* be a non-dictatorial solution to (1-3). Suppose d does not imply a neutral ASWF. Then there is a set of agents S such that $d^*_S(x, y)$ is non-zero for some but not all $(x, y) \in \mathcal{A}^2$. Hence, $d_1 = d^*_S, d_2 = d^*_{S^c}$ would be a non-dictatorial solution to (1, 2, 4–7).

Suppose then d implies a neutral ASWF. By the previous theorem we can choose d to be either the majority rule or the born loser rule. In the first case, we can build a 2 person ASWF by using a dummy voter with a fixed ordering from Ω and using the (3 person) majority rule. In the second case, we can build a 2 person ASWF by adding a dummy born loser. □

The following refinement to Maskin's result also follows directly from Theorem 9.

Theorem 11. *Let the number of agents be odd. Suppose Ω does not contain any Condorcet triples, and suppose there exist $\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}$ in Ω and $x, y, z \in \mathcal{A}$ with*

$$x \mathbf{p_1} z \mathbf{p_1} y, \;\; x \mathbf{p_2} y \mathbf{p_2} z, \;\; z \mathbf{p_3} x \mathbf{p_3} y.$$

Then, the majority rule is the **only** *anonymous, neutral ASWF on Ω.*

Proof.(Sketch) From the proof to Theorem 9, we know that if d_S corresponds to a neutral ASWF, and if there exist $\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}$ in Ω and $x, y, z \in \mathcal{A}$ with $x \mathbf{p_1} z \mathbf{p_1} y$, $x \mathbf{p_2} y \mathbf{p_2} z, z \mathbf{p_3} x \mathbf{p_3} y$, then d_S is monotonic. i.e., $d_S \leq d_T$ if $S \subset T$. By May's Theorem, it has to be the majority rule since the majority rule is the only ASWF that is anonymous, neutral and monotonic. □

Single-peaked Domains. The domain Ω is single-peaked with respect to a linear ordering $\mathbf{q}$ over $\mathcal{A}$ if $\Omega \subseteq \{\mathbf{p} \in \Sigma :$ for every triple (x, y, z) if $x\mathbf{q}y\mathbf{q}z$ then it is **not** the case that $x\mathbf{p}y$ and $z\mathbf{p}y\}$. The class of single-peaked preferences has received a great deal of attention in the literature. Here we show how the IP can be used to characterize the class of ASWF's on single peaked domains. We prove that the constraints (1–3), along with the non-negative constraints on the d variables, are *sufficient* to characterize the convex hull of the 0-1 solutions.

Theorem 12. *When Ω is single-peaked the set of non-negative solutions satisfying (1-3) is an integral polytope. All ASWF's are extreme point solutions of this polytope.*

Proof. (Sketch) It suffices to prove that every fractional solution satisfying (1-3) can be written as a convex combination of 0-1 solutions satisfying the same set of constraints. Let $\mathbf{q}$ be the linear ordering with respect to which Ω is single-peaked.

Let $d_S(\cdot)$ be a (possibly) fractional solution to the linear programming relaxation of (1-3). We round the solution d to the 0-1 solution d' in the following way:

- Generate a random number Z uniformly between 0 and 1.
- For $a, b \in \mathcal{A}$ with $a\mathbf{q}b$, and $S \subset E$, then
  - $d'_S(a, b) = 1$, if $d_S(a, b) > Z$, 0 otherwise;
  - $d'_S(b, a) = 1$, if $d_S(b, a) \geq 1 - Z$, 0 otherwise.

The 0-1 solution d'_S generated in the above manner clearly satisfies constraints (1). To verify that it satisfies constraint (2), consider a set $T \subseteq E$, an arbitrary pair of alternatives a, b, and suppose without loss of generality $a\mathbf{q}b$. From the linear programming relaxation, we know that either $d_T(a, b) > Z$ or $d_T(b, a) \geq 1 - Z$ (since the two variables add up to 1), but not both. Thus, exactly one of $d'_T(a, b)$ or $d'_T(b, a)$ is set to 1.

We show next that all the constraints in (3) are satisfied by the solution $d'_S(\cdot)$. Consider three alternatives a, b, c, and constraint (3) (with a, b, c replacing the role of x, y, z) can be re-written as:

$$d_{A\cup U\cup V}(a, b) + d_{B\cup U\cup W}(b, c) + d_{C\cup V\cup W}(c, a) \leq 2.$$

Suppose $a\mathbf{q}b\mathbf{q}c$. Then in constraints (3), by the single-peakedness property, we must have $A = V = \emptyset$. In this case, the constraint reduces to $d_U(a, b) + d_{B\cup U\cup W}(b, c) + d_{C\cup W}(c, a) \leq 2$.

We need to show that $d'_U(a, b) + d'_{B\cup U\cup W}(b, c) + d'_{C\cup W}(c, a) \leq 2$. By choosing the sets in constraints (3) in a different way, with $U' \leftarrow U$, $B' \leftarrow B$, $W' \leftarrow W \cup C$, $C' \leftarrow \emptyset$, we have a new inequality $d_{U'}(a, b) + d_{B'\cup U'\cup W'}(b, c) + d_{C'\cup W'}(c, a) \leq 2$, which is equivalent to $d_U(a, b) + 1 + d_{C\cup W}(c, a) \leq 2$. Hence we must have $d_U(a, b) + d_{C\cup W}(c, a) \leq 1$. Note that since $a\mathbf{q}b$ and $b\mathbf{q}c$, our rounding scheme ensures that $d'_U(a, b) + d'_{C\cup W}(c, a) \leq 1$. Hence $d'_U(a, b) + d'_{B\cup U\cup W}(b, c) + d'_{C\cup W}(c, a) \leq 2$.

To finish the proof, we need to show that constraint (3) holds for different orderings of a, b and c under $\mathbf{q}$; the above argument can be easily extended to handle all these cases to show that constraint (3) is valid. Integrality of the polytope follows directly from this rounding method. We omit the details here. □

The argument above shows the set of ASWF's on single-peaked domains (wrt $\mathbf{q}$) has a property similar to the generalized median property of the stable marriage problem (see Teo and Sethuraman [15]).

Theorem 13. *Let $f_1, f_2, \ldots, f_N$ be distinct ASWF's for the single-peaked domain Ω (with respect to $\mathbf{q}$). Define a function $F_k : \Omega^n \to \Sigma$ with the property:*

The set S under F_k is decisive for x over y if
$x\mathbf{q}y$, and S is decisive for x over y for at least $k+1$ *of the ASWF f_i's; or*
$y\mathbf{q}x$, and S is decisive for x over y for at least $N-k$ *of the ASWF f_i's.*
Then F_k is also an ASWF.

One consequence of Theorem 13 is that when Ω is single-peaked, it is Arrovian, since the dictatorial ASWF's can be used to construct non-dictatorial ASWF in the above manner. For instance, consider the case $n = 2$. Let f_1 and f_2 be the dictatorial rule associated with agent's 1 and 2 respectively. The function F_1 constructed above reduces to the following ASWF:

If $x\mathbf{q}y$, the social welfare function ranks x above y if and only if both agents prefer x over y.
If $y\mathbf{q}x$, the social welfare function ranks y above x if and only if none of the agents prefer x above y.

Generalized Majority Rule. Moulin [10] has introduced a generalization of the majority rule called the generalized majority rule. A Generalized majority rule (GMR) M for n agents is of the following form:

- Add n-1 dummy agents, each with a fixed preference drawn from Ω.
- x is ranked above y under M if and only if the majority (of real and dummy agents) prefer x to y.

Each instance of a GMR can be described algebraically as follows. Fix a profile $\mathbf{R} \in \Omega^{n-1}$ and let $R(x, y)$ be the number of orderings in $\mathbf{R}$ where x is ranked above y. Given any profile $\mathbf{P} \in \Omega^n$, GMR ranks x above y if the number of agents who rank x above y under $\mathbf{P}$ is at least $n - R(x, y)$. To check that GMR is an ASWF on single peaked domains, set

$$g_S(x, y) = 1 \text{ iff } |S| \geq n - R(x, y)$$

and zero otherwise. It is easy to check that g satisfies (1)-(3) when Ω is single peaked.

GMR has two important properties. The first is that it is anonymous and second that it is monotonic.

Definition 2. *An ASWF is* monotonic *if when one switches from the profile* **P** *to* **Q** *by raising the ranking of* $x \in A$ *for at least one agent, then* $f(\mathbf{Q})$ *will not rank* x *lower than it is in* $f(\mathbf{P})$.

Theorem 14 (Moulin). *An ASWF that is anonymous and monotonic on a single-peaked domain* Ω *must be a generalized majority rule*

Proof. Let d_S be a solution to (1)-(3), corresponding to an anonymous and monotonic ASWF on the domain Ω. Let $\mathbf{q}$ be the underlying order of alternatives. For each $(x, y) \in A^2$, by anonymity, $d_S(x, y)$ depends only on the cardinality of S. Monotonicity implies $d_S(x, y) \leq d_T(x, y)$ if $S \subseteq T$. Thus

$$d_S(x, y) = 1 \text{ if and only if } |S| \geq e(x, y)$$

for some number $e(x, y)$. To complete the proof we need to determine a profile $\mathbf{R} \in \Omega^{n-1}$ such that

$$n - R(x, y) = e(x, y) \ \forall (x, y) \in A^2.$$

Since $d_S(x, y) + d_{S^c}(y, x) = 1$, we have

$$e(x, y) + e(y, x) = n + 1$$

for all (x, y) and (y, x). Note that $e(x, y) \geq 1$ and $e(x, y) \leq n$. Furthermore, if $x\mathbf{q}y\mathbf{q}z$, then by (4) and (5), $d_S(x, y) \leq d_S(x, z)$ and hence $e(x, y) \geq e(x, z)$. Similarly, we have $e(y, x) \leq e(z, x)$, $e(z, y) \geq e(z, x)$ and $e(x, z) \geq e(y, z)$.

We use the geometric construction used in the earlier proof to construct the profile $\mathbf{R} \in \Omega^{n-1}$.

- To each (x, y) such that $x\mathbf{q}y$, associate the interval $[0, e(x, y)]$ and label it $l(x, y)$.
- To each (x, y) such that $y\mathbf{q}x$, associate the interval $[n + 1 - e(x, y), n + 1]$ and label it $l(x, y)$.

We construct preferences in $\mathbf{R}$ in the following way:

- For each $k = 1, 2, \ldots, n - 1$, if $l(x, y)$ covers the point $k + 0.5$, then the kth dummy voter ranks y over x. Otherwise the dummy voter ranks x over y.

Since the intervals $l(x, y)$ and $l(y, x)$ are disjoint and cover $[0, n+1]$ the procedure is well-defined. If $R(x, y)$ is the number of dummy voters who rank x above y in this construction it is easy to see that $n - R(x, y) = e(x, y)$, which is what we need. It remains then to to show that the profile constructed is in Ω^{n-1}.

Claim. The procedure returns a linear ordering of the alternatives.

Proof. Suppose otherwise and consider three alternatives x, y, z where the procedure (for some dummy voter) ranks x above y, y above z and z above x. Hence the intervals $l(x, y)$, $l(y, z)$ and $l(z, x)$ do not cover the point $k + 0.5$. From symmetry, it suffices to consider the following two cases:

- Case 1. Suppose $x\mathbf{q}y\mathbf{q}z$. Since $l(x,z)$ covers the point $k+0.5$ and $e(x,y) \geq e(x,z)$, $l(x,y)$ must cover the point $k+0.5$, a contradiction.
- Case 2. Suppose $y\mathbf{q}x\mathbf{q}z$. Now, there exists $\mathbf{p}$ and $\mathbf{p}'$ in Ω with $z\mathbf{p}x\mathbf{p}y$ and $x\mathbf{p}'y\mathbf{p}'z$, hence $l(z,x) \geq l(z,y)$. This is impossible as $l(z,y)$ covers the point $k+0.5$ but $l(z,x)$ does not.

Hence the ordering constructed is a linear order. □

Claim. The linear orderings constructed for the dummy voters correspond to orderings from Ω.

Proof. If not there exist k and $x\mathbf{q}y\mathbf{q}z$ with the kth dummy voter ranking y below x and z. i.e. $l(x,y)$ does not cover the point $k+0.5$ and $l(y,z)$ does. Hence $e(x,y) < e(y,z)$. Now, using $x\mathbf{q}y\mathbf{q}z$, we have

$$d_S(x,y) \leq d_S(x,z), d_S(x,z) \leq d_S(y,z).$$

So

$$e(x,y) \geq e(x,z), e(x,z) \geq e(y,z),$$

which is a contradiction. □

Muller-Satterthwaite Theorem. A *social choice function* maps profiles of preferences into a single alternative. These are objects that have received as much attention as social welfare functions. It is therefore natural to ask if the integer programming approach described above can be used to obtain results about social choice functions. Up to a point, yes. The difficulty is that knowing what alternative a social choice function will pick from a set of size two, does not, in general, allow one to infer what it will choose when the set of alternatives is extended by one. However, given the additional assumptions imposed upon a social choice function one can surmount this difficulty. We illustrate how with an example.

The analog of Arrow's impossibility theorem for social choice functions is the Muller-Satterthwaite theorem [11]. The counterpart of Unanimity and the Independence of Irrelevant Alternatives condition for social choice functions are called *pareto optimality* and *monotonicity.* To define them, denote the preference ordering of agent i in profile $\mathbf{P}$ by $\mathbf{p_i}$.

1. **Pareto Optimality:** Let $\mathbf{P} \in \Omega^n$ such that $x\mathbf{p}y$ for all $\mathbf{p} \in \mathbf{P}$. Then $f(\mathbf{P}) \neq y$.
2. **Monotonicity:** For all $x \in \mathcal{A}$, $\mathbf{P}, \mathbf{Q} \in \Omega^n$ if $x = f(\mathbf{P})$ and $\{y : x\mathbf{p_i}y\} \subseteq \{y : x\mathbf{q_i}y\}$ $\forall i$ then $x = f(\mathbf{Q})$.

We call a social choice function that satisfies pareto-optimality and monotonicity an Arrovian social choice function (ASCF).

Theorem 15 (Muller-Satterthwaite). *When $\Omega = \Sigma$, all ASCF's are dictatorial*[2].

[2] The more well known result about strategy proof social choice functions is due to Gibbard [3] and Satterthwaite [13]. It is a consequence of Muller-Satterthwaite [11].

Proof: For each subset S of agents and ordered pair of alternatives (x, y), denote by $[S, x, y]$ the set of all profiles where agents in S rank x first and y second, and agents in S^c rank y first and x second. By the hypothesis on Ω this collection is well defined.

For any profile $\mathbf{P} \in [S, x, y]$ it follows by pareto optimality that $f(\mathbf{P}) \in \{x, y\}$. By monotonicity, if $f(\mathbf{P}) = x$ for one such profile $\mathbf{P}$ then $f(\mathbf{P}) = x$ for all $\mathbf{P} \in [S, x, y]$.

Suppose then for all $\mathbf{P} \in [S, x, y]$ we have $f(\mathbf{P}) \neq y$. Let $\mathbf{Q}$ be any profile where all agents in S rank x above y, and all agents in S^c rank y above x. We show next that $f(\mathbf{Q}) \neq y$ too.

Suppose not. That is $f(\mathbf{Q}) = y$. Let $\mathbf{Q}'$ be a profile obtained by moving x and y to the top in every agents ordering but preserving their relative position within each ordering. So, if x was above y in the ordering under $\mathbf{Q}$, it remains so under $\mathbf{Q}'$. Similarly if y was above x. By monotonicity $f(\mathbf{Q}') = y$. But monotonicity with respect to $\mathbf{Q}'$ and $\mathbf{P} \in [S, x, y]$ implies that $f(\mathbf{P}) = y$ a contradiction.

Hence, if there is one profile in which all agents in S rank x above y, and all agents in S^c rank y above x, and y is not selected, then all profiles with such a property will not select y. This observation allows us to describe ASCF's using the following variables.

For each $(x, y) \in \mathcal{A}^2$ define a 0-1 variable as follows:

– $g_S(x, y) = 1$ if when all agents in S rank x above y and all agents in S^c rank y above x then y is never selected,
– $g_S(x, y) = 0$ otherwise.

If E is the set of all candidates we set $g_E(x, y) = 1$ for all $(x, y) \in \mathcal{A}^2$. This ensures pareto optimality.

Consider a $\mathbf{P} \in \Omega^n$, $(x, y) \in \mathcal{A}^2$ and subset S of agents such that all agents in S prefer x to y and all agents in S^c prefer y to x. Then, $g_S(x, y) = 0$ implies that $g_{S^c}(y, x) = 1$ to ensure a selection. Hence for all S and $(x, y) \in \mathcal{A}^2$ we have

$$g_S(x, y) + g_{S^c}(y, x) = 1 \ . \tag{9}$$

We show that the variables g_S satisfy the cycle elimination constraints. If not there exists a triple $\{x, y, z\}$, and set A, B, C, U, V, W such that the cycle elimination constraint is violated. Consider the profile $\mathbf{P}$ where each voter ranks the triple $\{x, y, z\}$ above the rest, and with the ordering of x, y, z depending on whether the voter is in A, B, C, U, V or W. Since $g_{A \cup U \cup V}(x, y) = 1$, $g_{B \cup U \cup W} = 1$, and $g_{C \cup V \cup W} = 1$, none of the alternatives x, y, z is selected for the profile $\mathbf{P}$. This violates pareto optimality, a contradiction.

Hence g_S satisfies constraints (1-3). Since $\Omega = \Sigma$, by Arrow's Impossibility Theorem, g_S corresponds to a dictatorial solution. □

4 Decomposability, Complexity and Valid Inequalities

A domain is called *decomposable* if and only if there is a non-trivial solution (not all 1's or all 0's) to the system of inequalities (1, 2, 4–7) for the case

$n = 2$. The main result of [5] (cf. Theorem 10) can be phrased as follows: *the domain Ω is non-dictatorial if and only if it is decomposable.* This result allows one to formulate the problem of deciding whether Ω is arrovian as an integer program involving a number of variables and constraints that is polynomial in $|\mathcal{A}|$. However, the set $\mathcal{A}$ is not the only input to the problem. The preference domain Ω is also an input. If Ω is specified by the set of permutations it contains, and if it has exponentially many permutations (say $O(2^{|\mathcal{A}|})$, the the straight forward input model needs at least $O(2^{|\mathcal{A}|})$ bits. Recall the number of decision variables for the integer program for 2-person ASWF's is polynomial in $|\mathcal{A}|$. Furthermore, the time complexity of verifying the existence of triplets in Ω can trivially be performed in time $O(n^3 2^{|\mathcal{A}|})$. Hence the decision version of the decomposability conditions can be solved in time polynomial in the size of the input.

Suppose, however, instead of listing the elements of Ω, we prescribe a polynomial time oracle to check membership in Ω. The complexity issue of deciding whether the domain is decomposable now depends on how we encode the membership oracle, and not on the number of elements in Ω. In this model, we exhibit an example to show that checking whether a triplet exists in Ω is already NP-hard.

Let G be a graph with vertex set V. Let Ω_G consist of all orderings of V that correspond to a Hamiltonian path in G. Given any triple $(u, v, w) \in V$, the problem of deciding if G admits a Hamiltonian path in which u precedes v precedes w is NP-complete[3]. Hence the problem of deciding whether there is a preference ordering $\mathbf{p}$ in Ω with $u\mathbf{p}v\mathbf{p}w$ is already NP-complete.

Thus, given an Ω specified by hamiltonian paths, it is already NP-hard just to write down the set of inequalities specified by the decomposability conditions!

One way to by-pass the above difficulties is to focus on ordering on triplets that are realized by some preferences in Ω. The input to the complexity question is thus the set of orderings on triplets ($O(n^3)$ size) that are admissible in Ω. We will focus on this input model for the rest of the paper.

Ignore, for the moment, inequalities of types (6) and (7). The constraint matrix associated with the inequalities of types (1, 2, 4, 5) and $0 \leq d(x, y) \leq 1 \ \forall (x, y) \in \mathcal{A}^2$ is totally unimodular. This is because each inequality can be reduced to one that contains at most two coefficients of opposite sign and absolute value of 1 [4]. Hence the extreme points are all 0-1. If one or more of these extreme points was different from the all 0's solution and all 1's solution we would know that Ω is Arrovian. If the only extreme points were the all 0's solution and all 1's solution that would imply that Ω is not Arrovian.

Thus difficulties with determining the existence of a feasible 0-1 solution different from the all 0's and all 1's solution have to do with the inequalities of the form (6) and (7). Notice that any admissible ordering (by Ω) of three

[3] If not, we can apply the algorithm for this problem thrice to decide if G admits a Hamiltonian cycle.

[4] It is well known that such matrices are totally unimodular. See for example, Theorem 11.12 in [1].

alternatives gives rise to an inequality of types (6) and (7). However some of them will be redundant. Constraints (6, 7) are not redundant only when they are obtained from a triplet (x, y, z) with the property:

There exists $\mathbf{p}$ such that $x\mathbf{p}y\mathbf{p}z$ but no $\mathbf{q} \in \Omega$ such that $y\mathbf{q}z\mathbf{q}x$ or $z\mathbf{q}x\mathbf{q}y$.

Such a triplet is called an *isolated triplet.*

Call the inequality representation of Ω, by inequalities of types (1, 2, 4, 5), the *unimodular representation* of Ω. Note that all inequalities in the unimodular representation are of the type $d(x, u) \le d(x, v)$ or $d(u, x) \le d(v, x)$. *Furthermore,* $d(x, u) \le d(x, v)$ *and* $d(u, y) \le d(v, y)$ *appear in the representation only if there exist* $\mathbf{p}, \mathbf{q}$ *with* $u\mathbf{p}x$ *and* $v\mathbf{p}x$ and $x\mathbf{q}u$ *and* $x\mathbf{q}v$.

This connection allows us to provide a graph-theoretic representation of the unimodular representation of Ω as well as a graph-theoretic interpretation of when Ω is not Arrovian.

With each *non-trivial* element of $\mathcal{A}^2$ we associate a vertex. If in the unimodular representation of Ω there is an inequality of the form $d_1(a, b) \le d_1(x, y)$ where (a, b) and $(x, y) \in \mathcal{A}^2$ then insert a *directed* edge from (a, b) to (x, y). Call the resulting directed graph D^{Ω}.

If (x, y) is a trivial pair (and hence $(x, y) \notin D^{\Omega}$), then $d_1(x, y)$ is automatically fixed at 1, and $d_1(y, x)$ fixed at 0. An inequality of the form $d_1(x, y) \le d_1(x, z)$ (or $d_1(z, y)$) *cannot* appear in the unimodular representation, for any alternative z in $\mathcal{A}$. Otherwise there must be some $\mathbf{p} \in \Omega$ with $y\mathbf{p}x$. Similarly, if (x, y) is trivial, $d_1(y, x) \ge d_1(z, x)$ (or $d_1(y, z)$) *cannot* appear in the unimodular representation, for any alternative z in $\mathcal{A}$. Thus fixing the values of $d_1(x, y)$ and $d_1(y, x)$ arising from a trivial pair (x, y) does not affect the value of $d_1(a, b)$ for $(a, b) \in D^{\Omega}$.

A subset S of vertices in D^{Ω} is *closed* if there is no edge directed out of S. That is, there is no directed edge with its tail incident to a vertex in S and its head incident to a vertex outside S. Notice that $d_1(x, y) = 1 \ \forall (x, y) \in S$ and 0 otherwise (and together with those arising from the trivial pairs) is a feasible 0-1 solution to the unimodular representation of Ω if S is closed. Hence every closed set in D^{Ω} corresponds to a feasible 0-1 solution to the unimodular representation. The converse is also true.

Theorem 16. *If* D^{Ω} *is strongly connected then* Ω *is non-Arrovian.*

Proof. The set of all vertices of D^{Ω} is clearly a closed set. The solution corresponding to this closed set is the ASWF where agent 1 is the dictator. The empty set of vertices is closed and this corresponds to agent 2 being the dictator. If D^{Ω} is strongly connected[5], these are the only closed sets in the graph. Since any ASWF must correspond to some closed set in D^{Ω}, we conclude that Ω is non-Arrovian. □

We note that verifying whether a directed graph is strongly connected can be done efficiently. See [1] for details. Note also that if Ω does not contain any isolated triplets, then Ω is Arrovian if and only D^{Ω} is not strongly connected.

[5] A directed graph is strongly connected if there is a directed cycle through every pair of vertices.

We describe next a sequential lifting method to derive valid inequalities for the problem to strengthen the LP formulation, using the directed graph D^Ω defined previously. We say that the node u *dominates* the node v if there is a directed path in D^Ω from v to u (i.e. $d(u) \geq d(v)$).

Sequential Lifting Method:

- For each isolated triplet (x, y, z), we have the inequality

$$1 + d(x, z) \geq d(x, y) + d(y, z). \tag{10}$$

- Let $D(x, y)$ (and resp. $D(y, z)$) denote the set of nodes in D^Ω that are dominated by the node (x, y) (resp. (y, z)) in D^Ω.
- For each node (a, b) in D^Ω, if

$$u \in D(a, b) \cap D(x, y) \neq \emptyset,\ v \in D(a, b) \cap D(y, z) \neq \emptyset,$$

then the constraint arising from the isolated triplet can be augmented by the following valid inequalities:

$$d(a, b) + d(x, z) \geq d(u) + d(v). \tag{11}$$

To see the validity of the above constraint, note that by the definition of domination, we have $d(x, y) \geq d(u), d(y, z) \geq d(v), d(a, b) \geq d(u), d(a, b) \geq d(v)$. If $d(a, b) = 0$, then $d(u) = d(v) = 0$ and hence (11) is trivially true. If $d(a, b) = 1$, then (11) follows from (10).

We have successfully verified that the sequential lifting method finds the convex hull of the set of all ASWF's whenever the number of alternatives is at most four. A natural question is if whether the sequential lifting method will gives rise to all facets even for the case $|\mathcal{A}| \geq 5$; we do not yet know, although we suspect the answer to be negative.

5 Conclusions

In this paper, we study the connection between Arrow's Impossibility Theorem and Integer Programming. We show that the set of ASWF's can be expressed as integer solutions to a system of linear inequalities. Many of the well known results connected to the impossibility theorem are direct consequences of the Integer Program. Furthermore, the polyhedral structure of the IP formulation warrants further study in its own right. We have initiated the study on this class of polyhedra by characterizing the polyhedral structure of ASWF's on single peaked domain. We have also demonstrated by an extensive computational experiment that the sequential lifting method proposed in this paper can be used to obtain the complete polyhedral description of ASWF's when the number of alternatives is small. Several interesting problems still remain:

1. Given a domain Ω specified by certain membership oracle, is it possible to check for existence of non-dictatorial ASWF's in polynomial time? Is the problem in the class NP?

2. The LP relaxation of our proposed IP formulation characterizes the ASWF's for single peaked domain. What are the domains that can be characterized by the LP relaxation given by the sequential lifting method?
3. Can the conditions for ASCF's be written down as a system of integer linear inequalities?

We leave the above questions for future research.

References

1. Ahuja, R., T. L. Magnanti and J. B. Orlin. (1993). **Network Flows: Theory, Applications and Algorithms**, Prentice-Hall, Englewood Cliffs, New Jersey, U.S.A.
2. Arrow, K. J. (1963). **Social Choice and Individual Values**, Wiley, New York.
3. Gibbard, A. (1973). "Manipulation of Voting Schemes: A general result", *Econometrica*, 41, 587-602.
4. P. C. Fishburn and J. S. Kelly (1997), Super Arrovian domains with strict preferences, SIAM J. Disc. Math., 11, pp. 83-95.
5. Kalai, E. and E. Muller. (1977). "Characterization of Domains Admitting Non-dictatorial Social Welfare Functions and Non-manipulable Voting Procedures", *J. of Economic Theory*, 16, 2, 457-469.
6. Kalai, E., E. Muller and M. Satterthwaite. (1979). "Social Welfare Functions when Preferences are Convex, Strictly Monotonic and Continuous", *Public Choice*, 34, 87-97.
7. Le Breton, M. and J. Weymark. (1996). "An Introduction to Arrovian Social Welfare Functions on Economic and Political Domains". In **Collective Decision Making: Social Choice and Political Economy**, edited by Norman Schofield, Kluwer Academic Publishers, Boston.
8. Maskin, E. (1995), "Majority Rule, Social Welfare Functions, and Game Forms", in K. Basu, P.K. Pattanaik and K. Suzumura, eds., **Choice, Welfare and Development** (Oxford: Clarendon Press).
9. Moulin, H. (1980). "On Strategy-Proofness and Single Peakedness", *Public Choice*, 35, 437-455.
10. Moulin, H. (1984). "Generalized Condorcet Winners for Single Peaked and Single Plateau Preferences", *Social Choice and Welfare*, 1, 127-147.
11. Muller, E. and M. Satterthwaite. (1977). "The Equivalence of Strong Positive Association and Strategy-Proofness", *Journal of Economic Theory*, 14, 412-418.
12. Nemhauser, G. and L. Wolsey. (1998). **Integer and Combinatorial Optimization**, John Wiley, New York.
13. Satterthwaite, M. (1975). "Strategy-Proofness and Arrow's Conditions: Existence and correspondence theorems for voting procedures and social welfare functions", *J. of Economic Theory*, 10, 187-217.
14. Sen, A. (1966). "A Possibility Theorem on Majority Decisions", *Econometrica*, vol. 34, Issue 2, 491-499.
15. Teo, C.P. and J. Sethuraman (1998) "Geometry of fractional stable matchings and its applications", *Mathematics of Operations Research*, Vol 23, Number 4, 874-891.