ELEN 4810 Final Exam

Monday, December 15, 2014, 9 AM - 12 PM. Two sheets of handwritten notes are allowed. No electronics of any kind are allowed. Please record your answers in the exam booklet. Raise your hand if you need additional scratch paper. Good luck!

Name:

Uni:
1. **Z-transform poles and zeros.** Consider a rational transfer function

\[ H(z) = \frac{1 + z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})} \]  

(a) What are the poles and zeros of \( H \)? *Be careful about \( z \to 0 \).*

(b) Assuming that the system is *causal*, sketch the pole-zero diagram, ROC, and compute the impulse response \( h[n] \).

(c) Assuming that the DTFT \( H(e^{j\omega}) \) exists, sketch the pole-zero diagram, ROC, and compute the impulse response \( h[n] \).
**Answer to Problem 1.** You can sketch the pole-zero diagrams and ROC on the axes below:

(a) The poles are $z = 1/2$ and $z = 2$. The zeros are $z = -1$ and $z = 0$. To see that $z = 0$ is a zero, either evaluate $\lim_{z \to 0} H(z)$, or note that

$$H(z) = \frac{z(z + 1)}{(z - 1/2)(z - 2)}.$$  

Before answering (b)-(c), we perform partial fraction expansion to write

$$H(z) = \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - 2z^{-1}},$$

with

$$A_1 = \frac{1 + z^{-1}}{1 - 2z^{-1}} \bigg|_{z=1/2} = \frac{1 + \frac{1}{2}}{1 - \frac{1}{2} \times \frac{1}{2}} = -1,$$

and

$$A_2 = \frac{1 + z^{-1}}{1 - \frac{1}{2}z^{-1}} \bigg|_{z=2} = \frac{1 + \frac{1}{2}}{1 - \frac{1}{2} \times \frac{1}{2}} = \frac{3/2}{3/4} = 2.$$  

Thus,

$$H(z) = -\frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{2}{1 - 2z^{-1}}.$$  

(b) If $h[n]$ is causal, ROC extends outward from largest magnitude pole: $\text{ROC} \{h\} = \{z \mid |z| > 2\}$. 
Inverting each of the pieces of \( H(z) \), we obtain

\[
h[n] = -\left(\frac{1}{2}\right)^n u[n] + 2^nu[n].
\]

(c) If \( H(e^{j\omega}) \) exists, the ROC contains the unit circle. Thus ROC\(\{h\} = \{z \mid 1/2 < |z| < 2\}. \) Inverting each of the pieces of \( H(z) \) under this assumption, we obtain

\[
h[n] = -\left(\frac{1}{2}\right)^n u[n] - 2^nu[-n - 1].
\]
2. Interpretation of LTI Systems. An LTI system is applied to a superposition of windowed sinusoids $x[n]$, shown below. Plotted below are the magnitude of $X(e^{j\omega})$, and the magnitude response, phase response, and group delay of the system.

Sketch the output $y[n]$. Explain your sketch in terms of the magnitude response, phase response and group delay.
**Answer to Problem 2.** The plot below shows the actual output of the system applied to $x[n]$:

![Plot](image)

Correct answers will approximate (to some extent) this output, noting that

- The packet with frequency $0.8\pi$ (highest frequency) is almost completely attenuated.
- The packet with frequency $0.4\pi$ (medium frequency) is scaled by $\approx 6.5$, and is not delayed ($\text{grd} \approx 0$).
- The packet with frequency $0.2\pi$ (low frequency) is scaled by $\approx 4$, and is delayed by $\approx 145$ samples.
3. **Bilinear Transform and Min-Phase All-Pass Decomposition.** Suppose we design a discrete-time IIR system by applying bilinear transformation to a continuous-time system $H_c(s)$ to produce a discrete-time system $H(z)$. The pole-zero diagram of $H_c(s)$ is shown below:

![Pole-zero diagram](image)

*Note: $H_c(s)$ does not have any poles or zeros at $\infty$.*

(a) Sketch the pole-zero diagram of $H(z)$.

(b) Is $h[n]$ real? Briefly justify.

(c) Can the discrete-time system be both causal and stable? Briefly justify.

(d) Now, suppose that we write $H(z) = H_{\text{min}}(z)H_{\text{ap}}(z)$ with $H_{\text{min}}$ minimum phase and $H_{\text{ap}}$ all-pass. Sketch the pole-zero diagrams of $H_{\text{min}}$ and $H_{\text{ap}}$. 

Answer to Problem 3. For the continuous time system $H_c(s)$, we have poles at $s = 1 + j$ and $s = -3/2$, and zeros at $s = -2$ and $s = 1/2$. Using the relationship $z = \frac{s+1}{1-s}$, the corresponding poles and zeros of $H(z)$ are

- $s = 1 + j \mapsto z = -1 + 2j$
- $s = -2 \mapsto z = -1/3$
- $s = -3/2 \mapsto z = -1/5$
- $s = 1/2 \mapsto z = 3.$

So, $H(z)$ has poles at $z = -1 + 2j$ and $z = -1/5$, and zeros at $z = -1/3$ and $z = 3$.

(b) No. If $h[n]$ is real, the non-real poles must occur in complex conjugate pairs. $-1 + 2j$ is a pole, but $-1 - 2j$ is not.

(c) No. If $h[n]$ is causal, the ROC extends outward from the largest magnitude pole – here ROC$\{h\} = \{z \mid \|z\| > \sqrt{5}\}$. This set does not include the unit circle, and so if the system is causal, it cannot be stable.

(d) The min-phase / all-pass decomposition “reflects” the poles and zeros outside the unit circle to their conjugate reciprocals within the unit circle. The minimum phase component $H_{\min}$ has poles at $z = \frac{1}{-1-2j}$ and $z = -1/5$ and has zeros at $z = -1/3$ and $z = 1/3$. The all-pass component $H_{ap}$ has poles at $z = -1 + 2j$ and $z = 1/3$ and zeros at $z = \frac{1}{-1-2j}$ and $z = 3$. The pole zero diagrams should also be sketched on the axes below.

Answer to Problem 3, continued.
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Sketch for part (a)

Poles and zeros of $H_{\text{min}}$

Poles and zeros of $H_{\text{ap}}$

Sketches for part (d)
4. Spectrogram and Sampling. A sinusoidal signal \( x_c(t) = \cos(\Omega t) \) is sampled with period \( T = 2 \) to produce a discrete-time signal \( x[n], n = 0, \ldots, 9,999 \). We compute the Short-Time Fourier Transform \( X[r, k] \) of \( x \), using a window \( w[n] \) of length 20, a step size \( R = 10 \), a DFT length \( N = 256 \). Below, we plot \( |X[r, k]| \) for \( k = 0, \ldots, 128 \), with two different choices of window \( w \).

(a) Suppose we are told that one of the above spectrograms was created using a rectangular window \( w_r[m] \) while one was created using a Hamming window \( w_h[m] \). Which spectrogram was created using the rectangular window? Which was created using the Hamming window? Briefly justify your answer.

(b) Estimate the original frequency \( \Omega \). Is this the only possible \( \Omega \) that could have generated this \( X[r, k] \)?
**Answer to Problem 4.** The following code generated the problem:

Omega = pi / 8; % true frequency
T = 2; % sampling period
nVec = 0:9999;
x = cos( nVec * T * Omega ); % sampled version

SR = spectrogram(x,rectwin(20)); % spectrogram with rectangular window
SH = spectrogram(x,hamming(20)); % spectrogram with Hamming window

% display SR on the right, SH on the left:
figure(1);
subplot(1,2,1);
imagesc(0:1:998,0:1:128,(abs(SH)));
xlabel('Time r');
ylabel('Discrete Frequency k');
set(gca,'YDir','normal');
title('|X[r,k]|, darker is larger');
subplot(1,2,2);
imagesc(0:1:998,0:1:128,(abs(SR)));
xlabel('Time r');
ylabel('Discrete Frequency k');
set(gca,'YDir','normal');
colormap('gray');
title('|X[r,k]|, darker is larger');
colormap(flipud(colormap));

(a) The left display uses the Hamming window; the right display uses the rectangular window. The Hamming window has a wider mainlobe, but much smaller sidelobes. The rectangular window has a narrower mainlobe, but its sidelobes decay very slowly, leading to the oscillatory patterns seen at right.

(b) Per the above code, the original frequency $\Omega$ was $\pi/8$. To estimate this, note that both spectrograms have a peak around $k \approx 32$. This corresponds to a discrete frequency $\omega = 2\pi k/256 = \pi/4$. The corresponding continuous frequency is $\Omega = \omega/T = \pi/8$. **It is ok if your estimate of the discrete frequency $\omega$ is not precise, as long as it is reasonably close and the methodology for going back to $\Omega$ is correct.**

No, the answer is not unique. Any $\Omega' = \Omega + \pi\ell$ for $\ell \in \mathbb{Z}$ will produce the same observation.
5. **Design of an FIR notch filter.** Consider a continuous-time signal \( x(t) \) which is a superposition of a desired signal \( x_0(t) \) and noise \( e(t) \):

\[
x(t) = x_0(t) + e(t).
\]

We assume that the “noise” \( e(t) \) is sinusoidal:

\[
e(t) = \cos(2\pi f_e t + \phi),
\]

where \( f_e = 100 \text{ kHz} \), and phase \( \phi \in [0, 2\pi) \). We obtain samples, at frequency \( f_s = 1.2 \text{ MHz} \):

\[
x[n] = x \left( \frac{n}{f_s} \right) = x_0 \left( \frac{n}{f_s} \right) + e \left( \frac{n}{f_s} \right) = x_0[n] + e[n].
\]

We wish to design a filter that will remove the noise component \( e[n] \), by choosing a system function

\[
H(z) = \frac{(z - z_1)(z - z_2)}{z^2},
\]

with impulse response \( h[n] \). Our goal is to choose the zeros \( z_1 \) and \( z_2 \) such that \( h * e = 0 \), so \( h * x = h * x_0 \).

(a) How should we choose \( z_1, z_2 \)?

(b) What is the impulse response \( h[n] \)?

(c) Does the resulting system have generalized linear phase?
**Answer to Problem 5.** First note that

\[ e[n] = \cos \left( 2\pi \frac{f_c}{f_s} n + \phi \right) = \cos \left( \frac{\pi}{6} n + \phi \right). \]

The discrete-time Fourier transform of \( e \) is

\[ E(e^{j\omega}) = \pi e^{j\phi} \delta \left( \omega - \frac{\pi}{6} \right) + \pi e^{-j\phi} \delta \left( \omega + \frac{\pi}{6} \right) \quad -\pi < \omega \leq \pi. \]

If we ensure that \( H(e^{j\pi/6}) = 0 \) and \( H(e^{-j\pi/6}) = 0 \), then \( h * e = 0 \). So, we position the zeros at \( z = e^{j\pi/6} \) and \( z = e^{-j\pi/6} \):

(a) We choose \( z_1 = e^{j\pi/6}, \ z_2 = e^{-j\pi/6} \).

(b) Dividing through by \( z^2 \), the system function is

\[ H(z) = 1 - (z_1 + z_2)z^{-1} + z_1z_2z^{-2} = 1 - 2\cos(\pi/6)z^{-1} + z^{-2}. \]

This corresponds to an impulse response

\[ h[n] = \begin{cases} 1 & n = 0 \\ -\sqrt{3} & n = 1 \\ 1 & n = 2 \\ 0 & \text{else} \end{cases} \]

(c) Yes. The system is a Type I Generalized Linear Phase system (symmetric, \( M \) even).
Scratch paper:
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