2.23  (a) (1) Suppose that $|x[n]| \leq B$ for all $n$. Then $|y[n]| = |\cos(\pi n) x[n]| \leq |x[n]| \leq B$. Since every bounded input $x$ produces a bounded output $y$, the system is stable. (2) Since $y[n]$ is a function of $x[n]$ only, the system is causal. (3) For arbitrary sequences $x_1$ and $x_2$,

$$T \{\alpha x_1 + \beta x_2\} [n] = \cos(\pi n) (\alpha x_1[n] + \beta x_2[n]) = \alpha \cos(\pi n) x_1[n] + \beta \cos(\pi n) x_2[n] = \alpha T \{x_1\} [n] + \beta T \{x_2\} [n],$$

and so the system is linear. (4) Not time invariant. For a counterexample, consider $x[n] = \delta[n]$. Then $y[0] = \cos(0) x[0] = 1$. But if $y' = T \{x'\}$, with $x'[n] = x[n-1] = \delta[n-1], y'[1] = \cos(\pi) x'[1] = -1 \times 1 = -1 \neq y[0]$. Since there exists inputs $x'[n] = x[n-1]$ whose corresponding outputs $y'[n] \neq y[n-1]$, the system is not time invariant.

(b) (1) Suppose that $|x[n]| \leq B$ for all $n$. Then $|y[n]| = |x[n]^2| \leq B$, and so $y$ is bounded. Since every bounded input produces a bounded output, the system is stable. (2) Notice that $y[-1] = x[(-1)^2] = x[1]$. Since $y[-1]$ depends on a future value $x[1]$, the system is not causal. (3) For arbitrary sequences $x_1$ and $x_2$,

$$T \{\alpha x_1 + \beta x_2\} [n] = (\alpha x_1[n] + \beta x_2[n]) \quad (1)$$

$$= \alpha x_1[n^2] + \beta x_2[n^2] \quad (2)$$

$$= \alpha T \{x_1\} [n] + \beta T \{x_2\} [n] \quad (3)$$

and so the system is linear. (4) The system is not time invariant. For a counterexample, notice that if $x[n] = \delta[n]$, $T \{x\} = \delta$ as well. But for $x'[n] = \delta[n-2]$, $T \{x'\} [n] = 0$ for all $n$, and so $T \{D_2 x\} \neq D_2 T \{x\}$.

(c) (1) stable. Notice that for every $n$, $\sum_{k=0}^{\infty} \delta[n-k] \leq 1$, since at most one of the terms in the summation is nonzero. Hence, $|x[n]| \sum_{k=0}^{\infty} \delta[n-k] | \leq |x[n]|$, and every bounded input $x$ produces a bounded output $T \{x\}$. (2) causal. $T \{x\} [n]$ depends only on $x[n]$. (3) linear. To keep the notation more concise, we can notice that $\sum_{k=0}^{\infty} \delta[n-k] = u[n]$, and simply write

$$T \{\alpha x_1 + \beta x_2\} [n] = (\alpha x_1[n] + \beta x_2[n]) u[n] \quad (4)$$

$$= \alpha x_1[n] u[n] + \beta x_2[n] u[n] \quad (5)$$

$$= \alpha T \{x_1\} [n] + \beta T \{x_2\} [n]. \quad (6)$$

(4) Not time invariant. Again, consider $x[n] = \delta[n]$. $T \{x\} [n] = \delta[n] u[n] = \delta[n]$. But $T \{D_{-1} x\} [n] = \delta[n+1] u[n] = 0 \neq D_{-1} T \{x\} [n]$.

(d) (1) Not stable. For a counterexample, consider the bounded input $x$ with $x[n] = 1$ for all $n$. Then $T \{x\} [n] = \sum_{k=n-1}^{\infty} 1 = +\infty$ is not bounded. (2) Not causal. The output at time $n$ depends
on values of $x[k]$ for $k > n$. (3) **linear** over all sequences $x$ for which the output is well-defined.

Again,

$$T\{\alpha x_1 + \beta x_2\}[n] = \sum_{k=n-1}^{\infty} (\alpha x_1[n] + \beta x_2[n])$$

(7)

$$= \alpha \sum_{k=n-1}^{\infty} x_1[n] + \beta \sum_{k=n-1}^{\infty} x_2[n]$$

(8)

$$= \alpha T\{x_1\}[n] + \beta T\{x_2\}[n].$$

(9)

(4) **Time invariant:**

$$T\{D_t x\}[n] = \sum_{k=n-1}^{\infty} (D_t x)[k]$$

(10)

$$= \sum_{k=n-1}^{\infty} x[k - \ell]$$

(11)

$$= \sum_{k' = n-\ell-1}^{\infty} x[k']$$

(12)

$$= T\{x\}[n - \ell]$$

(13)

$$= (D_t T\{x\})[n].$$

(14)

2.40  (a) Periodic, with period $N = 5$.

(b) Periodic: with period $N = 38$.

(c) **Not periodic:** Notice that $x[0] = 0$. If $x$ is $N$-periodic, then for all $n$, $x[n + N] = x[n]$. But for any $N \neq 0$, $x[0 + N] \neq 0$. So, the signal cannot be periodic.

(d) **Not periodic:** This one is trickier. In lecture, we showed that if $e^{j\omega n}$ is periodic with integer period $N$, then $\omega$ is an integer multiple of $2\pi/N$: $\omega = 2\pi k/N$, for some $k \in \mathbb{Z}$. For our signal $x[n] = e^{jn}$, this would require that $1 = 2\pi k/N$, or equivalently, $\pi = N/(2k)$. Since $\pi$ is irrational, it cannot be expressed as a quotient of two integers, and so this is impossible. Notice that this is an important difference between continuous-time and discrete-time complex exponentials: the continuous time signal $z(t) = e^{jt}$ is periodic, with period $2\pi$, but the discrete-time signal $x[n] = e^{jn}$ is not periodic.

**Computational Questions**

Please see the example solution code and corresponding figures in HW1_MATLAB_Soln.zip.