ELEN 4810 Homework 4

Analytical Questions

4.27 Because we sample at twice the bandlimit, the composite system is LTI when restricted to bandlimited inputs.

(a) The continuous-time frequency response is

\[ H_c(j\Omega) = \begin{cases} H_d(e^{j\Omega T}) & |\Omega| < \pi/T \\ 0 & \text{else} \end{cases} \]  

\[ = \begin{cases} e^{j\Omega T/2} - e^{-j\Omega T/2} & |\Omega| < \pi/T \\ 0 & \text{else} \end{cases}. \]  

(b) Here, \(x_d[n] = \frac{\sin(\Omega_M n T)}{\Omega_M n T} = \frac{\sin(n\pi)}{n\pi} \). Because the input is bandlimited, the composite system is LTI, and so \(Y_c(j\Omega) = H_c(j\Omega)X_c(j\Omega)\). In time domain, and using the expression for \(H_c\), we obtain

\[ y_c(t) = \frac{x_c(t + T/2) - x_c(t - T/2)}{T}. \]  

Plugging in \(x_c(t) = \frac{\sin(\Omega_M t)}{\Omega_M t}\) yields an explicit expression for \(y_c(t)\), namely,

\[ y_c(t) = \frac{\sin(\Omega_M t + \pi/2)}{\pi t + \pi/2} - \frac{\sin(\Omega_M t - \pi/2)}{\pi t - \pi/2}. \]  

Because \(y_c(t)\) is an ideal discrete-to-continuous reconstruction of \(y_d[n]\), we have \(y_d[n] = y_c(nT)\). An explicit expression can be obtained by plugging in:

\[ y_d[n] = \frac{1}{T} \left\{ \frac{\sin(n\pi + \pi/2)}{n\pi + \pi/2} - \frac{\sin(n\pi - \pi/2)}{n\pi - \pi/2} \right\} \]  

\[ = (-1)^n \frac{1}{T\pi(n^2 - 1/4)}. \]  

4.30 As follows:
8.28  (a) \( X[k] = \sum_{n=0}^{5} (6-n)W_6^{kn} \). This follows very directly from the formula for the DFT and the fact that \( x[n] = 6-n \) for \( 0 \leq n \leq 5 \). Any equivalent expression is ok.

(b) \( W[k] \) is a the DFT of a cyclic shift of \( x[n] \) by two samples. In your sketch, you should have \( w[0] = 2, w[1] = 1, w[2] = 6, w[3] = 5, w[4] = 4 \) and \( w[5] = 3 \).


d) \( x \) has length 6, and \( h \) has length 3. The linear convolution \( x * h \) will have length \( 3+6-1 = 8 \). So, we may choose any \( N \geq 8 \).

e) Length \( L \) cyclic convolution of \( h \) and \( x \) is equivalent to linear convolution of \( h \) with an \( L \)-periodized version of \( x \). Namely, if we set \( x_1[n] = x[n \text{ mod} L] \), then for \( n \in \{0, 1, \ldots, L - 1\} \), \( x_1 * h[n] = x @_L h[n] \). If we wish to minimize the number of samples that we need to add, you can check that we need only add \( M - 1 \) samples to the left of zero, setting

\[
x_1[n] = \begin{cases} x[n \text{ mod} L] & -(M - 1) \leq n \leq L - 1 \\ 0 & \text{else.} \end{cases}
\]

8.30  (a) Let \( \tilde{h}[n] = h[-n] \) be a time-reversed version of \( h \). Notice that \( \tilde{h} \) is supported from \( n = -31, \ldots, -18 \). The \( n \)-th element of the convolution \( y = x * h \) is just

\[
y[n] = \sum_{\ell} x[\ell] \tilde{h}[n - \ell].
\]
This is a dot product of $y$ with a version of $\hat{h}$ which has been shifted to the right by $n$ samples:

$$y[n] = \langle x, D_n \hat{h} \rangle.$$  \hspace{1cm} (10)

The shifted version $D_n \hat{h}$ is supported from $-31 + n, \ldots, -18 + n$. The smallest $n$ for which this overlaps with the support $21, \ldots, 31$ of $x$ occurs when

$$-18 + N_1 = 21,$$

while the largest $n$ for which $D \hat{h}$ overlaps with $x$ is

$$-31 + N_2 = 31.$$  \hspace{1cm} (12)

So, the output $y$ is supported from $N_1 = 39$ to $N_2 = 62$.

(b) For $n = 0, \ldots, 31$, let

$$\bar{x}_1[n] = x_1[n + 21 \text{ mod } 32]$$

$$\bar{h}_1[n] = h_1[n + 18 \text{ mod } 32].$$  \hspace{1cm} (13)

Notice that the linear convolution $\bar{x}_1 * \bar{h}_1[n]$ is simply $y[n + 39]$. Moreover, because length($\bar{x}_1$) + length($\bar{h}_1$) − 1 ≤ 32, linear convolution and cyclic convolution of these two signals are equivalent. So

$$\text{DFT}^{-1}\{\tilde{H}[k] \tilde{X}[k]\}[n] = y[n + 39], \hspace{1cm} n = 0, 1, \ldots, 32.$$  \hspace{1cm} (15)

By the cyclic shift property of the DFT,

$$Y[k] = \tilde{H}[k] \tilde{X}[k] \exp \left( -j \frac{2\pi k \times 39}{32} \right),$$

and so

$$\text{DFT}^{-1}\{Y\}[n] = \text{DFT}^{-1}\{\tilde{H} \tilde{X}\}[n - 39 \text{ mod } 32],$$  \hspace{1cm} (17)

giving

$$y_1[n] = y[(n - 39 \text{ mod } 32) + 39].$$  \hspace{1cm} (18)

(c) Both sequences are treated as length 32, we can set $N = 32 + 32 - 1 = 63$. For any $N < 63$, there will be spurious (incorrect) nonzero components in the output for $y[0], \ldots, y[63 - N - 1]$. 