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Theory and Methodology

On a cutting plane heuristic for the stable roommates problem and its applications

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Abstract

We propose a new cutting plane heuristic for the classical stable roommates problem. Our approach utilises a new linear programming formulation for the problem, and the underlying geometric properties of the fractional solution to construct the violated inequality. We test the approach on moderate size problems, with encouraging computational performance. To further illustrate the versatility of this approach, we also show how it can be suitably extended to handle variants of the basic stable roommates model. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Given a complete graph H with an even number of vertices, and with weights on its edges, the problem of finding a minimum weight perfect matching in H has been studied extensively. In fact, a complete polyhedral description of the convex hull of the set of perfect matching has been obtained by Edmonds [3]. In this paper, we study a variant of the perfect matching problem, commonly known as the stable roommates problem. In an instance of this problem, each node of the graph represents a college student who wishes to be assigned to a roommate from a list of possible candidates. The key difference from the perfect matching problem is that here, each student ranks all the other students in decreasing order of desirability. More formally, in the stable roommates problem, we have a set of $2n$ persons who need to be paired up as roommates. Each person has a preference list ranking *all* the other persons, and all preferences are *strict*. A matching is unstable when two people are not paired in the matching,

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but prefer each other to their assigned roommates. Such a pair is called a blocking pair. A matching is stable if and only if there is no blocking pair. We write $k <_i j$ if person i prefers person j to person k. This problem is a generalization of the famous stable marriage problem first studied by Gale and Shapley [5] in 1962. The problem, as stated above, is slightly different from some other papers in the literature, where the focus is on finding a stable (partial) matching in the graph H.

In sharp contrast to the stable marriage problem, the stable roommates problem need not have a solution. Knuth [10], in one of his 12 unsolved problems associated with stable matching, asked whether the feasibility of an instance of the stable roommates problem can be settled in polynomial time. His question had been answered by Irving [8] in the affirmative, and also recently by Subramanian [13] who related the problem to the more general network stability problem (cf. Ref. [13]).

Feder [4] considered a special version of the weighted stable roommates problem where the preference list of each person is given by a set $\{c(i,j)\}$ (with distinct values), and

$$c(i,j) < c(i,k) \text{ only if } k <_i j \quad \text{for all } i,j,k.$$

Using $c(i,j) + c(j,i)$ as the weight of the edge $\{i,j\}$, Feder showed that the optimal stable roommates problem is NP-hard in this case. Gusfield and Pitt [7], using their 2-approximation algorithm for a more general min-cost 2-SAT problem, showed that the optimal stable roommates problem (with the above cost function) can be approximated to within a factor of 2 from the optimal. It remains an interesting open problem whether one can do better than the bound of 2.

Abeledo and Rothblum [2] recently initiated the study of the polyhedral structure of the stable roommates problem. Generalizing Vande Vate's formulation (cf. Refs. [11,12,15]) for the marriage problem, they showed that many structural properties of the stable roommates problem can be derived via linear-algebraic arguments, mainly via linear programming duality. However, the proposed polytope has a major weakness it is not strong enough (to be used) to decide the existence of a stable matching (i.e. there are infeasible instances of the stable roommates problem for which the associated polytope is non-empty). Their formulation was recently improved by the authors [14], who identified a new class of valid inequalities for the problem. This new LP formulation, with an exponential number of constraints, is feasible if and only if the stable roommates problem has a solution. This opens up a new way to construct near optimal stable roommates solution (if one exists at all) via the LP approach. Furthermore, the LP relaxation can be solved via the ellipsoid algorithm. However, the separation routine for the (exponentially many) constraints requires solving a series of shortest path problems on an expanded graph and appears to be computationally prohibitive. In this paper, we adapt the proof technique in [14] to devise a new cutting-plane heuristic (coupled with a rounding procedure) to solve the optimal stable roommates problem. The heuristic combines the power of rounding and cutting-plane ideas into a conceptually simple approach. In fact, under some restrictions on the cost function, the heuristic always return a solution which is within two times of the optimal stable roommates solution. In this aspect, our heuristic can be viewed as a 2-approximation algorithm for the optimal stable roommates problem. The computational results show that the LP based method can be a viable approach to obtain reasonably good solutions for the optimal stable roommates problem in a reasonable amount of time. Although this approach does not seem to be as fast as some of the combinatorial methods (cf. Refs. [6,9]), its algebraic nature allows us to adapt the method to several variants of the stable roommates problem. To illustrate this, we show how the minimum regret stable roommates solution can be obtained via a simple linear programming formulation.

A different LP approach has also been proposed recently by Abeledo and Blum [1]. However, their work differs from ours in a variety of ways. (1) Their LP approach solves a series of linear programming problems in order to obtain a feasible solution to the roommates problem, whereas our approach works for the NP-hard *optimal* stable roommates problem. (2) The number of LPs solved in their approach is polynomially bounded, whereas our approach cannot guarantee a priori the number of cutting planes needed before the algorithm terminates.

The main contributions of this paper are summarized as follows.

- We propose a simple cutting plane heuristic for the optimal stable roommates problem, which always returns a solution whose cost is within a factor of two of the optimal.
- We perform an experiment to evaluate the performance of this heuristic on moderate-sized problems. Note that as far as we are aware, no such computational studies have been performed for the optimal stable roommates problem. The heuristic is shown to perform exceptionally well on randomly generated instances. However, on some specially constructed instances, the gap can be as large as 37%.
- We illustrate further applications of the LP approach to variants of the stable roommates problem. Interestingly, our heuristic can be used to solve the *minimum regret* stable roommates problem.

2. Formulation of the problem

Abeledo and Rothblum [2] studied the properties of fractional solutions to the stable roommates problem, which is also known as the stable matching problem in the literature. They consider the following relaxation:

(P_{FSM})

$$\sum_j x_{i,j} = 1, \quad \forall i, \tag{1}$$

$$\sum_{l:l<_j i} x_{l,j} + \sum_{l:l<_i j} x_{i,l} + x_{i,j} \leqslant 1, \quad \forall i,j, \tag{2}$$

where (P_{FSM}) denotes the set of fractional solutions that satisfies constraints (1) and (2) as shown. Note that when $x_{i,j} = 0$ or 1 for all i,j, the variable $x_{i,j}$ models the decision whether person i is matched to person j. Hence (P_{FSM}) contains in particular the set of integral stable roommates solutions. Constraints (2) must be valid, since otherwise, j is matched to someone inferior to i whereas i is matched to someone inferior to j and so (i,j) is a blocking pair. We call constraints (2) the *paired* inequalities.

By considering the dual of (P_{FSM}), Abeledo and Rothblum [2] proved an interesting result for (P_{FSM}).

Lemma 1. *Let x be a feasible solution in (P_{FSM}). Then*

$$x_{i,j} > 0 \textit{ implies } \sum_{k:k<_i j} x_{i,k} + \sum_{k:k \leqslant_j i} x_{k,j} = 1.$$

An analogous result holds for the stable marriage case and is the basis of some of the most beautiful results in that area (cf. Ref. [14] for a recent approach utilizing the above property). One of the drawbacks of the formulation in (P_{FSM}) is that one can construct infeasible instances of the stable roommates problem for which a polyhedron associated with (P_{FSM}) is non-empty. So (P_{FSM}) cannot be used to address the feasibility question for the stable roommates problem. A natural way to improve the above formulation is to include the odd-set constraints from the matching polytope $\sum_{i,j\in S} x_{i,j} \leqslant \lfloor |S|/2 \rfloor$ for every set S of odd cardinality. However, it turns out that even the odd-set constraints are not strong enough to guarantee feasibility.

Let $x_{u,v} = 1$ if (u,v) is an edge in the matching, 0 otherwise. Consider distinct nodes i,j,k such that j prefers k to i. For the matching to be stable, the following must be valid:

$$S(i,j,k) \equiv \frac{1}{2}\left(\sum_{l \leqslant_j i} x_{l,j} + \sum_{l \leqslant_k j} x_{l,k}\right) \leqslant \frac{1}{2},$$

since the above is dominated by the paired inequalities (2) in (P_{FSM}).

The above inequality can be extended to an odd cycle version. Suppose $i_0, i_1, \ldots, i_C$ (C even) are such that i_k prefers i_{k+1} to i_{k-1}, where the indices are taken modulo $(C+1)$. Then by adding up the above inequality, we have

$$\sum_{k=0}^{C} S(i_{k-1}, i_k, i_{k+1}) \leqslant \frac{C+1}{2}.$$

Note that the nodes in the cycle need *not* be distinct. The coefficients of all variables appearing in the left hand side are integral. Hence by rounding down the right hand side, we have the following *odd cyclic preference* inequality:

$$\sum_{k=0}^{C} S(i_{k-1}, i_k, i_{k+1}) \leqslant \left\lfloor \frac{C+1}{2} \right\rfloor.$$

Thus we obtain an improved formulation for the stable roommates problem:

(P_{SR})

$$\sum_j x_{i,j} = 1, \ \ \forall i, \tag{3}$$

$$\sum_{l:l<_j i} x_{l,j} + \sum_{l:l<_i j} x_{i,l} + x_{i,j} \leqslant 1, \ \ \forall i,j, \tag{4}$$

$$\sum_{k=0}^{C} S(i_{k-1}, i_k, i_{k+1}) \leqslant \left\lfloor \frac{C+1}{2} \right\rfloor,$$

$$i_{k-1} <_{i_k} i_{k+1}, \ \ k = 0, \ldots, C, \tag{5}$$

$$x_{i,j} \geqslant 0, \ \ \forall i,j, \tag{6}$$

where (P_{SR}) denotes the set of fractional solutions that satisfies constraints (3)–(6).

Teo and Sethuraman [14] showed the following theorem.

Theorem 2. *(P_{SR}) is non-empty if and only if the corresponding stable roommates problem is feasible.*

3. Cutting-plane heuristic

We propose an approach to construct good stable roommate solutions based on solving the resulting LP over the polytope (P_{SR}). Although the latter has an exponential number of constraints, we propose a heuristic which combines rounding and cutting-plane techniques to produce near-optimal solutions. In general, separating the odd cyclic preference inequalities involves computing $O(n^2)$ shortest-path solutions in an associated network with nodes corresponding to triplets (i,j,k) obtained from the roommates problem. This appears to be computationally prohibitive. In this section, we propose a more efficient routine to construct a stable roommate solution from the LP. The generic method is as follows.

Cutting-plane heuristic:

Step 1. Solve the LP over (P_{FSM}). If (P_{FSM}) is empty, the problem has no feasible stable roommates solution. Otherwise, proceed to Step 2.

Step 2. Round the fractional solution to obtain a partial matching and (possibly) a union of disjoint cycles.

Step 3. Test for odd cycles in an auxiliary graph associated with the solution obtained in Step 2. If no odd cycle is found, go to Step 4, else use the odd cycle to construct a violated inequality for the fractional solution. Add the violated inequality to the partial LP formulation and re-optimize. Repeat Step 2.

Step 4. If no odd cycle is found, round the current solution to obtain a stable roommates solution.

Consider the problem $\min\{cx\colon x \in (P_{\mathrm{SR}})\}$. Suppose the cost coefficients satisfy the following U-shape property:

for each i, there exists a person i_g such that
$c(i,j) < c(i,k)$ if $k <_i j <_i i_g$ or
$c(i,j) < c(i,k)$ if $i_g <_i j <_i k$.

In this case, we have the following theorem.

Theorem 3. *The cost of the solution obtained by the cutting-plane heuristic is within a factor of two of the optimal cost.*

We now elaborate on the above procedure and show that the method indeed produces a valid solution to the stable roommates problem. The proof of Theorem 3 would follow immediately from the way we construct the matching.

3.1. Solving (P_{FSM})

We solve the LP using CPLEX on a SUN SPARC 10 workstation. To speed up the computation, we first compute a feasible solution for the LP by reducing the problem associated with (P_{FSM}) to an appropriate stable marriage instance (described below); we use the Gale–Shapley algorithm to construct a solution easily. Let H' denote the set of nodes $\{i'\colon i \in H\}$, and let (H,H') be a complete equi-bipartite graph with $4n$ nodes (i.e. $2n$ on each partite set). Consider the following instance of the stable marriage problem. The sets H and H' assume the role of men and women respectively; the preference list of each node i in H over the set of nodes in H' is given by

$$k' <_i j' \text{ if and only if } c(i,j) < c(i,k).$$

Since $c(i,i')$ is not defined in the original stable roommates problem, we assume that $c(i,i') = \infty$. Hence i' is the worst partner for i in the stable marriage instance. Similarly, the preference list for each node i' in H' is given by

$$k <_{i'} j \text{ if and only if } c(i,j) < c(i,k).$$

A feasible stable marriage solution can be obtained in (H,H') easily using the Gale–Shapley's algorithm. However, we note that not all feasible marriage solution can be transformed to a feasible solution in (P_{FSM}). The main difficulty lies in the case when i is matched to i' by the Gale–Shapley algorithm. However, we claim that if this is the case, the original stable roommates problem has no feasible solution. To see this, first observe that since i' and i are matched to their worst possible partners, and since the Gale–Shapley algorithm produces a men-optimal solution, i must be matched to i' in every stable marriage solution. On the other hand, suppose H has a feasible stable roommates solution in which i is matched to j. By splitting (i,j) to (i,j') and (i',j), we obtain a stable marriage solution where i is not matched to i'. Thus if i is matched to i', we can conclude that the original stable roommates problem is infeasible. (The algorithm would have terminated in this case.) Otherwise, we recombine the stable marriage solution in the following way:

$$\begin{aligned} x(i,j) &= 1 && \text{if } i \leftrightarrow j', i' \leftrightarrow j;\\ x(i,j) &= 1/2 && \text{if } i \leftrightarrow j' \text{ or } i' \leftrightarrow j \text{ but not both};\\ x(i,j) &= 0 && \text{otherwise};\end{aligned}$$

and obtain a (possibly) half-integral solution to (P_{FSM}).

3.2. Rounding

For each node i, we arrange the $n-1$ intervals (left-open, right-closed), of lengths given by the $x_{i,j}$ for each $j \neq i$ to cover the interval (0,1], according to increasing preference of i. (We refer to this as the arrangement according to i.) Recall that $\sum_j x_{i,j} = 1$, and hence, these intervals, properly sequenced, cover the interval (0,1]. If i prefers j to k, then the interval corresponding to $x_{i,j}$ is placed to the right of that corresponding to $x_{i,k}$. Furthermore, each $x_{i,j}$ appears twice, once each in the arrangement according to i and j. See Fig. 1 for a geometrical representation of the above. Note also the geometrical implication of Lemma 1 with this arrangement.

If k is the index such that $x_{i,k}$, in the above arrangement covers the point 1/2 in the interval, then round $x_{i,k}$ to 1, and set $match[i] = k$, $prematch[k] = i$.

We construct a graph G with node set consisting of the set of roommates and a directed edge (i,j) if $x_{i,j}$ is rounded to 1 according to the arrangement by i.

Suppose for some $k \neq l$, $match[k] = i = match[l]$. By the way we round,

$$\sum_{p:p\leqslant_k i} x_{k,p} \geqslant 1/2, \qquad \sum_{p:p<_k i} x_{k,p} < 1/2;$$
$$\sum_{p:p\leqslant_l i} x_{l,p} \geqslant 1/2, \qquad \sum_{p:p<_l i} x_{l,p} < 1/2.$$

By Lemma 1, since

$$\sum_{p:p<_i k} x_{i,p} + \sum_{p:p\leqslant_k i} x_{k,p} = 1, \quad \sum_{p:p<_i l} x_{i,p} + \sum_{p:p\leqslant_l i} x_{l,p} = 1,$$

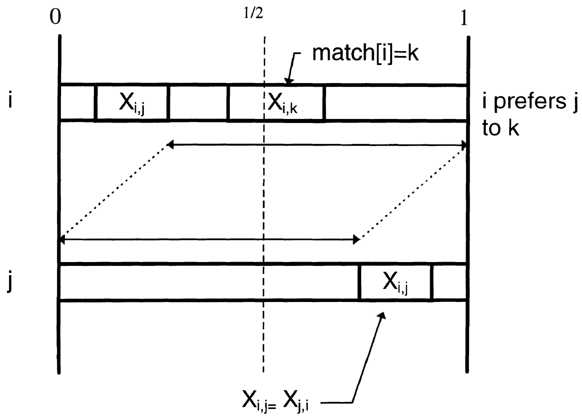

Fig. 1. Geometry of the fractional solution.

we have

$$\sum_{p:p<_i k} x_{i,p} \leqslant 1/2, \qquad \sum_{p:p\leqslant_i k} x_{i,p} > 1/2;$$
$$\sum_{p:p<_i l} x_{i,p} \leqslant 1/2, \qquad \sum_{p:p\leqslant_i l} x_{i,p} > 1/2,$$

hence $k = l$ which is a contradiction.

The above argument shows that, without loss of generality, we may assume that G consists of union of directed cycles and partial matching (i.e. when $match[i] = k$ and $match[k] = i$ for some nodes i, k).

3.3. Testing for violated inequalities

Suppose G contains an odd directed cycle $\mathscr{C} = (i_0, i_2, \ldots, i_C)$, where $match[i_k] = i_{k+1}$ (and $i_{C+1} = i_0$), then since

$$\sum_{p:p\leqslant_{i_{k-1}} i_k} x_{i_{k-1},p} \geqslant 1/2,$$

we have

$$\sum_{k=0}^{C} S(i_{k-1}, i_k, i_{k+1}) \geqslant \frac{C+1}{2},$$

which immediately gives rise to a violated odd cyclic preference inequality for the fractional solution. We add the cuts to the LP and re-optimize. Otherwise we proceed to identify more complicated odd cyclic preference inequalities from G. We add directed arcs (i, j) and (j, i) to the graph G if the following holds:

j prefers i to $match[j]$ but not $prematch[j]$; and

i prefers j to $match[i]$ but not $prematch[i]$.

Call the new graph G'. The set of edges in $E(G') \setminus E(G)$ is called the set of *obstructing pairs*. We now look for odd cycles in G' by a breadth-first search, and re-optimize the LP if we can find any. All odd cycles in G' would similarly give rise to violated odd cyclic preference inequalities for the fractional solution.

The main motivation for looking for odd cycles in G' arises in the proof of Theorem 2 (cf. Ref. [14]). Note also that the above procedure does not ensure that the fractional solution obtained from the LP satisfies *all* odd cyclic preference inequalities as our attention is only directed to those that can be obtained as odd cycles in G'. However, the next step shows that as long as G' does not contain any odd cycle, we can round the fractional solution to a feasible stable roommates solution.

3.4. Constructing a feasible stable roommates solution

Let x denote the fractional solution obtained after the previous step. The graph G' associated with x is then bipartite. The nodes fall into two separate sets A and B. Assign each node i in A to $match[i]$. (Equivalently, each node j in B is matched to $prematch[j]$). Note that the edge $(i, match[i])$ is in G'. Let M denote the perfect matching obtained in this way. We show that M is a feasible stable roommates solution.

Claim. *(i, j) is a blocking pair in M only if (i) (i, j) is an obstructing pair, and (ii) $\{i, j\} \subset A$.*

Proof. Let (i, j) be a blocking pair for the matching M. We prove the statement by considering several cases.

Case 1: Suppose $\{i, j\} \subset B$. By the definition of a blocking pair, $j >_i prematch(i)$ (the interval $x_{i,j}$ is on the right of interval $x_{i,prematch(i)}$ for node i, i.e., strictly in (1/2,1)), then by Lemma 1, for the arrangement by j, the subinterval spanned by $x_{i,j}$ lies strictly in (0,1/2). Hence $prematch(j) >_j i$ (interval $x_{j,prematch(j)}$ on the right of interval $x_{i,j}$ for node j). This contradicts the fact that (i, j) is a blocking pair.

Case 2: Suppose $i \in A, j \in B$. Then the matching M contains $(i, match(i)), (prematch(j), j)$. Now $i >_j prematch(j)$ implies $j <_i match(i)$ (by Lemma 1, and the same reasoning in Case 1). Hence (i, j) cannot be a blocking pair. Similarly, we can rule out the case when $i \in B, j \in A$.

The above two cases prove that $\{i, j\} \subset A$. So the matching M contains $(i, match(i))$, $(j, match(j))$. Since $\{i, j\}$ is a blocking pair, $j >_i match(i)$. If $j \geqslant_i prematch(i)$, then $i \leqslant_j match(j)$ by Lemma 1, contradicting the fact that (i, j) is a blocking pair. Hence we must have $prematch(i) >_i j >_i match(i)$. By symmetry, we also have *prematch*$(j) >_j i >_j$ *match*(j), i.e., (i,j) is an obstructing pair. □

However, by the construction of G', and since G' is bipartite, the set A does not contain any obstructing pair. We therefore conclude that there is no blocking pair in M, i.e. M is a stable roommates solution.

The proof of Theorem 3 then follows (see also Ref. [14]).

Proof of Theorem 3. Let (i,j) be an edge in M, and let i_k denote the index with the least cost among all $c(i,k)$ as k varies.

Suppose *match*$(i) = j$. In the arrangement according to i, the interval corresponding to $x^*_{i,j}$ spans $(a,b]$ where $a < 1/2$ and $b \geqslant 1/2$. Hence for the arrangement according to node j, $x^*_{i,j}$ spans $(1-b, 1-a]$ (by Lemma 1). If $i_k \leqslant_i j$, then since c satisfies the U-shape condition,

$$c(i,j) \times \frac{1}{2} \leqslant c(i,j)\left(\left(b - \frac{1}{2}\right) + \sum_{l:l>_i j} x^*_{i,l}\right) \leqslant c(i,j)x^*_{i,j} + \sum_{l:l>_i j} c(i,l)x^*_{i,l}.$$

On the other hand, if $i_k >_i j$, then

$$c(i,j) \times \frac{1}{2} \leqslant c(i,j)\left(\left(\frac{1}{2} - a\right) + \sum_{l:l<_i j} x^*_{i,l}\right) \leqslant c(i,j)x^*_{i,j} + \sum_{l:l<_i j} c(i,l)x^*_{i,l}.$$

Hence $c(i,j) \leqslant 2\sum_l c(i,l)x^*_{i,l}$.

Similarly, $c(j,i) \leqslant 2\sum_l c(j,l)x^*_{j,l}$. So we have $\sum_{(i,j)\in M}(c(i,j) + c(j,i)) \leqslant 2\sum(c(i,j) + c(j,i))x^*_{i,j}$ and the result follows. □

4. Computational experience

We have implemented and tested the procedure on various random instances of the problem. The priority of each person is generated randomly. $c(i,j)$ is uniformly distributed between 0 and 100. In this scheme, i prefers j to k if $c(i,j) < c(i,k)$. The cost of matching i to j is then $c(i,j) + c(j,i)$. We perform the experiments on problems with size upto 60 people. For each set of people, we perform the experiment on 500 random instances of the problem. Table 1 shows the performance of the algorithm.

The second and third columns give the number of instances where integral optimal solutions can be constructed directly by solving over (P_{FSM}) and (P_{SR}) respectively. The numbers in the brackets indicate the number of instances the formulations are feasible. The fourth and fifth columns list the means and standard deviations of the optimal solutions found by our technique. The last column lists the average performance of our rounding heuristic when integral solutions cannot be found by the LP.

We note that solving over (P_{FSM}) has limited applicability, because of the wide disparity between LP and IP feasibility. For example, in the case of 60 people, only 337 (out of 500) instances are IP feasible, whereas 495 (out of 500) are feasible for (P_{FSM}). Our cutting-plane heuristic solves 333 (out of 337) problems to optimality. This suggests that our approach is viable for problems of this size. In those few cases where an optimal solution could not be found, the rounding heuristic produces a near-optimal solution most of the time. Another interesting observation is that the standard deviation does not vary much with an

Table 1
Random instances

Size	(P_{FSM})	(P_{SR})	Mean	S.D.	Bound
10	405(460)	438(438)	275.57	56.58	–
20	348(465)	396(404)	414.94	60.48	1.05
30	311(480)	396(399)	517.11	65.73	1.06
40	280(490)	360(364)	602.20	63.99	1.02
50	270(497)	373(379)	680.10	63.33	1.02
60	246(495)	333(337)	744.39	63.16	1.01

increase in the problem size. But further testing is needed in order to validate this.

In order to better evaluate the rounding heuristic, we designed a second experiment, in which we generate only feasible stable roommate instances. This is achieved by fixing a matching (with costs $c(i,j) = 50$ for all (i,j) in the matching) and generating other costs in such a way that this matching is stable. This results in a drastic difference in performance of the LP based approach. The number of cuts needed to solve a problem in this class increases drastically. We perform the experiment on 100 such instances. Our results are tabulated in Table 2.

Here, the second and third columns list the number of instances which are solved to optimality by (P_{FSM}) and (P_{SR}) respectively. Again, we see that the cuts improve the formulation considerably. We have been able to solve more than 40% of these problems upto optimality. However, compared to the first experiment, the rounding heuristic does not perform as well. Among those that cannot be sloved to optimality, the average gap is close to 21% for the instances with size 60, and the maximum gap is about 37%.

5. Applications of the LP approach

Although feasibility issues of the stable roommates problem can be settled in $\mathrm{O}(n^2)$ time which is asymptotically optimal, we feel that it is still of interest to study the roommates problem from a polyhedral perspective; in part, because this approach allows us to use the the well-developed linear programming theory in elegant ways. It also allows us to unify the treatment of many of the known results. We have described various applications of this approach in our previous paper [14]. In this section, we describe further applications of the LP approach to the roommates problem.

First we look at the "minimum regret" version of the roommates problem. Here, in seeking a "minimum regret" stable matching, we try to make the person who is worst-off as happy as possible; a person's happiness is measured by the position of his roommate in his preference list. The regret of a person i, when matched to person j, is given by $r(i,j)$, where $r(i,j)$ indicates the rank of person j in i's list. It is well-known that the minimum regret version is solvable in $\mathrm{O}(n^2)$ (cf. Ref. [6]) which is asymptotically optimal. As an application of the LP approach, we show that this problem can be solved via a *single* LP.

The key tool needed is Theorem 3. Let

$$d(i,j) = r(i,j)^p,$$

where p is selected so that

$$r(i,j)^P > 4nr(k,l)^p, \text{ whenever } r(i,j) > r(k,l).$$

Theorem 4. *The solution obtained by the cutting plane heuristic for the problem*: $\min\{\mathrm{d}x : x \in (P_{\mathrm{SR}})\}$ *is the minimum regret stable roommates solution.*

In fact, the above theorem can be generalized further. For each person i, consider the following min–max deviation problem:

$$(\mathrm{MMD}) \ \min\left\{\max_i |r(i,j) - D(i)| : x_{ij} = 1\right\},$$

where $D(i)$ denote the desired rank of person i's roommate on his list, and the minimization is over all feasible vectors $\mathbf{x}$ that define a stable room-

Table 2
Feasible instances

Size	(P_{FSM})	(P_{SR})	Mean	S.D.	Avg/Max
10	35	92	472.18	31.74	1.09/1.17
20	7	71	915.78	72.58	1.10/1.19
30	1	60	1392.54	86.80	1.13/1.24
40	0	44	1900.86	91.16	1.16/1.30
50	0	44	2398.51	115.72	1.19/1.36
60	0	47	2811.09	123.50	1.21/1.37

mates solution. For the rest of this section, we will write (MMD) as an abbreviation for the min–max deviation problem.

The minimum regret problem can be reduced to the above, with $D(i) = 0$ for all i. We show next that (MMD) can also be solved in polynomial time via the LP approach.

Let

$$d(i,j) = |r(i,j) - D(i)|^p,$$

where p is selected so that

$$|r(i,j) - D(i)|^P > 4n|r(k,l) - D(k)|^p,$$

whenever

$$|r(i,j) - D(i)| > |r(k,l) - D(k)|.$$

Note that p is polynomially bounded, as we can set p to be $\log(4n)/\log((n-1)/(n-2)) \sim O(n\log(n))$. Furthermore, $d(i,j) > d(i,k)$ if and only if $|r(i,j) - D(i)| > |r(k,l) - D(k)|$, so that Theorem 3 holds for the cost function d.

Theorem 5. *The solution obtained by the cutting plane heuristic for the problem* $\min\{\mathrm{d}x{:}\, x \in (P_{\mathrm{SR}})\}$ *is an optimal stable roommates solution to* (MMD).

Proof. Suppose not. Let x^* (resp. x') be an optimal (resp. heuristic) solution to (MMD). Let (k,l) be a match with the maximum deviation in x^*, i.e. $|r(k,l) - D(k)|$ is maximum among all edges (i,j) with $x^*_{ij} = 1$. Furthermore, suppose $|r(i,j) - D(i)| > |r(k,l) - D(k)|$ for some (i,j) such that $x'(i,j) = 1$. Then $d(i,j) = |r(i,j) - D(i)|^P > 4n\,|r(k,l) - D(k)|^p = 4nd(k,l)$. In this case, $\mathrm{d}x' > d(i,j) > 4nd(k,j) > 2\mathrm{d}x^*$, which contradicts the fact that x' is within 2 times of the optimal solution for the problem $\min\{\mathrm{d}x{:}\, x \in (P_{\mathrm{SR}})\}$. □

A major issue in the study of the stable roommates problem is fairness. The egalitarian objective function attempts to optimize the total social welfare by minimizing the sum $\sum_i \sum_{j\neq i} r(i,j)x_{ij}$. The minimum regret model focused instead on the individual welfare, by attempting to minimize the maximum regret ($\max\{r(i,j)x_{ij}{:}\, i \neq j\}$) in the stable roommates solution. While the former is NP-hard, the latter is easy. In general, however, the choice of the ideal roommates solution should be based on a proper trade-off between the total welfare (egalitarian model) and the individual welfare (min-regret model). In the rest of this section, we propose an LP model to achieve this. Our model performs the trade-off of the two functions by using two parameters α and β as follows:

$$\min \quad \alpha\left(\sum_i \sum_{j\neq i} r(i,j)x_{ij}\right) + \beta\Psi$$

s.t.

$$\sum_j x_{i,j} = 1, \ \forall i, \tag{7}$$

$$\sum_{l<_j i} x_{l,j} + \sum_{l<_i j} x_{i,l} + x_{i,j} \leqslant 1, \ \forall i,j, \tag{8}$$

$$\sum_{j\neq i} r(i,j)x_{ij} \leqslant \Psi, \ \forall i, \tag{9}$$

$$\sum_{k=0}^{C} S(i_{k-1}, i_k, i_{k+1}) \leqslant \left\lfloor \frac{C+1}{2} \right\rfloor,$$

$$i_{k-1} <_{i_k} i_{k+1}, \ k = 0,\ldots,C, \tag{10}$$

$$x_{i,j} \geqslant 0, \ \forall i,j. \tag{11}$$

Let (P_{BEI}) denote the set of fractional solutions that satisfies constraints (7)–(11) in the above balanced egalitarian/individual formulation. By varying the parameters α and β, the above model allows us to find a stable roommates solution which has the right mix of the egalitarian and min-regret solutions.

This model is NP-hard, as it includes the egalitarian model as a special case. However, we show that the following procedure returns a near-optimal solution.

Step 1. Solve the LP over (P_{BEI}). If (P_{BEI}) is empty, the problem has no feasible stable roommates solution. Otherwise, proceed to Step 2.

Step 2. Round the fractional solution (as in Section 2) to obtain a stable roommate solution.

Theorem 6. *The solution obtained by the above procedure is a stable roommates solution whose cost is within twice the optimal cost.*

Proof of Theorem 5. Let M denote the stable matching obtained by the rounding procedure, and x^* the optimal fractional solution.

From the proof to Theorem 3, we have

$$\sum_{(i,j)\in M} (r(i,j) + r(j,i)) \leqslant 2\sum (r(i,j) + r(j,i))x^*_{i,j}.$$

This follows mainly from the fact that if $(i,j) \in M$, then the interval corresponding to $x^*_{i,j}$ spans $(a,b]$ where $a < 1/2$ and $b \geqslant 1/2$ or $a \leqslant 1/2$ and $b > 1/2$. This geometric property also implies

$$r(i,j) \leqslant 2\sum_{k\neq i} r(i,k)x^*_{i,k}.$$

Hence

$$\max_{(i,j)\in M} r(i,j) \leqslant 2\Psi,$$

and the result follows. □

In general, solving over (P_{BEI}) can be difficult due to the exponential number of odd cyclic constraints. Our heuristic can be easily adapted to handle this problem, by replacing its Step 1 by the following

Step 1(a). Solve the LP over (P_{BEI}), excluding the set of odd cyclic preference inequalities. If there is no fractional solution, the problem has no feasible stable roommates solution. Otherwise, proceed to Step 1(b).

Step 1(b). Round the fractional solution to obtain a partial matching and (possibly) a union of disjoint cycles.

Step 1(c). Test for odd cycles in an auxiliary graph associated with the solution obtained in Step 1(b). If no odd cycle is found, go to Step 2, else use the odd cycle to construct a violated inequality for the fractional solution. Add the violated inequalities to the partial LP formulation and re-optimize. Repeat Step 1(b).Again, we have the following theorem.

Theorem 7. *The solution obtained by the above cutting plane heuristic is a stable roommates solution with cost within two times of the optimal cost.*

6. Conclusions

In this paper, we study the stable roommates problem from a polyhedral perspective. We propose a reasonably efficient cutting plane heuristic for the classical NP-hard egalitarian stable roommates problem. Our computational results on randomly generated instances show that its typical performance bound is within 6% of the optimal for randomly generated instances. For a different class of problem instances, we show that the observed worst case gap between the LP relaxation and its optimal can be as large as 37%.

We describe how the LP approach can be used to generate near optimal solutions for a general stable roommates model that incorporates the egalitarian and min-regret objective function. For the pure min-regret version (and its generalization, the min–max deviation problem), we show that in fact the LP approach can be used to construct an optimal solution. We also propose a new LP model to address fairness issues in the roommates problem. Our approach gives rise to a 2-approximation algorithm for this problem.

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