Supplementary Notes for ELEN 4810 Lecture 1
DT Signals + A Brief Review of Complex Arithmetic

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Disclaimer: These notes are intended to be an accessible introduction to the subject, with no pretense at completeness. In general, you can find more thorough discussions in the textbook. Please let me know if you find any typos.

Reading suggestions: Oppenheim and Schafer Chapter 1 and Section 2.1

Homework: HW0 is out. It should be very easy!

Optional (nongraded) homework suggestion: If you are not familiar with Matlab, you may wish to take a few minutes to work through Yenson’s introduction (available on the course website). There is also an introduction available through the website for the Oppenheim and Schafer book.

1 Discrete-Time Signals

Throughout the course, we use
\[ Z = \{ \ldots, -2, -1, 0, 1, 2, 3, \ldots \} \]  (1.1)

to denote the integers. Note that \( Z \) includes negative numbers. We use \( \mathbb{R} \) for the real numbers, and \( \mathbb{C} \) for the complex numbers. We will review the complex numbers in detail below.

A discrete-time signal is a sequence of numbers \( x[n] \), one for each integer \( n \in \mathbb{Z} \):
\[ \ldots, x[-2], x[-1], x[0], x[1], x[2], \ldots \]  (1.2)

The textbook occasionally uses the notation
\[ \{x[n]\}_{n \in \mathbb{Z}} \]  (1.3)

to refer to the sequence as a whole; this notation is a bit cumbersome, but for consistency we will adopt it. Typically, we simply refer to “the sequence \( x[n] \),” in the same way that we might refer to “the function \( f(t) \).”
Figure 1: A discrete-time signal $x[n]$. We can visualize a discrete-time signal via a stem plot, as in Figure 1. At each integer $n$, we have a value $x[n]$. We usually refer to the argument $n$ as “time,” although it could have other physical meanings.\(^1\)

We will always use square brackets $x[n]$ for a discrete-time signal. When we talk about continuous time signals

$$f(t) \quad t \in \mathbb{R},$$

we will always use curvy parenthesis “(·)” It is important to note that a discrete time signal $x[n]$ is only defined for $n \in \mathbb{Z}$. In Figure 1, we don’t imagine that the signal $x$ takes on zero values off of the integers – the value $x[1/2]$ is simply not defined.\(^2\)

Obtaining Discrete-Time Signals from Continuous-Time Signals. Many discrete-time signals $x[n]$ that are useful for applications can be viewed as sampled versions of a continuous time signal $x_a(t)$.\(^3\) That is to say, we set

$$x[n] = x_a(nT) \quad n \in \mathbb{Z}. \quad (1.5)$$

Here, $T$ is the sampling period, with units of time (e.g., seconds). The sampling frequency is $f = 1/T$, with units of frequency (e.g., Hz). Figure 2 shows a continuous time signal, and its sampled version.

Often we are interested in processing the sampled signal $x[n]$ to learn something about the continuous signal $x_a(t)$. So, it is very important to understand the relationship between the two. In

\(^1\)Imagine, for example, generating a measurement $x[n]$ by measuring the temperature at various points along a straight line. Here, $n$ really means distance. However, when we talk in general about one-dimensional discrete-time signals, we prefer to refer to $n$ as time.

\(^2\)Actually, discrete-time signals are much simpler mathematical objects than continuous-time signals. In undergraduate signals and systems, you were probably given cryptic warnings about the fact that the Dirac delta $\delta(t)$ is not a function, but rather a functional/distribution. Most conceptual difficulties of this nature vanish when we work in discrete time.

\(^3\)Here, the $a$ in $x_a(t)$ stands for analog.
some situations – for example, when \(x_a(t)\) is bandlimited\(^4\) – it is possible to exactly reconstruct \(x_a(t)\) from the samples \(x[n]\). Understanding when and how this is possible will be a major topic in the course.

**What is \(x[n]\)?**  Thus far, we have been vague about the nature of the values \(x[n]\) – we simply told you that they are “numbers”. Depending on the application, \(x[n]\) might be modeled as real numbers \(x[n] \in \mathbb{R}\), or they might be quantized to live in some smaller set \(x[n] \in \mathbb{Q}\). For example, many popular image representation formats use eight bits per pixel, and the signal is comprised of numbers \(x[n] \in \mathbb{Q} = \{0, 1, 2, \ldots, 255\}\). Thus, the range of \(x[n]\) is somewhat application-dependent. However, for developing analytical tools and insights, it is often very convenient to assume that our signal \(x[n]\) is comprised of complex numbers \(x[n] \in \mathbb{C}\).

## 2 Complex Arithmetic in Signal Processing

In signal processing, we often work over the complex numbers, rather than the reals. This may seem somewhat strange at first glance, since the mathematical models that we make for the world are often real-valued. For example, when we talk about sound, we talk about air pressure as a function of time. Air pressure can be plausibly modeled as a single real number. When we talk about images, we are talking about the number of photons that hit a given photosensitive surface, per unit time, as a function of space. Again, the number of photons hitting in a given time is an integer; the rate at which photons impinge on the surface can be plausibly modeled as a real number.

So, why then, are complex numbers so prevalent in the analysis of signal processing systems? The answer is simple: complex numbers provide a convenient system for making calculations – in particular, for making calculations with sinusoids. Sinusoids are important for human perception

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\(^4\)A *bandlimited* signal is one whose Fourier transform vanishes outside of some interval \([-\Omega, \Omega]\) – put informally, “the signal only contains frequencies below \(\Omega\.” We will review properties of the Fourier transform beginning in Lecture 3.
our ears are naturally attuned to distinguish oscillations at different frequency. They also play a very fundamental mathematical role, as the eigenfunctions of linear time-invariant systems.

Much of the utility of complex numbers derives from the magic of Euler’s formula,

\[ e^{j\theta} = \cos(\theta) + j\sin(\theta), \]

which we will re-derive below. This allows us to compute with sinusoids in a very convenient way, by using complex exponentials. Before reminding you of this relationship, we briefly recall the properties of the complex numbers.

### 2.1 The complex numbers

The complex numbers \( \mathbb{C} \) consist of pairs \( z = (z_r, z_i) \), which are sometimes referred to as the real and imaginary parts. Following the convention in electrical engineering, we reserve the symbol \( j \) for the imaginary unit (or square root of \(-1\)). We often write the pair \( z = (z_r, z_i) \) as

\[ z = z_r + jz_i. \]

The complex number \( z = z_r + jz_i \) can be visualized as a two dimensional vector, whose horizontal component is \( z_r \) and whose vertical component is \( z_i \).

It is useful to have notation for the operators that extract the real and imaginary parts of the complex number \( z \). For \( z = z_r + jz_i \), we write

\[ \text{Re}[z] = z_r, \quad \text{Im}[z] = z_i. \]

The complex numbers form a field – that is to say, we can add them and multiply them in ways that conform to our intuition. In particular, if \( w = w_r + jw_i \in \mathbb{C} \) and \( z = z_r + jz_i \in \mathbb{C} \), then the sum \( w + z \) is simply

\[ w + z = (w_r + z_r) + j(w_i + z_i). \]
In terms of the vector picture, we simply add the vectors \( w \) and \( z \) tip-to-tail.

Complex multiplication works in a similarly intuitive way, as long as we remember that \( j^2 = -1 \). Namely,

\[
    wz = (w_r + jw_i)(z_r + jz_i) \\
    = w_r z_r + jw_r z_i + jw_i z_r + j^2 w_i z_i \\
    = (w_r z_r - w_i z_i) + j(w_r z_i + w_i z_r). 
\]

The length \( |z| = \sqrt{z_r^2 + z_i^2} \) of the vector \((z_r, z_i)\), is sometimes known as the magnitude or modulus of the complex number \( C \), and plays a very important role in studying sums of many complex numbers.

### 2.2 Infinite sums and convergence

If we have a sequence of \( k \) complex numbers \( z_1, z_2, \ldots, z_k \), we can sum them:

\[
    z_1 + z_2 + \cdots + z_k. \tag{2.6}
\]

It is sometimes more convenient to write this as

\[
    z_1 + z_2 + \cdots + z_k = \sum_{i=1}^{k} z_i. \tag{2.7}
\]

If the sequence \( z_1, z_2, \ldots, z_i, \ldots \) is defined for every integer \( i > 0 \), we can define the partial summation

\[
    S_k = \sum_{i=1}^{k} z_i. \tag{2.8}
\]

The infinite summation

\[
    \sum_{i=1}^{\infty} z_i = \lim_{k \to \infty} S_k, \tag{2.9}
\]

is defined as the limit of the partial summations, *whenever the limit exists*.\(^5\)

Similarly, if \( z_i \) is defined for every \( i \in \mathbb{Z} \), we can define a two-sided partial summation

\[
    S_{k,k'} = \sum_{i=-k}^{k'} z_i. \tag{2.10}
\]

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\(^5\)It is easy to make examples for which the limit does not exist. Consider, e.g.,

\[
    z_i = \begin{cases} 
        1 & i \text{ even}, \\
        -1 & i \text{ odd}. 
    \end{cases}
\]

Then

\[
    S_k = \begin{cases} 
        0 & k \text{ even}, \\
        -1 & k \text{ odd}, 
    \end{cases}
\]

the limit does not exist, and \( \sum_{i=1}^{\infty} z_i \) is not defined. These kinds of technicalities become important when we start to analyze signals in transform domains, such as the (discrete-time) Fourier domain, because the transforms of interest are defined via infinite summations. If the infinite summation is not defined, the transform is not defined for that input signal.
The infinite summation
\[ \sum_{i=-\infty}^{\infty} z_i \] (2.11)
is defined whenever we can interchange taking the limit with respect to \( k \) and taking the limit with respect to \( k' \):
\[ \lim_{k \to \infty} \lim_{k' \to \infty} S_{k,k'} = \lim_{k' \to \infty} \lim_{k \to \infty} S_{k,k'} . \] (2.12)
Whenever these two limits exist and are equal, we define \( \sum_{i=-\infty}^{\infty} z_i \) to be equal to the common value.

A simple sufficient condition to ensure that the limit exists is to check that the sequence is absolute summable:
\[ \sum_{i=-\infty}^{\infty} |z_i| < +\infty , \] (2.13)
then \( \sum_{i=-\infty}^{\infty} z_i \) is well-defined. (Ex: show this.)

### 2.3 Euler’s formula

The exponential of a complex number \( x \in \mathbb{C} \) can be defined through the power series
\[ \exp(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!} . \] (2.14)
It is not too difficult to show that the infinite summation above is convergent for any \( x \in \mathbb{C} \).

**Theorem 2.1.** For any \( \theta \in \mathbb{R} \), \( e^{j \theta} = \cos(\theta) + j \sin(\theta) \).

**Proof.** To understand why this is true, we recall the power series representations \( \cos \) and \( \sin \). Namely, for any \( t \in \mathbb{R} \),
\[ \cos(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{(2k)!} . \] (2.15)
and
\[ \sin(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+1}}{(2k+1)!} . \] (2.16)
Plugging in \( t = j \theta \) to (2.14), and recalling that \( j^2 = -1 \), we see that
\[ \exp(j \theta) = 1 + (j \theta) + \frac{(j \theta)^2}{2!} + \frac{(j \theta)^3}{3!} + \frac{(j \theta)^4}{4!} + \frac{(j \theta)^5}{5!} + \frac{(j \theta)^6}{6!} + \cdots \]
\[ = 1 + j\theta - \frac{\theta^2}{2!} - j \frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j \frac{\theta^5}{5!} - \frac{\theta^6}{6!} + \cdots \]
\[ = \left( 1 - \frac{\theta^2}{2} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots \right) + \left( j \theta - j \frac{\theta^3}{3!} + j \frac{\theta^5}{5!} + \cdots \right) \]
\[ = \cos(\theta) + j \sin(\theta) , \] (2.17)
as desired.

2.4 Polar representation

Using Euler’s formula, we can form a “polar” representation of the complex number \( z = z_r + jz_i \). The magnitude is

\[
|z| = \sqrt{z_r^2 + z_i^2}.
\]  
(2.18)

In the vector picture in Figure 3, the magnitude is the length of the vector \((z_r, z_i)\). The phase is

\[
\angle(z) = \tan^{-1}(z_i/z_r).
\]  
(2.19)

This is the angle between the vector \((z_r, z_i)\) and the horizontal axis. In terms of these quantities, the polar representation is

\[
z = |z|e^{j\angle(z)}.
\]  
(2.20)

The polar form is very convenient for multiplying complex exponentials. Given \( z = |z|e^{j\angle(z)} \) and \( w = |w|e^{j\angle(w)} \), the product \( zw \) is simply

\[
zw = |z||w|e^{j(\angle(z) + \angle(w))}.
\]  
(2.21)

That is to say,

\[
|zw| = |z||w|,
\]  
(2.22)

and

\[
\angle(zw) = \angle(z) + \angle(w).
\]  
(2.23)

When multiplying complex numbers, magnitudes multiply and phases add.

2.5 Complex conjugation

The conjugate of a complex number \( z = z_r + jz_i \in \mathbb{C} \) is given by

\[
z^* = z_r - jz_i.
\]  
(2.24)

That is to say, we flip the sign of the imaginary part. Pictorially, we flip the vector \((z_r, z_i)\) about the horizontal axis. It is also easy to check that after conjugation, the polar representation \( z = |z|e^{j\theta(z)} \) becomes

\[
z^* = |z|e^{-j\angle(z)}.
\]  
(2.25)

That is to say conjugation does not change the magnitude, but multiplies the phase by \(-1\):

\[
\angle(z^*) = -\angle(z).
\]  
(2.26)

From (2.25), we can see that

\[
zz^* = |z|^2,
\]  
(2.27)

or equivalently, \( |z| = \sqrt{zz^*} \). We can also notice that the product of a complex number \( z \) and its conjugate \( z^* \) is always real.
Similarly, the sum of $z$ and $z^*$ is always real:

$$z + z^* = z_r + jz_i + z_r - jz_i. \tag{2.28}$$

From this, we get

$$\text{Re}[z] = \frac{z + z^*}{2}, \tag{2.29}$$

and

$$\text{Im}[z] = \frac{z - z^*}{2j}. \tag{2.30}$$

If any of the above material on complex numbers is rusty, please take some time to review it. The textbook contains numerous worked problems with answers, which you can use to check your understanding.