ELEN 4810 Final Exam

Monday, December 21, 2015, 9 AM - 12 PM. Two sheets of handwritten notes are allowed. No electronics of any kind are allowed. Please record your answers in the exam booklet. Raise your hand if you need additional scratch paper. Good luck!

Name:

Uni:
1. The \textbf{Z-transform}. Consider a rational system function

\[ H(z) = \frac{z^{-2}}{(1 - 2z^{-1})(1 - jz^{-1})(1 + jz^{-1})} \]  \hspace{1cm} (1)

(a) What are the poles and zeros of \( H \)?

(b) Could this be a stable system? Could the impulse response \( h[n] \) be real-valued?

(c) Assuming that system is causal, specify the Region of Convergence (ROC) and determine the impulse response \( h[n] \).
**Answer to Problem 1.**

(a) \( H(z) \) has poles at \( z = 2, z = j, z = -j \). It has a zero of multiplicity 2 at \( z = \infty \); it also has a zero at \( z = 0 \).

(b) No, it cannot be stable. It has poles on the unit circle, and so the region of convergence cannot contain the unit circle. Yes, it is real-valued: all poles and zeros that have non-zero imaginary part occur in complex conjugate pairs.

(c) We use partial fraction expansion to simplify \( H(z) \). We write

\[
H(z) = \frac{A}{1-2z^{-1}} + \frac{B}{1-jz^{-1}} + \frac{C}{1+jz^{-1}},
\]

with

\[
A = \left. \frac{[(1-2z^{-1})H(z)]}{z=2} \right.
\]

\[
= \frac{1}{(1-j/2)(1+j/2)}
\]

\[
= \frac{1}{4}\frac{1}{1+1/4}
\]

\[
= \frac{1}{5}
\]

and

\[
B = \left. \frac{[(1-jz^{-1})H(z)]}{z=j} \right.
\]

\[
= \frac{1}{(1+2j) \times 2}
\]

\[
= -\frac{1}{2+4j}
\]

and

\[
C = \left. \frac{[(1+jz^{-1})H(z)]}{z=-j} \right.
\]

\[
= \frac{1}{(1-2j) \times 2}
\]

\[
= -\frac{1}{2-4j}
\]

Inverting term-by-term, we obtain

\[
h[n] = A \times 2^n u[n] + Bj^n u[n] + C(-j)^n u[n]
\]

\[
= u[n] \left( \frac{2^n}{5} - \frac{j^n}{2+4j} - \frac{(-j)^n}{2-4j} \right).
\]

[[Note: this is indeed real-valued, as can be seen by noticing that the final two terms are complex conjugates of each other.]]
2. **Pole-Zero Diagrams.** Consider the following list of filter types:

A. Lowpass filter  
B. Highpass filter  
C. Bandpass filter  
D. Allpass filter

and the following list of system properties:

1. FIR system  
2. IIR system  
3. Stable and causal system  
4. Minimum phase system  
5. FIR, symmetric: \( h[n] = h[M - n] \).  
6. FIR, antisymmetric: \( h[n] = -h[M - n] \).

For each of the following pole-zero diagrams, indicate which of the filter types A-D best describes the system, and which, if any, of the system properties 1-6 apply. *You do not need to justify your answer. For example, for an IIR low-pass filter, you can simply write “A, 2”.  

(a).

(b).
Answer to Problem 2. (a) D, 2, 3
(b) A, 1, 3, 5
(c) C, 2, 3
3. Linear Systems and STFT. Recall the definition of the Short-Time Fourier Transform

\[ X[r, k] = \text{DFT}_N \{w[n]x[n + rR]\}[k] \]

We compute \(X[r, k]\) for a signal \(x[n]\), with the following parameters: \(N = 256\), \(R = 2\), and \(w[n]\) a Hamming window of length 64. Below, we show the magnitude \(|X[r, k]|\) for \(r = 0, \ldots, 268\) and \(k = 0, \ldots, 128\).

![Magnitude Spectrogram](image)

In the above figure, the three largest values of \(|X[r, k]|\) occur at (i) \(r = 21, k = 64\), (ii) \(r = 96, k = 32\), and (iii) \(r = 171, k = 96\). At each of these peaks, \(|X[r, k]| = 1\).

The signal \(x[n]\) is passed through a linear, time-invariant system with frequency response \(H(e^{j\omega})\) to produce an output \(y[n]\). The magnitude response \(|H(e^{j\omega})|\) and group delay \(\text{grd}H(e^{j\omega})\) are shown below:

![Magnitude and Group Delay](image)

We compute the Short-Time Fourier Transform \(Y[r, k]\), with the same parameters. On the axis below, please sketch the spectrogram \(|Y[r, k]|\). Please label the approximate location \((r, k)\), and height \(|Y[r, k]|\) of any peaks in the spectrogram.

[Please briefly explain the reasoning behind your sketch.]
Answer to Problem 3. Please sketch your answer below:

Solution: we include the true output of the system below:

The reasoning leading to this answer is as follows: the high frequency component at \( r \approx 171 \) in the input is centered about discrete frequency \( k = 96 \). This corresponds to \( \omega = 2\pi k/N = 3\pi/4 \). The magnitude response is zero at \( \omega = 3\pi/4 \), and so this component is almost completely attenuated.
The low-frequency component at \( r = 96, k = 32 \) corresponds to frequency \( \omega = 2\pi k/N = \pi/4 \). \(|H(e^{j\pi/4})| \approx 3\), and so this component will be amplified by a factor of \( \approx 3 \). The group delay at \( \pi/4 \) is very close to zero, and so this component will occur at roughly \( r = 96 \) in the output.

Finally, the medium frequency component at \( r = 21, k = 64 \) corresponds to frequency \( \omega = \pi/2 \). The magnitude response at this point is roughly 1.5, and so this component will be amplified by roughly 1.5. The group delay is roughly 150 samples, and so this component will be delayed by \( n = 150 \) samples. Since the spectrogram step is \( R = 2 \), this corresponds to a delay in \( r = n/R \) by roughly 75. This component will occur at \( k = 64 \) and \( r \approx 21 + 75 = 91 \) in the output.

[[Note: minor imprecisions in the estimates of these quantities are fine, as long as the reasoning is correct.]]
4. **Inverse Systems.** Consider an LTI system with impulse response $h[n] = \delta[n] + 3\delta[n - 1]$. Let $H(e^{j\omega})$ denote its frequency response.

Recall that an inverse system for $h$ with impulse response $h_i[n]$ satisfies $h_i \ast (h \ast x)[n] = x[n]$ for any input $x$ with $\text{ROC}(x) \cap \text{ROC}(h) \cap \text{ROC}(h_i) \neq \emptyset$, and $\text{ROC}(h) \cap \text{ROC}(h_i) \neq \emptyset$.

(a) Find the impulse response $h_i[n]$ of a stable inverse system for $h[n]$. Is your answer causal?

(b) Find an impulse response $g[n]$ of a system whose magnitude response is the same as $|H(e^{j\omega})|$ (i.e., $|G(e^{j\omega})| = |H(e^{j\omega})|$ for all $\omega$), but which has a causal, stable inverse system.

(c) Now suppose, more generally, that $h[n] = \delta[n] + \beta\delta[n - 1]$, with $\beta \in \mathbb{C}$. For what choices of $\beta$ does the system have a causal, stable inverse system?
Answer to Problem 4. (a) Our system has

\[ H(z) = 1 + 3z^{-1}. \]  

(15)

An inverse system will have Z transform

\[ H_i(z) = \frac{1}{1 + 3z^{-1}} \]  

(16)

This system has a pole at \( z = -3 \). There are two possible ROC: \( |z| < 3 \) and \( |z| > 3 \). Only \( |z| < 3 \) contains the unit circle, and so only \( |z| < 3 \) corresponds to a stable system. With this ROC, the inverse Z transform of \( H_i(z) \) is

\[ h_i[n] = -(-3)^n u[-n - 1]. \]  

(17)

The ROC goes inward, and so this system is not causal.

(b) Notice that \( H(z) = 1 + 3z^{-1} \) has a zero at \( z = -3 \). Because of this, it does not possess a causal, stable inverse. To obtain a system with the same magnitude response which does have a causal, stable inverse, we perform minimum phase all-pass decomposition. We write

\[ H(z) = \underbrace{\frac{1 + 3z^{-1}}{1 + \frac{1}{3}z^{-1}}}_{\text{Min phase}} \quad \text{All-pass, } |H_{ap}(e^{j\omega})| = \frac{1}{3} \forall \omega \]

\[ = \underbrace{\frac{1 + 3z^{-1}}{3 + z^{-1}}}_{G(z)} \quad \text{All-pass, } |H_{ap}(e^{j\omega})| = 1 \forall \omega \]  

(18)

(19)

Notice that \( |G(e^{j\omega})| = |H(e^{j\omega})| \) for all \( \omega \), and that

\[ g[n] = 3\delta[n] + \delta[n - 1]. \]  

(20)

(c) In this more general setting, \( H(z) \) has all of its zeros strictly inside the unit circle whenever \( |\beta| < 1 \). \( H(z) \) has a causal, stable inverse if and only if all of its zeros are strictly inside the unit circle.
5. Filter Design. We design a lowpass filter, with passband cutoff $\omega_p = 0.25\pi$ and stopband $\omega_s = 0.35\pi$.

(a) We design a Type I Generalized Linear Phase FIR filter using the Parks-McClellan algorithm. The impulse response $h[n]$ is nonzero for $0 \leq n \leq M$, with $M = 8$. One of the four pictures below shows

$$A(e^{j\omega}) = H(e^{j\omega})e^{j\omega M/2},$$

for the filter that we designed. Which one is it? [Please briefly explain your answer].

(b) In part (a), we designed a Type I Generalized Linear Phase FIR filter. Why should we not use a Type III Generalized Linear Phase FIR filter for this task?

(c) List one potential advantage and one potential disadvantage of our design, compared to a low-pass elliptic filter designed via bilinear transformation. You may assume that the elliptic filter will be designed to have minimum order, for the same $\omega_p$, $\omega_s$, and a tolerance $\delta_p = \delta_s = \delta$ which matches that of the FIR filter designed in part (a).
**Answer to Problem 5.** (a) The Parks McClellan algorithm designs $A(e^{j\omega})$ to match the desired magnitude response over the interval $0 \leq \omega \leq \pi$. To determine which of the four possibilities is optimal, we count alternations of each over this interval. The correct design should possess $L + 2$ or $L + 3$ alternations, with $L = M/2 = 4$. So, we are looking for either 6 or 7 alternations.

$A_1$ has 4 alternations over $[0, \pi]$. $A_2$ has 5 alternations over $[0, \pi]$. $A_3$ has 6 alternations over $[0, \pi]$. Finally, $A_4$ has 8 alternations over $[0, \pi]$. **The only possibility is $A_3(e^{j\omega})$.**

(b) A Type III system has a canonical zero at $z = 1$. This means that $H(e^{j\omega})$ will be 0 at $\omega = 0$. This is not appropriate for a lowpass filter!

(c) Our FIR filter is designed to have generalized linear phase; an elliptic filter does not. However, the elliptic filter will have lower order, and hence be more efficiently implementable.

(a) Suppose we wish to design a *minimum phase* system $H(z)$ by performing bilinear transformation on a continuous-time system with rational transfer function $H_c(s)$. What properties should we ensure that the poles and zeros of $H_c(s)$ satisfy, in order to guarantee that $H(z)$ will be minimum phase?

(b) Suppose that we perform bilinear transformation, to convert a continuous time system with transfer function $H_c(s) = 1/s$ (an *integrator*) to a discrete time system with system function $H(z)$. What are the poles and zeros of $H(z)$?

(c) What is the difference equation relating the input $x$ and output $y$ of the discrete time system $H(z)$ designed in part (b)?
Answer to Problem 6. (a) A minimum phase $H(z)$ has all poles and zeros strictly inside the unit circle (i.e., $|z| < 1$). When we perform bilinear transformation, every point $z$ inside the unit circle is the image of some point $s$ in the left half plane. Thus $H(z)$ is minimum phase if and only if all poles and zeros of $H_c(s)$ satisfy $\Re[s] < 0$.

(b) Under bilinear transformation, we have

$$H(z) = \frac{z - 1}{z + 1} \quad \text{(21)}$$

$$= \frac{z + 1}{z - 1}. \quad \text{(22)}$$

This system has a pole at $z = 1$ and a zero at $z = -1$.

(c) We have $Y(z) = H(z)X(z)$, and so

$$Y(z)(z - 1) = X(z)(z + 1), \quad \text{(23)}$$

Since $zX(z)$ is the $Z$ transform of $x[n + 1]$, we have

$$y[n + 1] - y[n] = x[n + 1] + x[n] \quad \text{(24)}$$

If we want to put this in a slightly cleaner form, we can rewrite it as

$$y[n] = y[n - 1] + x[n] + x[n - 1] \quad \text{(25)}$$
Scratch paper:
Scratch paper: