Supplementary Notes for ELEN 4810 Lecture 6
The Discrete Time Fourier Transform

John Wright
Columbia University
September 26, 2016

Disclaimer: These notes are intended to be an accessible introduction to the subject, with no pretense at completeness. In general, you can find more thorough discussions in Oppenheim’s book. Please let me know if you find any typos.

Reading suggestions: Oppenheim and Schafer Section 2.6-2.8

Homework: HW2 is due Monday October 3.

In this lecture, we build on the observation that complex exponentials are eigenfunctions of linear time-invariant systems. We discuss the discrete-time Fourier transform, and a few of its simple properties.

1 The Discrete-Time Fourier Transform

In the previous lecture notes, we saw that if we took a stable linear, time-invariant system $T$ with impulse response $h[n]$ (necessarily satisfying $\sum_{k=-\infty}^{\infty} |h[k]| < +\infty$), and applied it to a complex exponential input

$$x[n] = \exp(j\omega n),$$

(1.1)
then the output

\[ y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] \]

\[ = \sum_{k=-\infty}^{\infty} \exp(j\omega n) \exp\{j\omega (k-n)\} h[n-k] \]

\[ = \exp(j\omega n) \sum_{k=-\infty}^{\infty} \exp\{-j\omega k\} h[k] \]

\[ = \exp(j\omega n) H(e^{j\omega}) \]

\[ = x[n] H(e^{j\omega}). \]  

(1.2)

is a scalar multiple \( H(e^{j\omega}) \) of the input \( x[n] \). The function \( H(e^{j\omega}) \) is tremendously useful for understanding the properties of the system.

**The Discrete Time Fourier Transform for absolute-summable \( x[n] \)**

Motivated by the above example, we introduce a *Discrete-Time Fourier Transform* (DTFT). This transform takes a discrete-time signal \( x[n] \), and produces a “frequency domain” representation, which we denote \( X(e^{j\omega}) \). Here, \( \omega \in \mathbb{R} \) is a continuous (i.e., real-valued) frequency; for each frequency \( \omega \in \mathbb{R} \), we have one complex number \( X(e^{j\omega}) \in \mathbb{C} \). We sometimes write the relationship between \( x \) and \( X \) as

\[ x[n] \xrightarrow{\text{DTFT}} X(e^{j\omega}). \]

(1.3)

The DTFT \( X(e^{j\omega}) \) is a function of frequency \( \omega \). It may seem strange to write it as \( X(e^{j\omega}) \) instead of the usual notation for a function of \( \omega \) (e.g., \( g(\omega) \)). The reason for the notation \( X(e^{j\omega}) \) will become clear in about a month when we study the \( Z \)-transform; for now, just remember that \( X(e^{j\omega}) \) is a function of \( \omega \).

**Definition.** Let \( x[n] \) be a signal, with

\[ \sum_{n=-\infty}^{\infty} |x[n]| < +\infty. \]  

(1.4)

The *Discrete-Time Fourier Transform (DTFT)* of \( x \) is a function \( X(e^{j\omega}) \) of frequency \( \omega \in \mathbb{R} \), defined by

\[ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \exp(-j\omega n). \]  

(1.5)

Notice that when \( x \) is absolute summable (i.e., (1.4) holds), the sequence \( \tilde{x}[n] = x[n] \exp(-j\omega n) \) is also absolute summable, and so the summation defined in (1.5) converges, and \( X(e^{j\omega}) \) is well-defined for every \( \omega \in \mathbb{R} \).

1In fact, when \( x \) is absolute summable, slightly more can be said: it is actually possible to prove that the partial sums in (1.5) converge uniformly, and that the discrete time Fourier transform \( X(e^{j\omega}) \) is continuous in \( \omega \). We prove this in Theorem A.1 of the appendix of these notes.
Periodicity. In the Lecture 2 notes, we discussed some curious properties of discrete-time sinusoids and complex exponentials. The most important fact was that if $L \in \mathbb{Z}$ is an integer, the complex exponentials
\begin{align*}
q[n] &= \exp(-j\omega n) \\
q'[n] &= \exp(-j(\omega + 2\pi L)n) \\
&= \exp(-j\omega n)\exp(-j2\pi n) \\
&= \exp(-j\omega n) \\
&= q[n]
\end{align*}
are exactly the same sequence! This means that
\begin{equation}
X(e^{j(\omega + 2\pi L)}) = \sum_{n=-\infty}^{\infty} x[n] \exp\{-j(\omega + 2\pi L)n\} = \sum_{n=-\infty}^{\infty} x[n] \exp\{-j\omega n\} = X(e^{j\omega}). \tag{1.6}
\end{equation}

In particular, for all $\omega$, $X(e^{j\omega}) = X(e^{j(\omega + 2\pi)})$:

**Proposition 1.1** (Periodicity of the DTFT). The DTFT $X(e^{j\omega})$ is a $2\pi$-periodic function of $\omega$.

This means that if we know the value of the $X(e^{j\omega})$ over some interval of length $2\pi$, we know the value of $X(e^{j\omega})$ over all $\omega$. We therefore typically restrict our attention to the interval $-\pi < \omega \leq \pi$. Occasionally, we may work with the interval $0 \leq \omega < 2\pi$ instead.

**The Inverse DTFT.** If $x$ is absolutely summable, its discrete time Fourier transform $X(e^{j\omega})$ exists and is continuous. In this situation, we can recover $x[n]$ from $X(e^{j\omega})$ via the inversion formula
\begin{equation}
x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \exp(j\omega n) \, d\omega. \tag{1.7}
\end{equation}

This integral is called the Inverse Discrete Time Fourier Transform, and allows us to move from the Fourier domain back to the time domain. You can think of this expression as representing the sequence $x[n]$ as a superposition of complex exponential “basis functions” of the form $\exp(j\omega n)$.

We prove the relationship (1.7) below:

**Theorem 1.2.** Suppose that
\begin{equation}
\sum_{n=-\infty}^{\infty} |x[n]| < +\infty. \tag{1.8}
\end{equation}

Then for every $n \in \mathbb{Z}$,
\begin{equation}
x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \exp(j\omega n) \, d\omega. \tag{1.9}
\end{equation}
Proof. We first note that for any integer $k$,
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(j\omega k) \, d\omega = \delta[k]. \tag{1.10}
\]
Now, write
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \omega \exp(-j\omega n) \, d\omega = \sum_{k=-\infty}^{\infty} x[k] \exp(-j\omega k) \exp(j\omega n) \, d\omega
\]
\[
= \sum_{k=-\infty}^{\infty} x[k] \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp(j\omega(n-k)) \, d\omega \quad \text{(def'n of DTFT)}
\]
\[
= \sum_{k=-\infty}^{\infty} x[k] \delta[n-k] \quad \text{(by (1.10))}
\]
\[
= x[n]. \tag{1.11}
\]
In applying the dominated convergence theorem\(^2\), we have used that the partial sums of
\[
\sum_{k=-\infty}^{\infty} x[k] \exp(-j\omega(n-k))
\]
are all dominated by the integrable (uniform) function $g(\omega) = \sum_{k=-\infty}^{\infty} |x[k]|$.

The DTFT and inverse DTFT allow us to move back and forth between the time and frequency domains in a relatively straightforward way.

Examples. To illustrate the idea, we compute the DTFT of several simple example signals $x[n]$. The DTFT $X(e^{j\omega})$ is complex-valued; typically, we plot it in polar form, by plotting the magnitude spectrum $|X(e^{j\omega})|$ and phase spectrum $\angle X(e^{j\omega})$ individually. Several examples follow below.

- **The unit impulse.** If $x[n] = \delta[n]$, $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} = 1$, for every $\omega$. Thus, the DTFT of the unit impulse is the constant function $X(e^{j\omega}) = 1$. The magnitude spectrum $|X(e^{j\omega})| = 1$; the phase spectrum is $\angle X(e^{j\omega}) = 0$.

- **One sided exponential.** If $x[n] = \alpha^n u[n]$, $x$ is absolute summable for $|\alpha| < 1$. Then $X(e^{j\omega}) = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \frac{1}{1-\alpha e^{-j\omega}}$. The magnitude and phase spectrum are plotted in Figure 1 for the special case $\alpha = 1/2$.

\(^2\)If your background in analysis is such that the phrase “dominated convergence theorem” is meaningless to you, the issue in a nutshell is as follows. In the proof, we interchanged integration over $\omega$ and summation over $k$. This is only possible under specific assumptions on the function being integrated. Fortunately, when $\sum_{n} |x[n]|$ is finite, these conditions are met, and the proof proceeds without major difficulty. It is very much possible to appreciate the discrete time Fourier transform as an engineering tool without a detailed understanding of the analytical issues associated with its definition and proof. For further analytical work (beyond what will be covered in this class), a detailed understanding is essential.
• **Discrete-time box.** If \( x[n] = \begin{cases} 1 & 0 \leq n \leq L - 1, \\ 0 & \text{else} \end{cases} \), then \( X(e^{j\omega}) = \begin{cases} 1 - e^{-j\omega L} & \omega \neq 0 \\ \frac{1}{L} & \omega = 0 \end{cases} \). The magnitude and phase spectra are plotted in Figure 2 for the case \( L = 10 \).

The text contains large tables of DTFT’s for additional example signals.

**The Discrete Time Fourier Transform for \( x[n] \) that are not absolute-summable**

Above, we proved that the discrete time Fourier transform is well-defined and its inverse function is valid, for any signal \( x \) whose magnitudes are summable. In practice, we will need to work with signals that are not absolute summable. To see why, consider the following example:

An important example. Take some \( \omega_c \in (0, \pi) \), and define a function \( H_{lp}(e^{j\omega}) \) by

\[
H_{lp}(e^{j\omega}) = \begin{cases} 1 & -\omega_c \leq \omega \leq \omega_c, \\ 0 & \omega_c < |\omega| \leq \pi. \end{cases}
\]  

(1.12)
Figure 2: DTFT of the box sequence $x[n] = 1, \ 0 \leq n \leq 9, \ x[n] = 0 \text{ else}$

This is the frequency response of an *ideal low-pass filter*, with cutoff $\omega_c$. Let us take this specification, and formally push it through the inverse DTFT expression to obtain a sequence $h_{lp}[n]$:

$$h_{lp}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_{lp}(e^{j\omega}) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega$$

$$= \begin{cases} \frac{\omega_c}{\pi} n = 0 \\ \frac{1}{2\pi} \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{n} \text{ else} \end{cases}$$

$$= \begin{cases} \frac{\omega_c}{\pi} n = 0, \\ \frac{\sin(\omega_c n)}{n\pi} \text{ else} \end{cases}$$

(1.13)

The entries of $h_{lp}[n]$ decay to zero as $|n| \to \infty$, but they decay slowly – the $n$-th term is approximately proportional to $1/n$. It is not too difficult to show that $\|h_{lp}\|_\ell^1 = +\infty$ – *the ideal lowpass filter is not absolute summable*. We certainly want to be able to work with a low-pass filter in frequency domain! To do so, we will need to extend the class of signals we can handle.
Although $h_{lp}[n]$ is not absolute summable, it is square summable:

$$
\sum_{n=-\infty}^{\infty} |h_{lp}[n]|^2 = \frac{\omega_c}{\pi} + \sum_{n=1}^{\infty} \frac{\sin^2(\omega_c n)}{n^2 \pi^2} + \sum_{n=-\infty}^{-1} \frac{\sin^2(\omega_c n)}{n^2 \pi^2} \\
= \frac{\omega_c}{\pi} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin^2(\omega_c n)}{n^2} \\
\leq \frac{\omega_c}{\pi} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \\
= \frac{\omega_c}{\pi} + \frac{1}{3} < +\infty,
$$

(1.14)

where in the final line we’ve used that $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6$, a famous result of Euler. For square summable sequences such as $h_{lp}[n]$, we can apply the discrete-time Fourier transform. However, we have to be careful about what to expect. To get some intuition, in Figure 3, we plot the partial sums

$$H_N(e^{j\omega}) = \sum_{n=-N}^{N} h_{lp}[n] e^{-j\omega n}
$$

(1.15)

You can see that away from the discontinuity in $H_{lp}$, $H_N(e^{j\omega})$ appears to converge to $H_{lp}(e^{j\omega})$. However, at $\omega = \pm \omega_c$, it does not converge. Near $\omega_c$, we observe ripples, characteristic of the Gibbs phenomenon. In Figure 3, we plot the approximation error

$$\int_{-\pi}^{\pi} \left| H_N(e^{j\omega}) - H_{lp}(e^{j\omega}) \right|^2 d\omega,
$$

(1.16)

as a function of $N$. You can observe that the approximation error does converge to zero as $N$ increases.

**Square summable signals.** The behavior of the DTFT of the low-pass filter is representative of how the DTFT behaves for general signals $x[n]$ satisfying

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < +\infty,
$$

(1.17)

i.e., square summable signals. Every absolute summable sequence is square summable. However, some square summable sequences are not absolute summable – $h_{lp}[n]$ described above is one example.

For sequences that are not absolute summable, the DTFT may not be well-defined for all $\omega$. We saw this above for the ideal lowpass filter. Another simple example is $x[n] = (1/n) u[n-1] \exp(j\omega_0 n)$. What is the value of $X(e^{j\omega})$?

Although the DTFT of a sequence which is not absolute summable may not be well-defined for all $\omega$, if the sequence is square summable, the DTFT is always well-defined “in the sense of energy:” if we let

$$X_k(e^{j\omega}) = \sum_{n=-k}^{k} x[n] \exp\{-j\omega n\}
$$

(1.18)
Figure 3: Truncating the ideal low-pass filter. Left: ideal lowpass filter $h_{lp}[n]$, restricted to $n = -N, \ldots, N$. Center: magnitude of Fourier transform $|H_N(e^{j\omega})|$ for the truncated lowpass filter. Notice that at the discontinuity $\pm \omega_c$, the magnitude oscillates rapidly. This is an example of the Gibbs phenomenon. Right: $L^2$ error $\int_{\omega=-\pi}^{\pi} |H_N(e^{j\omega}) - H(e^{j\omega})|^2 d\omega$, as a function of $N$. Notice that the $L^2$ error converges to zero as $N$ increases.

then there is a function $X(e^{j\omega})$ satisfying

$$
\lim_{k \to \infty} \int_{\omega=-\pi}^{\pi} |X_k(e^{j\omega}) - X(e^{j\omega})|^2 d\omega = 0. \tag{1.19}
$$

Moreover, the inverse DTFT formula remains valid.

I will not prove either of these facts in lecture. However, I’ve provided proofs in the appendix to these notes, in case you’d like to develop a more mathematical understanding of what is going on. For our purposes, it is enough to have a qualitative understanding of the issues – in particular, the fact that when $\|x\|_1 = +\infty$, the DTFT summation may not exist for certain $\omega$, and that this lack of convergence shows up in finite approximations to the DTFT as a Gibbs-like phenomenon. This fact will have important practical implications when we try to design filters, or study signals (such as musical notes) whose frequency content changes with time.

DTFT for certain useful sequences that are not even square-summable. We can go one step further, and try to take the transform to certain signals $x[n]$ that are not even square summable. For example, we could try to take the DTFT of the constant signal $x[n] = 1$, or of a complex exponential $x[n] = \exp(j\omega_0 n)$. In this situation, the summation defined in the DTFT does not converge. Nevertheless, we find it useful to make a formal definition of the DTFT for these and other specific cases. For $x[n] = \exp(j \omega_0 n)$, we define the Fourier transform to be

$$
X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 + 2\pi k), \tag{1.20}
$$

where $\delta(\cdot)$ is the (continuous) Dirac delta. This just takes a scaled Dirac delta $2\pi \delta(\omega - \omega_0)$, and extends it to be $2\pi$-periodic.
For our purposes, the statement \( \text{DTFT}[\exp(j\omega_0 n)] = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 + 2\pi k) \) is not a mathematical claim – it is a definition. However, it happens to work out, in the sense that if we formally apply the inverse DTFT to \( X(e^{j\omega}) \), we recover

\[
\begin{align*}
x[n] &= \frac{1}{2\pi} \int_{-\pi}^{\pi} 2\pi \sum_k \delta(\omega - \omega_0 + 2\pi k) \exp(j\omega n) d\omega \\
&= \exp(j\omega n),
\end{align*}
\]

as desired. Really making this mathematically rigorous requires us to delve into the theory of generalized functions, which is substantially beyond the scope of this course. For our purposes we simply accept (1.20) as a definition.

Using the relationships

\[
\begin{align*}
\cos(\omega_0 n) &= \frac{\exp(j\omega_0 n) + \exp(-j\omega_0 n)}{2}, \\
\sin(\omega_0 n) &= \frac{\exp(j\omega_0 n) - \exp(-j\omega_0 n)}{2j},
\end{align*}
\]

(1.22)

and formal manipulations, we can also obtain formal expressions for the DTFT of \( x[n] = \cos(\omega_0 n) \) and \( x[n] = \sin(\omega_0 n) \).

## 2 Symmetry properties of the DTFT

The most basic symmetry property of the DTFT states that conjugation in time domain is equivalent to a conjugation and flip about zero in the frequency domain:

**Proposition 2.1.** Let \( x \) be a signal with DTFT \( X(e^{j\omega}) \). Then the DTFT of the complex conjugate signal \( \bar{x} = x^* \) is \( \bar{X}(e^{j\omega}) = X^*(e^{-j\omega}) \).

**Proof.** We simply calculate

\[
\begin{align*}
\bar{X}(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \bar{x}[n]e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} x^*[n]e^{-j\omega n} \\
&= \left( \sum_{n=-\infty}^{\infty} x[n]e^{j\omega n} \right)^* \\
&= X(e^{-j\omega})^*,
\end{align*}
\]

(2.1)

as claimed.

This property is useful because it implies several facts about the discrete-time Fourier transform of a real signal \( x[n] \). Namely,

**Proposition 2.2.** Let \( x[n] \) denote a real signal, with discrete-time Fourier transform \( X(e^{j\omega}) \). Then \( X \) is conjugate-symmetric: \( X(e^{j\omega}) = X^*(e^{-j\omega}) \).
If $x$ is real, $x^* = x$. Applying the previous proposition gives the result.

In particular, this implies that if $x$ is a real signal, the real part of $X(e^{j\omega})$ is even (symmetric), and the imaginary part is odd (antisymmetric). It also implies that the magnitude is even, and the phase is odd.

## 3 Basic DTFT relationships

### Linearity

**Proposition 3.1.** The discrete time Fourier transform is linear: if $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$ are the DTFT’s of signals $x_1$ and $x_2$, respectively, and $\alpha, \beta \in \mathbb{C}$ are any scalars, then

$$\alpha X_1(e^{j\omega}) + \beta X_2(e^{j\omega})$$

is the DTFT of the combination $\alpha x_1 + \beta x_2$.

**Proof.** Exercise: use the definition.

### Time Shifts

**Proposition 3.2.** Let $x[n]$ denote a signal with discrete time Fourier transform $X(e^{j\omega})$, and let $\bar{x}[n] = x[n - n_0]$, for some $n_0 \in \mathbb{Z}$. Then the DTFT $\bar{X}(e^{j\omega})$ of $\bar{x}$ is given by

$$\bar{X}(e^{j\omega}) = e^{-j\omega n_0} X(e^{j\omega}).$$

**Proof.** Calculate

$$\bar{X}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n - n_0] e^{-j\omega n}$$

$$= e^{-j\omega n_0} \sum_{n=-\infty}^{\infty} x[n - n_0] e^{-j\omega (n - n_0)}$$

$$= e^{-j\omega n_0} X(e^{j\omega}),$$

as claimed.

### Convolution in time

**Theorem 3.3.** Suppose that $y = x * h$, where $x$ and $h$ are absolute summable, with discrete time Fourier transforms $X(e^{j\omega})$ and $H(e^{j\omega})$, respectively. Then the discrete time Fourier transform $Y(e^{j\omega})$ of $Y$ satisfies

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}).$$
Proof. We calculate

\[ Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} \]
\[ = \sum_{n=-\infty}^{\infty} \left( \sum_{k=-\infty}^{\infty} x[k]h[n-k] \right) e^{-j\omega n} \]
\[ = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k] \exp(-j\omega k) h[n-k] \exp(-j\omega(n-k)) \]
\[ = \sum_{k=-\infty}^{\infty} x[k] \exp(-j\omega k) \sum_{n=-\infty}^{\infty} h[n-k] \exp(-j\omega(n-k)) \]
\[ = H(e^{j\omega}) \sum_{k=-\infty}^{\infty} x[k] \exp(-j\omega k) \]
\[ = H(e^{j\omega}) X(e^{j\omega}), \quad (3.7) \]
as claimed.

Thus, convolution in time becomes (pointwise) multiplication in the frequency domain. This property is one of the reasons that the DTFT is such a powerful tool for studying linear time invariant systems.

Modulation in time.

Theorem 3.4. Suppose that \( y[n] = x[n]w[n] \), and that \( Y(e^{j\omega}), X(e^{j\omega}) \) and \( W(e^{j\omega}) \) are the DTFT’s of \( y, x, \) and \( w \), respectively. Then

\[ Y(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) W(e^{j(\omega-\theta)}) d\theta. \quad (3.8) \]

Proof. We calculate

\[ Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} w[n] x[n] e^{-j\omega n} \]
\[ = \sum_{n=-\infty}^{\infty} w[n] \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) e^{j\theta n} d\theta \ e^{-j\omega n} \]
\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) \sum_{n=-\infty}^{\infty} w[n] e^{-j(\omega-\theta)n} d\theta \]
\[ = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) W(e^{j(\omega-\theta)}) d\theta, \quad (3.9) \]
as claimed.

Thus, multiplication (modulation) in time domain becomes (continuous-time) convolution in the frequency domain.
**Inner products and Parseval’s theorem.** We close our discussion of basic properties of the DTFT with a result showing that the DTFT preserves inner products, in the following sense. If we define, for two summable sequences \( x \) and \( y \),
\[
\langle x, y \rangle = \sum_{n=-\infty}^{\infty} x[n] y^*[n],
\]
and for any “sufficiently nice” functions \( X(e^{j\omega}) \) and \( Y(e^{j\omega}) \),
\[
\langle X, Y \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega,
\]
then whenever \( X \) and \( Y \) are the DTFT’s of \( x \) and \( y \), respectively,
\[
\langle x, y \rangle = \langle X, Y \rangle.
\]
That is to say,

**Theorem 3.5.** For all \( x \) and \( y \) with discrete time Fourier transforms \( X(e^{j\omega}) \) and \( Y(e^{j\omega}) \),
\[
\sum_{n=-\infty}^{\infty} x[n] y^*[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega.
\]

**Proof.**
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \right) Y^*(e^{j\omega})d\omega
\]
\[
= \sum_{n=-\infty}^{\infty} x[n] \frac{1}{2\pi} \int_{-\pi}^{\pi} Y^*(e^{j\omega})e^{-j\omega n}d\omega
\]
\[
= \sum_{n=-\infty}^{\infty} x[n] \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(e^{j\omega})e^{j\omega n}d\omega \right)^*
\]
\[
= \sum_{n=-\infty}^{\infty} x[n] y^*[n],
\]
as desired.

Applying this result with \( y[n] = x[n] \), we obtain an important corollary, known as Parseval’s theorem. Parseval’s theorem states that the DTFT preserves the energy of the signal, up to a multiplication by \( 1/2\pi \):

**Theorem 3.6.** For any square summable signal \( x \),
\[
\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2d\omega.
\]

**Proof.** Apply Theorem 3.5 with \( y = x \).
A Optional Appendix: Continuity of the DTFT, theory for square-summable sequences

As in previous lecture notes, in this appendix, we record some results and derivations that may be interesting to the theorists and completists amongst you. This is not required reading, but could enhance your understanding.

Continuity of the DTFT for absolute-summable $x$

Theorem A.1. Suppose that $\|x\|_{\ell_1} = \sum_{k=-\infty}^{\infty} |x[k]| < +\infty$. Then $X(e^{j\omega})$ is a continuous function of $\omega$.

Proof. Because $X(e^{j\omega})$ is $2\pi$ periodic, it is enough to prove that $X(e^{j\omega})$ is continuous on the interval $\omega \in [-\pi, \pi]$. We will show that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $|\omega - \omega'| < \delta$, $|X(e^{j\omega}) - X(e^{j\omega'})| < \varepsilon$. Because $\|x\|_{\ell_1} < +\infty$, there exists an integer $N$ such that

$$\sum_{|k| > N} |x[k]| < \varepsilon/4. \quad (A.1)$$

Moreover, there exists an $\eta$ (depending on $N$) such that for every $\omega, \omega' \in [-\pi, \pi]$ and every integer $k$ of size $|k| \leq N$,

$$|e^{-j\omega k} - e^{-j\omega' k}| \leq \eta |\omega - \omega'|. \quad (A.2)$$

Choose $\delta \leq \frac{\varepsilon}{2\eta \|x\|_{\ell_1}}$, so that for all $\omega, \omega'$ satisfying $|\omega - \omega'| \leq \delta$,

$$|X(e^{j\omega}) - X(e^{j\omega'})| = \left| \sum_{k=-\infty}^{\infty} x[k] e^{j\omega k} - \sum_{k=-\infty}^{\infty} x[k] e^{j\omega' k} \right| \leq \sum_{k=-\infty}^{\infty} |x[k]| \left| e^{j\omega k} - e^{j\omega' k} \right| \leq \sum_{|k| > N} |x[k]| \left| e^{j\omega k} - e^{j\omega' k} \right| \leq \eta \delta \|x\|_{\ell_1} + 2\varepsilon/4 \leq \varepsilon, \quad (A.6)$$

by our choice of $\delta$.

$L^2$ convergence of the DTFT for square-summable $x$

We prove that the DTFT of a square-summable sequence converges in “mean square sense:”

Theorem A.2. Let $x[n]$ be a sequence, with $\sum_{n=-\infty}^{\infty} |x[n]|^2 < +\infty$. Set $X_N(e^{j\omega}) = \sum_{n=-N}^{N} x[n] e^{-j\omega n}$. Then there exists function $X(e^{j\omega})$ which is square-integrable over $[-\pi, \pi]$, such that

$$\lim_{N \to \infty} \int_{-\pi}^{\pi} \left| X_N(e^{j\omega}) - X(e^{j\omega}) \right|^2 d\omega = 0. \quad (A.8)$$
Proof. We first note that for \( k \) an integer,
\[
\int_{\omega=-\pi}^{\pi} e^{j\omega k} d\omega = \begin{cases} 
2\pi & k = 0 \\
0 & \text{else}
\end{cases}
\] (A.9)

For functions \( f(\omega), g(\omega) \) which are square integrable on \([-\pi, \pi]\), define
\[
\langle f, g \rangle = \int_{\omega=-\pi}^{\pi} f(\omega)g^*(\omega)d\omega.
\] (A.10)

Write
\[
\|f\|_{L^2} = \sqrt{\langle f, f \rangle} = \sqrt{\int_{\omega=-\pi}^{\pi} |f(\omega)|^2 d\omega}.
\] (A.11)

Notice that for \( N > N' \),
\[
\|X_N(e^{j\omega}) - X_{N'}(e^{j\omega})\|_{L^2}^2 = \left\| \sum_{N' < |n| \leq N} x[n]e^{-j\omega n} \right\|_{L^2}^2
\] (A.12)
\[
= \left\langle \sum_{N' < |n| \leq N} x[n]e^{-j\omega n}, \sum_{N' < |m| \leq N} x[m]e^{-j\omega m} \right\rangle
\] (A.13)
\[
= \sum_{N' < |n| \leq N} \sum_{N' < |m| \leq N} x[n]x^*[m] \langle e^{-j\omega n}, e^{-j\omega m} \rangle
\] (A.14)
\[
= 2\pi \sum_{N' < |n| \leq N} |x[n]|^2
\] (A.15)
\[
\leq 2\pi \sum_{|n| > N'} |x[n]|^2.
\] (A.16)

Since \( \sum_n |x[n]|^2 \) is finite, as \( N' \to \infty, \sum_{|n| > N'} |x[n]|^2 \to 0 \). Hence, the sequence \( X_N \) is a Cauchy sequence. Because \( L^2([-\pi, \pi]) \) is complete, the sequence \( X_N \) converges in \( L^2 \) to a limit \( X(e^{j\omega}) \). \( \square \)