In this lecture, we discuss the \( \mathcal{Z} \)-transform, a powerful generalization of the discrete-time Fourier transform. In our study of sampling, as well as some of the application in examples in the homework, we’ve gotten substantial mileage out of the DTFT. The DTFT provides an excellent tool for studying sampling and reconstruction of continuous time signals, and for understanding the properties of stable LTI systems. The reason is that a stable LTI system has absolute summable impulse response (\( \sum_n |h[n]| < +\infty \)), and so the DTFT \( H(e^{j\omega}) \) exists, and even satisfies several stronger properties such as continuity.

However, for unstable systems, or systems which are not known to be stable ahead of time, the DTFT is less appropriate – it may not even exist! In this lecture, we study a powerful generalization of the DTFT, known as the \( \mathcal{Z} \)-transform. The \( \mathcal{Z} \)-transform allows us to work with many systems which may not be known to be stable ahead of time. We will see that even if the system turns out to be stable, the \( \mathcal{Z} \)-transform can give additional insight into how our design decisions affect the structure of \( H(e^{j\omega}) \).

A word of warning: the \( \mathcal{Z} \) transform is a powerful tool, but it raises some technical issues that are more subtle than anything else we see in this class. We will try to tackle these clearly and carefully in the lecture; please feel free to ask questions!

1 Definition and Examples

Consider the following function \( X(z) \) of a complex variable \( z \in \mathbb{C} \):

\[
X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}.
\] (1.1)
The function $X(z)$ is well-defined for any $z$ for which the summation in (1.1) converges. A sufficient condition for $X(z)$ to exist is that the series $x[n]z^{-n}$ is absolute summable, i.e.,

$$
\sum_{n=-\infty}^{\infty} |x[n]z^{-n}| < +\infty.
$$

(1.2)

We make a special notation for set of $z \in \mathbb{C}$ for which (1.2) holds: we call this the region of convergence (ROC). Formally:

$$
\text{ROC} \{x\} = \left\{ z \in \mathbb{C} \mid \sum_{n=-\infty}^{\infty} |x[n]z^{-n}| < +\infty \right\}.
$$

(1.3)

The $Z$-transform of a sequence $x[n]$ is defined as the pair $(X, \text{ROC} \{x\})$, where $\text{ROC} \{x\} \subseteq \mathbb{C}$ and $X : \text{ROC} \{x\} \rightarrow \mathbb{C}$. That is to say, the $Z$-transform consists of both the function $X(z)$, and the region of convergence. In notation:

$$
x \xleftarrow{Z} X(z), \text{ROC} \{x\}.
$$

(1.4)

A first example. Consider $x[n] = u[n]$. Notice that

$$
\sum_{n=-\infty}^{\infty} |u[n]z^{-n}| = \sum_{n=0}^{\infty} (|z|^{-1})^n = \begin{cases} \frac{1}{1-|z|^{-1}} & |z| > 1 \\ +\infty & \text{else} \end{cases}
$$

(1.5)

Thus, $\text{ROC} \{x\} = \{ z \mid |z| > 1 \}$. The ROC consists of the portion of the complex plane that lies strictly outside the unit circle. Moreover, for $z$ in the ROC,

$$
\sum_{n=-\infty}^{\infty} u[n]z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}},
$$

and so we have

$$
u[n] \xleftarrow{Z} \frac{1}{1-z^{-1}}, \; |z| > 1.
$$

(1.7)

Note I. **The ROC is rotationally symmetric.** The condition (1.2) for the ROC $\{x\}$ depends only on the magnitude of $z$:

$$
\sum_{n=-\infty}^{\infty} |x[n]z^{-n}| = \sum_{n=-\infty}^{\infty} |x[n]| |z|^{-n},
$$

(1.8)

and so if $z$ is in the ROC, every $z'$ satisfying $|z'| = |z|$ is also in the ROC. Put another way: the ROC is invariant under rotation around the origin of the complex plane.

Note II (a technicality): **ROC vs. “Region where the summation converges”**. The ROC as defined in (1.3) is not the same as

"The region on which the summation $\sum_{n=-\infty}^{\infty} x[n]z^{-n}$ converges."

(1.9)
Absolute summability is a sufficient condition for the summation to converge, but it is not a necessary one. To make matters even more confusing, some sources take (1.9) as the definition of the region of convergence. We (and the text) prefer (1.3), because it produces simpler regions, agrees with common usage in signal processing, and because absolute convergence implies a wealth of additional good properties of \( X(z) \) on \( \text{ROC} \{ x \} \).

**Note III** (another technicality): **does the ROC contain \( \infty \)?** The ROC as defined in (1.3) is a subset of the complex numbers \( \mathbb{C} \). It is typically useful to extend it to the “extended complex numbers” \( \bar{\mathbb{C}} = \mathbb{C} \cup \{ \infty \} \), by asserting that \( \infty \in \text{ROC} \{ x \} \) if and only if
\[
\lim_{|z| \to \infty} \sum_{n=-\infty}^{\infty} |x[n]z^{-n}| < +\infty.
\] (1.15)

For example, consider
\[
x[n] = \begin{cases} 
1 & n = 1, \\
0 & \text{else}.
\end{cases}
\] (1.16)

Clearly, \( X(z) = \sum_{n} x[n]z^{-n} = z^{-1} \). Notice that for any \( z \neq 0 \), \( \sum_{n} |x[n]z^{-n}| = |z|^{-1} < +\infty \). So, the region of convergence is the entire complex plane, except for \( z = 0 \). Moreover, \( \lim_{|z| \to \infty} |z|^{-1} = 0 \), and so we say that the region of convergence also contains \( \infty \).

In contrast, now consider what happens if
\[
x[n] = \begin{cases} 
1 & n = -1, \\
0 & \text{else}.
\end{cases}
\] (1.17)

In this case, \( X(z) = \sum_{n} x[n]z^{-n} = z \). Now, the ROC is the entire complex plane. However, \( \lim_{|z| \to \infty} |z| = +\infty \), and so \( \infty \) is not in the ROC: the ROC contains all of the extended complex numbers, except for \( \infty \).

**Because the summation** \( \sum_{n=-\infty}^{\infty} z^{-n} \) **contains terms of the form** \( z \) **and** \( z^{-1} \), **we have to take special care with** \( z = \infty \) **and** \( z = 0 \) **in determining the ROC.**

---

\(^1\)For example, if we take
\[
x[n] = \begin{cases} 
\frac{1}{n} & n \geq 1, \\
0 & \text{else}
\end{cases}
\] (1.10)

it is not difficult to show that
\[
\text{ROC} \{ x \} = \{ z \mid |z| > 1 \}.
\] (1.11)

However, if \( z = -1 \),
\[
\sum_{n=-\infty}^{\infty} x[n]z^{-n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}
\] (1.12)
\[
= \sum_{n=1,3,5,\ldots} \frac{-1}{n} + \frac{1}{n+1}
\] (1.13)
\[
= \sum_{n=1,3,5,\ldots} \frac{-1}{n(n+1)}.
\] (1.14)

This summation converges to a finite value. So, \( z = -1 \) is in the set defined by (1.9), but is not in \( \text{ROC} \{ x \} \), as defined in (1.3).
2 Relationship to the DTFT

If the region of convergence contains the unit circle \( \{ e^{j\omega} \mid \omega \in \mathbb{R} \} \), then

\[
X(z)\big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \quad (2.1)
\]

is finite if and only if \( |az^{-1}| < 1 \), which occurs iff \( |z| > |a| \). So

\[
X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a} \quad (3.3)
\]

Thus,

\[
x[n] = a^n u[n] \leftrightarrow \frac{z}{z - a}, \quad |z| > |a| \quad (3.4)
\]

Notice that the ROC of \( x \) contains the unit circle if and only if \( |a| < 1 \), in which case evaluating \( X(z) \) around the unit circle reproduces the familiar DTFT relationship

\[
x[n] = a^n u[n] \quad (|a| < 1) \leftrightarrow \text{DTFT} \quad \frac{1}{1 - ae^{-j\omega}}. \quad (3.5)
\]

When \( |a| \geq 1 \), the DTFT does not exist, but the \( Z \)-transform does exist (within its region of convergence).

3 Examples

Example I. For \( a \in \mathbb{C} \), take \( x[n] = a^n u[n] \). Let us first determine ROC \{x\}. Note that

\[
\sum_{n=-\infty}^{\infty} |x[n]z^{-n}| = \sum_{n=0}^{\infty} |az^{-1}|^n \quad (3.1)
\]

is finite if and only if \( |az^{-1}| < 1 \), which occurs iff \( |z| > |a| \). So

\[
\text{ROC } \{x\} = \{ z \in \mathbb{C} \mid |z| > |a| \}. \quad (3.2)
\]

In this case, the ROC contains \( z = +\infty \). For \( z \in \text{ROC } \{x\} \), evaluating the geometric summation \( \sum_{n=0}^{\infty} (az^{-1})^n \), we obtain that

\[
X(z) = \frac{1}{1 - az^{-1}} \quad (3.3)
\]

Thus,

\[
x[n] = a^n u[n] \leftrightarrow \frac{z}{z - a}, \quad |z| > |a|. \quad (3.4)
\]

Notice that the ROC of \( x \) contains the unit circle if and only if \( |a| < 1 \), in which case evaluating \( X(z) \) around the unit circle reproduces the familiar DTFT relationship

\[
x[n] = a^n u[n] \quad (|a| < 1) \leftrightarrow \text{DTFT} \quad \frac{1}{1 - ae^{-j\omega}}. \quad (3.5)
\]

When \( |a| \geq 1 \), the DTFT does not exist, but the \( Z \)-transform does exist (within its region of convergence).

Example II. Now let us again take \( a \in \mathbb{C} \), and

\[
x[n] = \begin{cases} 
-a^n & n \leq -1 \\
0 & \text{else}
\end{cases} \quad (3.6)
\]

\[
= -a^n u[-n - 1]. \quad (3.7)
\]
What is \( \text{ROC} \{x\} \)? We calculate

\[
\sum_{n=-\infty}^{\infty} |x[n]z^{-n}| = \sum_{n=-\infty}^{-1} |a^n z^{-n}| = \sum_{n=-\infty}^{-1} |az^{-1}|n
\]

which is finite if and only if \(|a^{-1}z| < 1\). Hence \(z \in \text{ROC} \{x\}\) if and only if \(|z| < |a|\). For \(z \in \text{ROC} \{x\}\),

\[
X(z) = -\sum_{n=1}^{\infty} (a^{-1}z)^n
\]

Thus,

\[
x[n] = -a^n u[-n-1] \quad \longleftrightarrow \quad \frac{z}{z-a}, \quad |z| < |a|.
\]

The ROC contains the unit circle if and only if \(|a| > 1\), in which case we reproduce another familiar DTFT relationship:

\[
x[n] = -a^n u[-n-1], (|a| > 1) \quad \longleftrightarrow \quad \frac{1}{1-ae^{-j\omega}}.
\]

The important observation from these two examples is that the functional form of the \(Z\) transform \(X(z)\) was exactly the same – the only difference was the region of convergence. To fully specify the \(Z\) transform, we need to specify both \(X(z)\) and \(\text{ROC} \{x\}\).

Example III. Let us now take \(x[n] = \begin{cases} a^n & 0 \leq n < N \\ 0 & \text{else} \end{cases}\), where \(N \in \mathbb{Z}\). Then

\[
\text{ROC} \{x\} = \{z \mid |z| > 0\}.
\]

Again, the ROC contains \(\infty\). For \(z \in \text{ROC} \{x\}\),

\[
X(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \begin{cases} \frac{1-(az^{-1})^N}{1-az^{-1}} & az^{-1} \neq 1 \\ N & az^{-1} = 1. \end{cases}
\]
We sometimes rewrite the first case as
\[ X(z) = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z - a}. \] (3.19)
This is a rational function of \( z \). It is worth noting that in the previous two examples, \( X(z) \) was a rational function as well.

4 Properties of the ROC

We have seen that the region of convergence
\[ \text{ROC} \{x\} = \left\{ z \in \mathbb{C} \left| \sum_{n=-\infty}^{\infty} |x[n]z^{-n}| < +\infty \right. \right\} \] (4.1)
is crucial in specifying the \( Z \)-transform of a signal \( x \). We next study the properties of \( \text{ROC} \{x\} \) for general sequences \( x \), without making strong assumptions on the functional form of \( X(z) \). In the next section, we will see that for rational \( X(z) \), much more can be said.

- **Connectivity.** \( \text{ROC} \{x\} \) is connected: if \( r = |z| \) and \( z \in \text{ROC} \{x\} \), and \( r' = |z'| \) for \( z' \in \text{ROC} \{x\} \), then for any \( r'' \in [r, r'] \), \( r'' \in \text{ROC} \{x\} \).

  **Proof.** We can quickly check this by calculating:
  \[
  \sum_{n=-\infty}^{\infty} |x[n]r''^{-n}| = \sum_{n=-\infty}^{0} |x[n]| |r''|^{-n} + \sum_{n=1}^{\infty} |x[n]| |r''|^{-n} \leq \sum_{n=-\infty}^{0} |x[n]| |r'|^{-n} + \sum_{n=1}^{\infty} |x[n]| |r|^{-n} \leq \sum_{n=-\infty}^{\infty} |x[n]|z'^{-n} + \sum_{n=-\infty}^{\infty} |x[n]|z^{-n} < +\infty. \] (4.2)

- **Finite duration sequences.** If \( x[n] \) has finite duration - i.e., \( x[n] = 0 \) for \( n < N_1 \) and \( x[n] = 0 \) for \( n > N_2 \), for some \( N_1, N_2 \), then \( \text{ROC} \{x\} \) is the entire complex plane \( \mathbb{C} \), with the possible exception of 0 and \( \infty \). Convince yourself that 0 \( \in \text{ROC} \{x\} \) if and only if \( x[n] = 0 \) for \( n > 0 \), and that \( \infty \in \text{ROC} \{x\} \) if and only if \( x[n] = 0 \) for \( n < 0 \).

- **Right-sided sequences.** If \( x[n] \) is right sided - i.e., \( x[n] = 0 \) for \( n < N \), for some \( N \in \mathbb{Z} \), then \( \text{ROC} \{x\} \) extends outward: if \( z \in \text{ROC} \{x\} \), then \( z' \in \text{ROC} \{x\} \) for any \( z' \in \mathbb{C} \) such that \( |z'| \geq |z| \).
Proof.

\[
\sum_{n=-\infty}^{\infty} |x[n] z'^{-n}| = \sum_{n=N}^{0} |x[n]| |z'|^{-n} + \sum_{n=1}^{\infty} |x[n]| |z'|^{-n} \quad (4.6)
\]

\[
\leq \sum_{n=N}^{0} |x[n]| |z'|^{-n} + \sum_{n=1}^{\infty} |x[n]| z^{-n} \quad (4.7)
\]

\[
< +\infty. \quad (4.8)
\]

For \( z' = \infty \), we have to be a little more careful: we also have to check that \( N \geq 0 \). In particular, if \( x[n] \) is a causal sequence, the ROC extends outward; if it is nonempty, it includes \( \infty \).

- **Left-sided sequences.** If \( x[n] \) is left-sided – i.e., \( x[n] = 0 \) for \( n > N \), for some \( N \in \mathbb{Z} \), the ROC \( \{x\} \) extends inward: if \( z \in \text{ROC} \{x\} \), then \( z' \in \text{ROC} \{x\} \) for any \( z' \) such that \( 0 < |z'| \leq |z| \).

Proof.

\[
\sum_{n=-\infty}^{\infty} |x[n] z'^{-n}| = \sum_{n=-\infty}^{0} |x[n]| |z'|^{-n} + \sum_{n=1}^{N} |x[n]| z'^{-n} \quad (4.9)
\]

\[
\leq \sum_{n=-\infty}^{0} |x[n]| |z'|^{-n} + \sum_{n=1}^{N} |x[n]| z'^{-n} \quad (4.10)
\]

\[
< +\infty. \quad (4.11)
\]

Here, we have to take special care with \( z = 0 \). If \( N \leq 0 \), then the ROC extends inward to zero; if \( x[n] \neq 0 \) for some \( n > 0 \), then the ROC does not contain \( z = 0 \).

5 **Rational \( X(z) \); poles and zeros**

In all of our examples up to this point \( X(z) \) has been rational function on the region of convergence:

\[
X(z) = \frac{P(z)}{Q(z)} \quad (5.1)
\]

with \( P \) and \( Q \) polynomials in \( z \). In fact, practically all of the \( Z \) transforms of our interest here are rational – including \( Z \) transforms arising from common sequences, and \( Z \) transforms arising in the solution of difference equations. If \( X(z) \) is rational, we can deploy the fundamental theorem of algebra, and factor the numerator \( P(z) \) and denominator \( Q(z) \), giving

\[
X(z) = \alpha \frac{\prod_{i=1}^{d} (z - \zeta_i)}{\prod_{k=1}^{d'} (z - \rho_k)}, \quad (5.2)
\]
where \( \alpha \in \mathbb{C}, d = \deg(P), d' = \deg(Q), \) the \( \zeta_i \) are the roots of \( P \), and the \( \rho_j \) are the roots of \( Q \). In particular, knowing the roots of \( P \) and \( Q \) (with multiplicity) tells us the function \( X(z) \) up to a single (nonzero) scalar \( \alpha \). The roots \( \{\zeta_i\} \) and \( \{\rho_j\} \) are extremely useful for determining the properties of \( X(z) \), and hence of \( x[n] \). We give them special names. Before continuing, though, it is worth noting that the polynomials \( P(z) \) and \( Q(z) \) in (5.1) are not uniquely defined. We can always create another pair of polynomials whose quotient is \( X(z) \), by multiplying the numerator and denominator by a common factor – e.g., setting \( \tilde{P}(z) = P(z)(z - \beta) \) and \( \tilde{Q}(z) = Q(z)(z - \beta) \). Conversely, if \( P \) and \( Q \) have a common root \( \beta \), the factor \( (z - \beta) \) can be removed from both. When we talk about the roots of \( P \) and \( Q \), we assume that \( P \) and \( Q \) have no common divisors of degree one or higher – \( \{\zeta_i\} \cap \{\rho_j\} = \emptyset \).

The zeros of a rational function \( X(z) \) are defined as the roots of the numerator \( P(z) \) in a rational expression \( X(z) = P(z)/Q(z) \) in which \( P(z) \) and \( Q(z) \) have no common roots. We also say that \( “X(z) \) has a zero at \( \infty” \) whenever \( \lim_{|z| \to \infty} |X(z)| = 0 \).

The poles of a rational function \( X(z) \) are defined as the roots of the denominator \( Q(z) \) in a rational expression \( X(z) = P(z)/Q(z) \) in which \( P(z) \) and \( Q(z) \) have no common roots. By convention, we also say that \( “X(z) \) has a pole at \( \infty” \) whenever \( \lim_{|z| \to \infty} |X(z)| = +\infty \).

If the numerator \( P(z) \) has a repeated root \( \zeta_i \) (i.e., \( P(z) = \tilde{P}(z)(z - \zeta_i)^\ell \) with \( \tilde{P} \) a polynomial), we say that \( P(z) \) has a repeated zero of order \( \ell \) (or multiplicity \( \ell \)) at \( \zeta_i \). Similarly, if \( Q(z) \) has a repeated root \( \rho \), we say that \( \rho \) is a pole (or multiple) pole.

As \( z \) approaches a pole \( \xi_j \), the magnitude \( |X(z)| \) approaches \(+\infty\). Clearly, for a rational \( Z \)-transform \( X(z) \), there can be no poles in the region of convergence. Depending on the situation, the zeros \( \zeta_i \) may lie inside or outside the region of convergence. In introducing the \( Z \)-transform, we defined \( X(z) \) over ROC \( \{x\} \). When \( X(z) \) has rational form \( X(z) = P(z)/Q(z) \), we mean that

\[
\forall z \in \text{ROC } \{x\}, \quad \sum_{n=-\infty}^{\infty} x[n]z^{-n} = X(z) = \left( \frac{P(z)}{Q(z)} \right).
\]  

Outside of the region of convergence, there is no guarantee that the summation on the left hand side converges. However, once we know that \( X(z) \) is rational, we can use the functional form \( X(z) = P(z)/Q(z) \) to think about \( X(z) \) as a function of a general complex variable \( z \in \mathbb{C} \), defined over \( \mathbb{C} \setminus \{\xi_j\} \). When we talk about the poles and zeros of a rational function \( X(z) \), we consider \( X(z) \) to be defined over the entire complex plane. The poles and zeros (as defined above) are the poles and zeros of this function.

**Note IV (another technicality): we only do this for rational \( X(z) \).** The text briefly defines the poles and zeros of a general \( Z \)-transform \( X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \) to be the points at which \( X(z) \) is infinite and zero, respectively. This definition is not consistent with the definition for rational \( X(z) \) given above.\(^2\) In this course, we will be interested almost exclusively in \( X(z) \) which are rational. This set of \( Z \)-transforms is large enough to allow us to solve linear constant coefficient difference equations, and also includes the \( Z \)-transforms of most common sequences. So, we will not dwell on the definition of poles and zeros for general functions \( X(z) \). Just be careful: repeating the message of the previous section The poles and zeros of \( X(z) \) are the poles and zeros of the rational function \( X(z) = P(z)/Q(z) \).

\(^2\)If the rational function \( X(z) \) has a zero \( \zeta_i \) outside ROC \( \{x\} \), it can happen that \( \sum_{n} x[n]z^{-n} \) does not converge, or is infinite at \( \zeta_i \).
ROC’s for rational $X(z)$. For rational $X(z)$, knowing the poles and zeros determines the functional form $X(z)$ up to a scalar multiplication. The location of the poles also puts strong constraints on the possible regions of convergence of $X(z)$. The most obvious constraint is that the ROC cannot contain any poles. In addition, we can strengthen our statements about the ROC for left-sided and right-sided sequences:

- **Finite duration sequences.** The $Z$-transform of a finite duration sequence is always rational. The ROC of a finite duration sequence is the entire complex plane, possibly except for $z = 0$ and $z = +\infty$.

- **Right-sided sequence, rational $X(z)$**. If $x[n] = 0$ for $n < N$ for some $N \in \mathbb{Z}$ ( $x$ is right-sided), and $X(z)$ is rational, ROC $\{x\}$ extends outward from the largest-magnitude finite pole $\xi_i$. Depending on $N$, the ROC may or may not contain $\infty$.

- **Left-sided sequence, rational $X(z)$**. If $x[n] = 0$ for $n > N$ for some $N \in \mathbb{Z}$ ( $x$ is left-sided), and $X(z)$ is rational, ROC $\{x\}$ extends inward from the smallest-magnitude nonzero pole $\xi_i$, up to (and possibly including) $z = 0$.

- **Two-sided sequence, rational $X(z)$**. If $x$ is two-sided (there does not exist $N$ such that $x[n] = 0$ for all $n < N$ and there also does not exist $N$ such that $x[n] = 0$ for all $n > N$), and $X(z)$ is rational, then ROC $\{x\}$ will be a ring in the complex plane, bounded on the interior and exterior by poles, and not containing any poles. We often draw the poles and zeros of $X(z)$ on the complex plane, with zeros marked by an o and poles marked by a x. Repeated poles and zeros are typically marked with multiple o symbols and multiple x symbols, according to multiplicity. We call a drawing of the poles and zeros in the complex plane a pole-zero diagram. Given the pole-zero diagram, we know $X(z)$ up to a scalar multiple $\alpha \in \mathbb{C}$. Moreover, together with the properties of the ROC, the pole-zero diagram strongly constrains the sequences $x[n]$ that could correspond to $X(z)$; which one actually does depends on the ROC itself. In the lecture, we illustrate this through a few simple examples.

**Example IV.** Let us consider another example

$$x[n] = \left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n].$$

Then

$$X(z) = \sum_{n=-\infty}^{\infty} \left[\left(\frac{1}{2}\right)^n u[n] + \left(-\frac{1}{3}\right)^n u[n]\right] z^{-n}$$

$$= \sum_{n=0}^{\infty} (1/2)^n z^{-n} + \sum_{n=0}^{\infty} (-1/3)^n z^{-n}$$

$$= \frac{1}{1 - \frac{1}{2} z^{-1}} + \frac{1}{1 + \frac{1}{3} z^{-1}} \quad |z| > \max \left\{\frac{1}{2}, \frac{1}{3}\right\}$$

$$= \frac{2z(z-1/12)}{(z-1/2)(z+1/3)} \quad |z| > \max \left\{\frac{1}{2}, \frac{1}{3}\right\}. \quad (5.7)$$

Here, $X(z)$ has zeros at $z = 0$ and $z = 1/12$, and poles at $z = 1/2$ and $z = -1/3$. $x[n]$ is a right-sided sequence; ROC $\{x\}$ extends outward from the outermost pole (at $1/2$).
Example V. We discuss the possible ROC (and signal properties) associated with two pole zero diagrams. In the first, there are poles at $z = a < b < c$, $b < 1$, and $c > 1$. Here, there are four possible regions of convergence. One corresponds to a right-sided sequence, one corresponds to a left-sided sequence, and the other two correspond to two-sided sequences. Exactly one of these (ROC \( \{ z \mid b < |z| < c \} \)) corresponds to a stable sequence; its DTFT exists.

Example VI. In the second example, we consider a pole-zero diagram in which there is a repeated zero at $z = 0$ (of multiplicity two) and poles at $z = 1/2$ and $z = -2$. Here, there are three possible ROC’s: (i) $|z| < 1/2$, which corresponds to a left-sided sequence, which is unstable, and whose DTFT does not exist (in $\ell^1$ sense). (ii) $1/2 < |z| < 2$, which corresponds to a stable sequence (whose DTFT exists), and which is two-sided. (iii) $|z| > 2$ which corresponds to an unstable sequence, whose DTFT does not exist in a strong sense, which is right-sided.

6 $\mathcal{Z}$-transform properties

Example IV above illustrates two important facts about the $\mathcal{Z}$-transform. The first is that because the definition involves a summation, the $\mathcal{Z}$-transform is linear in the sequence $x[n]$. However, the second point is that in manipulating $\mathcal{Z}$-transforms, we need to be very careful to keep track of the region of convergence.

Proposition 6.1 (Linearity). Suppose that $x_1[n] \longleftrightarrow X_1(z), R_1$ and $x_2[n] \longleftrightarrow X_2(z), R_2$, then for any $\alpha, \beta \in \mathbb{C}$,

$$\alpha x_1 + \beta x_2 \longleftrightarrow (\alpha X_1 + \beta X_2), R,$$

with $R_1 \cap R_2 \subseteq R$.\hspace{1cm} (6.1)

The next property generalizes the time-shifting property of the DTFT:

Proposition 6.2 (Time-shifting). Suppose that $x[n] \longleftrightarrow X(z), R$. Then

$$x[n - n_0] \longleftrightarrow z^{-n_0} X(z), R',$$

where $R'$ is identical to $R$, with the possible addition or deletion of $z = 0$ and $z = \infty$.\hspace{1cm} (6.2)

Finally, we provide a generalization of the convolution property of the DTFT. The proof is identical in structure to the proof of the convolution property of the DTFT – we just need to take additional care to track the regions of convergence.

Proposition 6.3 (Convolution). Suppose that $x_1[n] \longleftrightarrow X_1(z), R_1$, and $x_2[n] \longleftrightarrow X_2(z), R_2$. Then

$$x_1 * x_2 \longleftrightarrow X_1(z)X_2(z), R,$$

with $R_1 \cap R_2 \subseteq R$.\hspace{1cm} (6.3)

Proof. Let

$$y[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]$$

\hspace{1cm} (6.4)
Let us compute the $Z$-transform.

$$Y(z) = \sum_{n=-\infty}^{\infty} z^{-n} \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]$$  \hspace{1cm} (6.5)

$$= \sum_{k=-\infty}^{\infty} x_1[k]z^{-k} \sum_{n=-\infty}^{\infty} x_2[n-k]z^{-(n-k)}$$  \hspace{1cm} (6.6)

$$= X_2(z) \sum_{k=-\infty}^{\infty} x_1[k]z^{-k} \quad z \in R_2$$  \hspace{1cm} (6.7)

$$= X_2(z)X_1(z), \quad z \in R_2, \ z \in R_1.$$  \hspace{1cm} (6.8)

This is the desired result. $\square$