Road Traffic Modeling with PDEs and Cellular Automata

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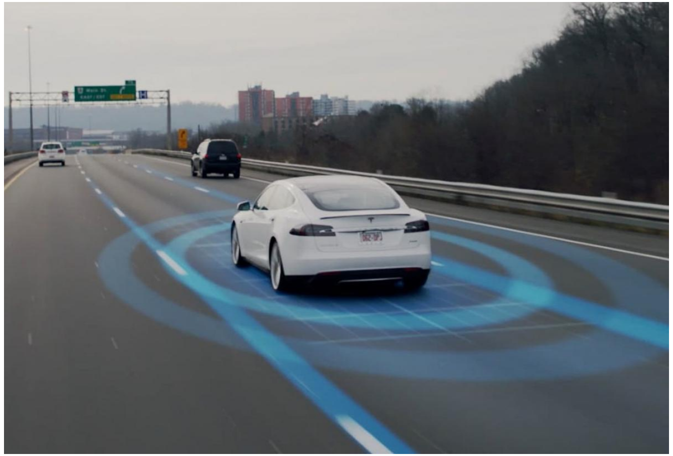
Outline

- Interest and Background
  - Traffic Inefficiencies, Autonomous Cars
- History of Highway Traffic Modeling
- Macroscopic Models
  - Intro to Conservation Laws, Characteristic Curves, the Riemann Problem
  - Lighthill-Whitham-Richards Model, Payne-Whitham, Aw-Rascle
  - Properties of Burger’s Equation, Shocks, Cole-Hopf Transformation
- Microscopic Models
  - Cellular Automata - Nagel, Schreckenberg
- Demo: Modeling a highway obstruction using cellular automata
- Future of the Field and Ourselves

Motivation and Background

- Traffic jams and accidents
- Autonomous driving R&D by Tesla, Google, Uber, etc.



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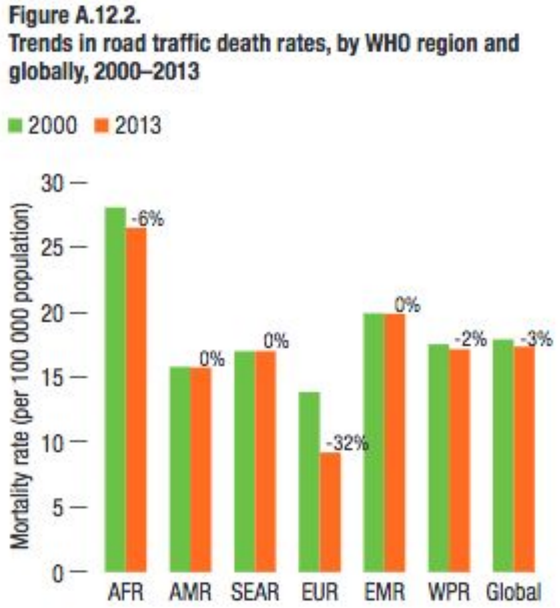


http://www.who.int/gho/publications/world_health_statistics/2016/whs2016_AnnexA_RoadTraffic.pdf?ua=1&ua=1

History of Highway Traffic Modeling

- 1955: Lighthill and Whitham apply hydrodynamic theory to traffic modelling; Richards introduces shocks soon after (collectively, the LWR Model)
- 1970s: Payne and Whitham (PW) add another equation to LWR, analogous to fluid momentum
- 1992: Nagel and Schreckenberg pioneer microscopic traffic model using cellular automata to represent individual cars
- 1993: Newell develops shortcut method for one-link hydrodynamic model
- 2000: Aw-Rascle Model modifies PW for more realistic behavior

Macroscopic Models

Conservation Laws

- Scalar Conservation Laws: Any system of the form

$$u_t + F(u)_x = 0$$

$u(x, t)$: a linear density (mass / length)

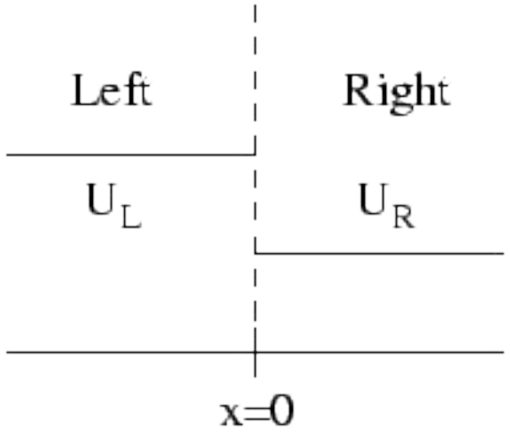
$F(u)$: flux, more generally equal to uv where v is velocity of flow

- Where does this equation come from? Where does it get its name?
- Conservation of mass, energy, cars, ….

Cooper Pg. 31 - 33

Sidenote: Riemann Problem

- The Riemann Problem: any conservation law problem along with initial data featuring a single discontinuity separating two constant states

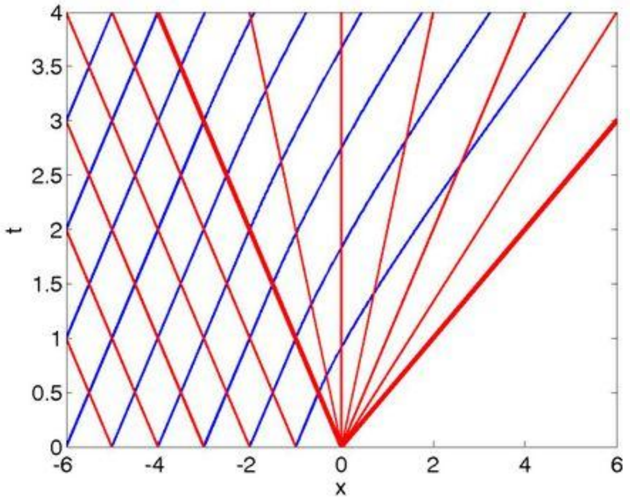


http://www.uv.es/astrorela/simulacionnumerica/img525.png

Characteristic Curves

- Definition: any solution *x(t)* to the equation

$$\frac{dx}{dt} = F'(u(x,t))$$



http://www.wikiwaves.org/files/thumb/0/0d/Characteristics_dam_break.jpg/500px-Characteristics_dam_break.jpg

Cooper Pg. 31 - 33

Characteristic Curves

- Important properties for scalar conservation laws:
  1) Along any characteristic curve *x(t)*:

     $$\frac{d}{dt}(u(x(t), t) = 0$$

  2) As a consequence, characteristics are of the form

     $$x(t) = x_0 + F'(u(x_0, 0))t$$

  3) From (2), we can say that information will travel at speed $F'(u(x_0, 0))$
     - E.g. any disturbance in *u(x, t)* will propagate at this speed

Cooper Pg. 31 - 33

Conservation Law Example: LWR Model

- Lighthill-Whitham-Richards Model: Linear "flow" of a lane of traffic on a highway

$$\rho_t + (\rho v)_x = 0$$

$$\rho(x) = \text{density of cars at position } x$$

$$v(x) = \text{velocity of cars at position } x$$

- If we are given a relationship $v = v(\rho)$, then we have a scalar conservation law in ρ

https://upload.wikimedia.org/wikipedia/commons/3/3e/I-80_Eastshore_Fwy.jpg

LeVeque Pg. 41 - 42

Conservation Law Example: LWR Model

- A simple example of a density-velocity relationship:

$$v(\rho) = v_{max}(1 - \rho/\rho_{max})$$

- Agrees with intuition
  - Unobstructed traffic: $\rho = 0 \implies v(\rho) = v_{max}$
  - As density increases: $\rho = \rho_{max} \implies v(\rho) = 0$

LeVeque Pg. 41 - 42

Payne-Whitham Model

- Proposed a two-equation traffic model comprised of LWR and another equation representing “momentum” conservation

$$\rho_t + (\rho v)_x = 0$$

$$v_t + v\, v_x = \frac{V(\rho) - v}{\tau} - \frac{(c_0^2 \rho)_x}{\rho}$$

Term	Meaning
$V(\rho)$	Equilibrium Speed
τ	Relaxation Time
$(V(\rho) - v)/\tau$	Relaxation

- Criticism: PW model flow is **isotropic**. Not realistic w.r.t. cars, because cars do not respond to speed of preceding cars - only those in front of them

Kachroo Pg. 34 - 37

Aw-Rascle Model

- Modification of PW to make traffic flow anisotropic:

$$\rho_t + (\rho v)_x = 0$$

$$[v + p(\rho)]_t + v[(v + p(\rho))]_x = \frac{V(\rho) - v}{\tau}$$

- The function $p(\rho)$ is akin to a fluid pressure, sometimes taken as

$$p(\rho) = c_0^2 \rho^\gamma$$

where $\gamma > 0$ and $c_0 = 1$

Kachroo Pg. 34 - 37

Burgers' Equation

- A mathematical model of the motion of a viscous compressible gas

$$u_t + uu_x = \nu u_{xx}$$

Where:

$u =$ speed of the gas

$\nu =$ kinematic viscosity

$x =$ spatial coordinate

$t =$ time

Solutions of Burgers' Equation

- When viscosity is not zero:

$$u(x,t) = \frac{u_R + u_L}{2} - \frac{u_L - u_R}{2} \tanh\left(\frac{(x - st)(u_L - u_R)}{4\nu} \right)$$

- When viscosity is zero (i.e. solution of inviscid Burgers' equation):

$$u_t + [\tfrac{1}{2}u^2]_x = 0$$

$$s = \frac{f(u_L) - f(u_R)}{u_L - u_R} = \frac{\text{jump in } f(u)}{\text{jump in } u}.$$

  - Observe that this is an example of a conservation law with $F(u) = \frac{1}{2}u^2$

Cameron Pg. 3

Weak Solutions of Burgers' Equation

Some definition of weak solution:

- a smooth function is a weak solution if and only if it solves the viscous Burger's equation
- weak solutions can be discontinuous
- discontinuous functions which satisfy the associated integral equation can be weak solutions
- We say u(x,t) is a weak solution of the conservation law with any φ(x, t) with compact support such that

$$\int_0^\infty \int_{-\infty}^\infty \phi_t u + \phi_x f(u) dx dt = 0$$

Weak solution with Riemann Problem:

Initial condition:

$$u(x,0) = \left\{ \begin{array}{ll} u_L & x < 0 \\ u_R & x > 0 \end{array} \right\}$$

Cameron Pg. 6

Characteristics of Inviscid Burgers' Equation

The characteristics of $u_t + [\frac{1}{2}u^2]_x = 0$ are given by:

$$\frac{dx}{dt} = u(x,t).$$

With the following solution:

$$x(t) = u(x(0),0)t + x(0) = u_0(x_0)t + x_0, \quad \text{where} \quad x_0 = x(0), \quad u_0(x) = u(x,0).$$

- the characteristics are straight lines
- they may intersect
- they do not necessarily cover the entire (x, t) space

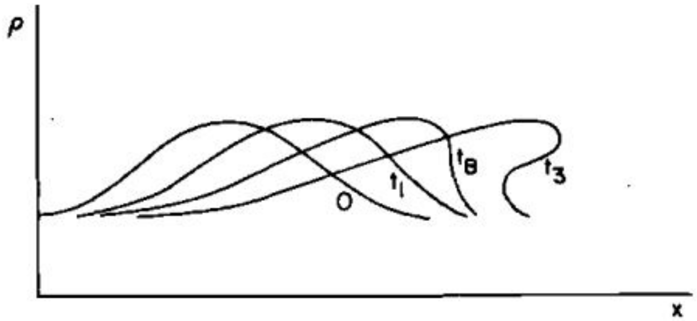
Shocks in Inviscid Burgers' and Wave Break Time

Consider two characteristic curves $x(t) = u_0(x_1)t + x_1,\ x(t) = u_0(x_2)t + x_2$

If the function $u_0'(x)$ is negative at some point, then Inviscid Burger's Equation will develop a shock (a point at which the function is multivalued) exactly at time

$$T_b = -\frac{1}{\min_{x\in\mathbb{R}} u_0'(x)}$$

- Shocks describe a very thin area with a rapid "break," and all weak solutions are discontinuous after this point
- Resulting discontinuity travels at a known speed



http://www.azimuthproject.org/azimuth/files/Shock_Formation_Burgers_Equation.png

Cole-Hopf Transformation (Diffusion Equation)

Transform Burgers' equation

$$u_t + uu_x = \nu u_{xx}$$

Into the heat equation

$$\phi_t = \nu\phi_{xx}.$$

Solution with infinite domain:

$$\phi(x,t) = \frac{1}{2(\pi\nu t)^{\frac{1}{2}}}\int_{-\infty}^{+\infty} \mathrm{e}^{\frac{-(x-\xi)^2}{4\nu t}} u_0(\xi)\,\mathrm{d}\xi.$$

Solution with finite domain:

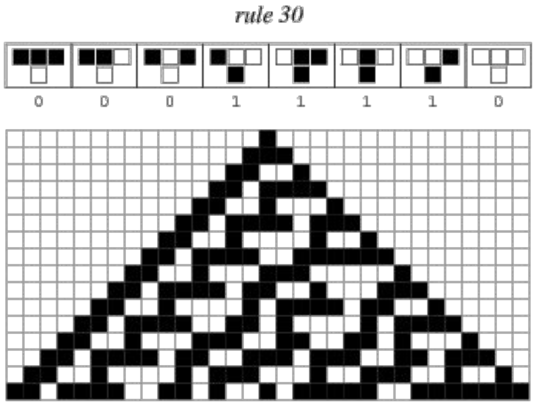
$$u(x,t) = \frac{2\nu\pi}{l}\frac{\sum_{n=1}^{\infty}\mathrm{e}^{\frac{-\nu n^2\pi^2}{l^2}t} nA_n \sin\left(\frac{n\pi x}{l}\right)}{A_0 + \sum_{n=1}^{\infty}\mathrm{e}^{\frac{-\nu n^2\pi^2}{l^2}t} A_n \cos\left(\frac{n\pi x}{l}\right)}$$

Gorguis Pg. 127 - 129

Microscopic Models

Cellular Automata

- A discrete model consisting of:
  - An array of cells in a finite number of states
  - A “rule” (i.e. update function) that determines the change in state for each cell after one time step
- Notable figures: Ulam, Von Neumann, Conway, Wolfram

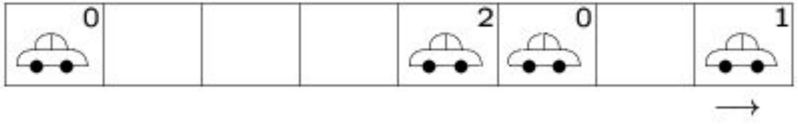


https://en.wikipedia.org/wiki/Cellular_automaton

http://mathworld.wolfram.com/images/eps-gif/ElementaryCARule030_700.gif

Nagel-Schreckenberg Model

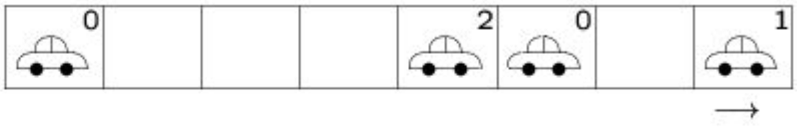
- Kai Nagel and Michael Schreckenberg at the University of Cologne, Germany
- “A Cellular Automaton Model for Freeway Traffic” (1992)
- Four-component stochastic cellular model in discrete time:
  - Acceleration
  - Reducing speed in reaction to cars in front
  - Randomization of velocity (i.e. a slowdown probability)
  - Car speed based on velocity (# of cells to advance a vehicle)
- One car per cell, velocity $v \in \{0, \ldots, 5\}$



http://www.thp.uni-koeln.de/~as/Mypage/konfig.gif

Nagel-Schreckenberg Model

- "It has been shown that a discrete model approach for traffic flow is not only computationally advantageous, but that it contains some of the important aspects of the fluid-dynamical approach to traffic flow such as the transition from laminar to start-stop traffic in a natural way… Thereby, it retains more elements of individual behavior of the driver."



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Nagel, et al. (1992)

Nagel-Schreckenberg Model

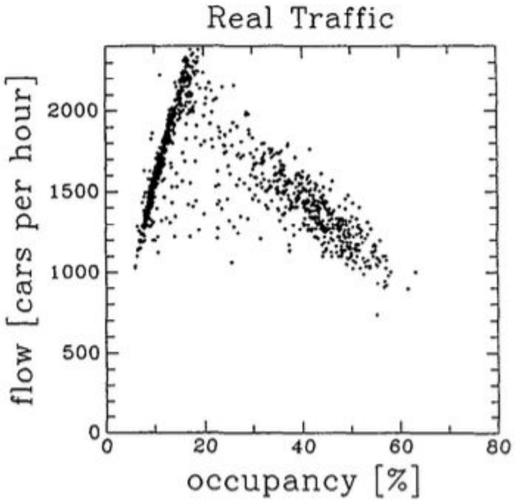

Fig.5. — Traffic flow q (in cars per hour) *vs.* occupancy (in cars per hour) from measurements in reality. Occupancy is the percentage of the road which is covered by vehicles (after [17]).

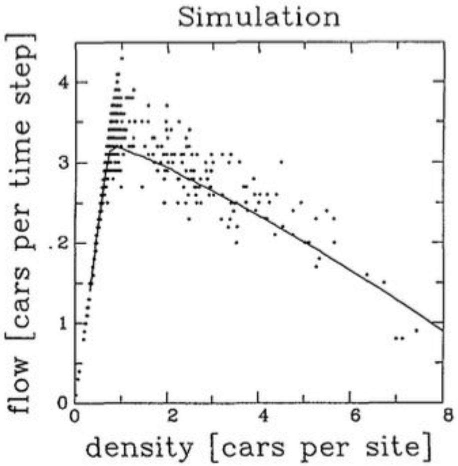

Fig.4. — Traffic flow q (in cars per time step) *vs.* density (in cars per site) from simulation results ($L = 10^4$). Dots are averages over 100 time steps, the line represents averages over 10^6 time steps.

Nagel, et al. (1992)

Demo

- https://github.com/morethanoneanimal/Nagel-Schreckenberg-simulation
  - A Python project with customizable simulations of n-lane road traffic with obstacles