Disclaimer: These notes are intended to be an accessible introduction to the subject, with no pretense at completeness. In general, you can find more thorough discussions in Oppenheim's book. Please let me know if you find any typos.

Reading suggestions: Oppenheim and Schafer Section 2.6-2.7

Homework: HW2 is released, due Wednesday September 28.

In this lecture, we will motivate our coming journey into the Fourier domain, and briefly review the (continuous time) Fourier series and Fourier transform.

1 The Fourier Domain: A Motivation

In the coming lectures, we will study frequency-domain representations of discrete time signals and systems. To motivate this journey, let us consider a BIBO stable LTI system, with impulse response $h$. Because $h$ is stable, $\|h\|_{\ell_1} = \sum_k |h[k]| < +\infty$. Let us consider the system output, when the input consists of the complex exponential

$$x[n] = \exp(j\omega n).$$

We calculate

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} \exp(j\omega k)h[n-k]$$

$$= \exp(j\omega n) \sum_{k=-\infty}^{\infty} \exp\{j\omega (k-n)\} h[n-k]$$

$$= x[n]H(e^{j\omega}),$$

$$n = 1.$$
where we define
\[ H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] \exp(-j\omega n). \]  
(1.3)

Notice that because \( h \) is absolutely summable, this summation converges, and \( H(e^{j\omega}) \) is well-defined. Thus, when \( x[n] = \exp(j\omega n) \), the output \( y \) is a scalar multiple of the input \( x \):
\[ y = H(e^{j\omega}) x. \]  
(1.4)

In language inspired by linear algebra, \( x \) is an eigenfunction of the system. In fact, we will see that fairly general families of signals can be expressed as superpositions of complex exponentials. Understanding how the system acts on these eigenfunctions can yield quite a bit of insight into its behavior on “arbitrary” inputs.

2 Review: Fourier Series and the Fourier Transform

Fourier series. Fourier series give a way of expressing a complicated periodic function as a superposition of much simpler periodic functions, such as complex exponentials (or, if the input is real, sinusoids). Suppose that \( x(t) \) is a (continuous-time) periodic signal with period \( T \), so
\[ x(t) = x(t + T), \quad \forall t. \]  
(2.1)

For \( k \in \mathbb{Z} \), the complex exponentials
\[ e_k(t) = \exp\left(\frac{2\pi}{T} kt\right) \]  
(2.2)

are also periodic with period \( T \). Moreover, if we define the inner product\(^1\)
\[ \langle f, g \rangle = \int_{-T/2}^{T/2} f(t)g^*(t)dt, \]  
(2.3)

then
\[ \langle e_k, e_\ell \rangle = \begin{cases} T & k = \ell \\ 0 & \text{else}. \end{cases} \]  
(2.4)

So, \( \{e_k \mid k \in \mathbb{Z}\} \) is an orthogonal set.

We look for an approximation of our periodic input \( x(t) \) in terms of this set. Namely, we are interested in approximations of the form
\[ x(t) \approx \sum_{k=-n}^{n} c_k e_k(t). \]  
(2.5)

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\(^1\)The seemingly arbitrary conjugation on \( g \) comes from the definition of an inner product on a complex vector space: an inner product must satisfy \( \langle f, g \rangle = \langle g, f \rangle^* \) for every \( f \) and \( g \). This ensures that for every \( f, \langle f, f \rangle = \langle f, f \rangle^* \), and so \( \langle f, f \rangle \) is real. In our particular choice of inner product for functions \( f \) defined on \([-T/2, T/2]\) — namely, \( \langle f, g \rangle = \int_{-T/2}^{T/2} f(t)g^*(t)dt \), this ensures that \( \langle f, f \rangle = \int_{-T/2}^{T/2} |f(t)|^2 dt = \|f\|_{t}^2 \) is the energy of \( f \) over the interval \([-T/2, T/2]\). See also discussion in the previous lecture.
How should we choose the coefficients $c_k$? To compute $c_\ell$, we can take the inner product of both sides with $e_\ell$. Because the $e_k$ are orthogonal, we get
\[
\langle x, e_\ell \rangle \approx \sum_{k=-n}^{n} c_k \langle e_k, e_\ell \rangle
\]
(2.6)
\[
= T c_\ell.
\]
(2.7)
This strongly suggests taking
\[
c_\ell = \frac{1}{T} \langle x, e_\ell \rangle
\]
(2.8)
It is not too difficult to show that when the $c_\ell$ are chosen in this manner, we obtain the best possible approximation to $x(t)$ of the form (2.5), in terms of the $L^2$ norm (energy). Moreover, if the input $x$ is “nice enough”, then this approximation becomes accurate when the number of terms is large enough. For example,

**Theorem 2.1.** Let $x : \mathbb{R} \to \mathbb{C}$ be continuous, periodic with period $T$, and piecewise continuously differentiable. Set
\[
c_k = \frac{1}{T} \langle x, e_k \rangle,
\]
(2.9)
and
\[
\hat{x}_n = \sum_{k=-n}^{n} c_k e_k.
\]
(2.10)
Then for every $t$,
\[
\lim_{n \to \infty} \hat{x}_n(t) = x(t).
\]
(2.11)
Moreover, the convergence is uniform, in the sense that
\[
\lim_{n \to \infty} \max_t |\hat{x}_n(t) - x(t)| = 0.
\]
(2.12)

Thus, if $x$ is nice enough, not only does $\hat{x}_n(t)$ converge to $x(t)$ for every $t$, we can actually provide a uniform bound on error at a given $n$, which holds over all $t$. If $x$ is not so nice – e.g., discontinuous, then this does not happen. A classical example of this is the Gibbs phenomenon in the Fourier representation of a discontinuous function. The Fourier series approximation converges much more slowly in the vicinity of a discontinuity. For functions that are “less nice” – for example, including finitely many discontinuities – the Fourier series still converges, in that the energy of the error goes to zero:

**Theorem 2.2.** Let $x : \mathbb{R} \to \mathbb{C}$ be bounded, periodic with period $T$, and Riemann integrable\(^2\) over the interval $-T/2 \leq t \leq T/2$. Set
\[
\varepsilon_n = \int_{t=-T/2}^{T/2} |x(t) - \hat{x}_n(t)|^2 \, dt.
\]
(2.13)
Then $\lim_{n \to \infty} \varepsilon_n = 0$.

This is sometimes referred to as “convergence in $L^2$”, and does not imply pointwise convergence.

\(^2\)A bounded function $x$ is Riemann integrable if and only if it has only finitely many discontinuities.
The Fourier transform. The Fourier transform is an (audacious) extension of the Fourier series, from periodic functions \(x(t)\) to “arbitrary” functions \(x : \mathbb{R} \to \mathbb{C}\). Since \(x\) is defined over the entire real line (and is not periodic), we extend our inner product to the entire real line, by writing
\[
\langle f, g \rangle = \int_{t=-\infty}^{\infty} f(t)g^*(t) \, dt,
\]
when \(f\) and \(g\) satisfy appropriate conditions to ensure that this integral exists.

Rather than using a countable collection of basis functions \(e_k\), corresponding to frequencies \(\Omega = \frac{2\pi k}{T}\), we allow the frequency to be arbitrary, and write
\[
\psi_{\Omega}(t) = \exp(j\Omega t).
\]

We set
\[
X(j\Omega) = \langle x, \psi_{\Omega} \rangle = \int_{t=-\infty}^{\infty} x(t) \exp(-j\Omega t) \, dt.
\]
This is the Fourier transform of \(x\). Whether it even exists depends on the properties of the input \(x\). For example, if \(x\) has finite \(L^1\) norm:
\[
\int_{t=-\infty}^{\infty} |x(t)| \, dt < +\infty,
\]
then the Fourier transform \(X(j\Omega)\) exists.

As with the Fourier series, under certain assumptions on \(x\), it is possible to reconstruct \(x\) using a superposition of the functions \(\psi_{\Omega}\). The reconstruction formula is
\[
x(t) = \frac{1}{2\pi} \int_{\Omega=-\infty}^{\infty} X(j\Omega) \exp(j\Omega t) \, d\Omega.
\]
For example, if \(x\) is continuous, then the above equation holds for every \(t\). If both \(x(t)\) and \(X(j\Omega)\) have finite \(L^1\) norm, then it holds for “almost all” \(t\).

The picture. Table 1 shows describes the Fourier domain representations of various types of signals. Our study of discrete time signals will lead us to consider in some depth the discrete time Fourier transform, which gives a frequency domain representation of a discrete time signal \(x[n]\).

| Fourier Series | Continuous time, periodic \(x(t)\) | Series \(\{c_k | k \in \mathbb{Z}\}\) |
|---------------|----------------------------------|-------------------------------|
| Fourier Transform | Continuous time \(x(t)\) | Continuous frequency \(X(j\Omega), \Omega \in \mathbb{R}\) |
| Discrete Time Fourier Transform | Discrete time \(x[n]\) | Discrete frequency, periodic \(X(e^{j\omega})\) |
| Discrete Fourier Transform | Discrete time finite or periodic \(x[n]\) | Discrete finite / periodic \(X[k]\) |

Table 1: Fourier representations of various types of signals.