In this lecture, we continue our discussion of the Z-transform, focusing on issues that arise when we try to invert the transform to recovery \( x[n] \).

1 Inverting the Z-transform

In this section, we describe approaches to computing the inverse Z transform. The most conceptually straightforward approach is to use the following inversion formula, which asserts that \( x[n] \) can be obtained by integrating \( X(z) \) over any closed contour \( C \) in the ROC:

\[
x[n] = \frac{1}{2\pi j} \oint_C X(z)z^{n-1} \, dz.
\] (1.1)

In this formula, \( C \) can be any closed contour encircling \( 0 \) and lying entirely in the ROC.

This formula looks intimidating. To make it more comfortable, consider the case in which the ROC contains the unit circle \( \{e^{j\omega} \mid \omega \in \mathbb{R}\} \). Taking the particular contour \( C = e^{j\omega} \bigg|_{-\pi}^{\pi} \), and noting that for \( z = e^{j\omega}, \, dz = je^{j\omega} \, d\omega \), we can express the contour integral as

\[
x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} \, d\omega,
\] (1.2)

which is just the familiar expression for the inverse DTFT. When the DTFT does not exist (i.e., the unit circle is not in the ROC), we can still invert the Z-transform by computing a similar integral around a different circle which does lie in the ROC.
It may seem strange that in the inversion formula (1.1), $C$ can be any closed contour that encloses zero and lies inside the ROC. The fact that the particular choice of contour does not matter follows from a remarkable result in complex analysis known as the Cauchy integral theorem. Proving this is beyond our scope. However, we will briefly justify (1.1) based on the related Cauchy integral formula, which asserts that

$$\frac{1}{2\pi j} \oint_C z^{-k} dz = \begin{cases} 1 & k = 1 \\ 0 & \text{else} \end{cases}.$$  

(1.3)

Based on this, consider the right hand side of (1.1):

$$\frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz = \frac{1}{2\pi j} \oint_C \sum_k x[k] z^{n-k-1} dz$$

$$= \sum_k x[k] \frac{1}{2\pi j} \oint_C z^{n-k-1} dz$$

$$= \sum_k x[k] \delta[n-k]$$

$$= x[n].$$  

(1.4)

Above, the interchange of integration and summation is justified by the fact that $C$ lies within the ROC, and hence the summation converges absolutely.

**Easier ways to deduce the inverse.** In practice, contour integration is complicated, and it is desirable to avoid it when possible. The most obvious way around it, called the "inspection method" in the text, is to use a table of known $Z$ transforms in conjunction with the properties of the $Z$-transform to try to guess the inverse.

A more structured approach is to use partial fraction expansion to write $X(z)$ as a superposition of simpler functions, and then invert each of these simpler functions individually. For our applications, we are almost exclusively interested in $Z$-transforms with a rational functional form. In this situation, we can use polynomial division and partial fraction expansion to express $Z$ as a superposition of simpler functions. Since the $Z$ transform is linear, we can then invert it by summing the inverses of these simpler functions. To make this more concrete, let us assume that

$$X(z) = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{\ell=0}^N a_\ell z^{-\ell}}$$

(1.5)

$$= \frac{b_0 \prod_{k=1}^m (1 - \zeta_k z^{-1})}{a_0 \prod_{\ell=1}^N (1 - \rho_\ell z^{-1})}$$

(1.6)

$$= \frac{P(z^{-1})}{Q(z^{-1})}.$$  

(1.7)

We have chosen here to express $X(z)$ as a quotient of polynomials in $z^{-1}$, rather than polynomials in $z$. This is for consistency with the text, which mostly describes rational $Z$-transforms as polynomials in $z^{-1}$. However, everything we will say about this situation translates into equivalent statements about polynomials in $z$, with very minor modifications.
In (1.6), the $\zeta_k$ are the zeros of $X(z)$, while the $\rho_\ell$ are the poles. $P$ is a polynomial of degree $M$; $Q$ is a polynomial of degree $N$. We first show how to express $X(z)$ as a sum of simpler terms when $M < N$ and the poles $\rho_1, \ldots, \rho_N$ are distinct:

**Proposition 1.1** (Partial fraction expansion, distinct poles). Suppose that $X(z)$ has the form (1.6), with $M < N$ and $\rho_1, \ldots, \rho_N$ distinct. Then

$$X(z) = \sum_{\ell=1}^{N} \frac{A_\ell}{1 - \rho_\ell z^{-1}}.$$  \hspace{1cm} (1.8)

with

$$A_\ell = \left. [(1 - \rho_\ell z^{-1})X(z)] \right|_{z=\rho_\ell}. \hspace{1cm} (1.9)$$

**Proof.** By placing everyone the over a common denominator, we obtain

$$\sum_{\ell=1}^{n} \frac{A_\ell}{1 - \rho_\ell z^{-1}} = \sum_{\ell=1}^{N} A_\ell \prod_{k \neq \ell} (1 - \rho_k z^{-1}) \prod_{\ell=1}^{N} (1 - \rho_\ell z^{-1}). \hspace{1cm} (1.10)$$

Both $X(z)$ and the above expression consist of polynomials of degree at most $N - 1$ divided by the polynomial $\prod_{\ell=1}^{n} (1 - \rho_\ell z^{-1})$. It is a basic fact in algebra that if two polynomials of degree at most $N - 1$ agree at $N$ distinct points, they are equal everywhere.\(^1\) Our choice of coefficients $A_\ell$ ensures that the two expressions agree at the $N$ points $\rho_1, \ldots, \rho_N$, and hence that $X(z) = \sum_{\ell=1}^{N} \frac{A_\ell}{1 - \rho_\ell z^{-1}}$. \(\blacksquare\)

The above proposition gives a very simple way of expanding certain rational functions $X(z)$ as sums of simpler functions, which are then easily inverted. However, it requires that in the expression $X(z) = P(z^{-1})/Q(z^{-1})$, $\deg(P) < \deg(Q)$. If this condition does not hold, we can use polynomial long division to write $X(z)$ as a sum of a polynomial in $z^{-1}$ and a rational function $R(z^{-1})/Q(z^{-1})$ with $\deg(R) < \deg(Q)$:

**Proposition 1.2** (Polynomial division). Suppose $X(z) = \frac{P(z^{-1})}{Q(z^{-1})}$, with $P$ and $Q$ polynomials. Then there is a unique expression

$$X(z) = D(z^{-1}) + \frac{R(z^{-1})}{Q(z^{-1})},$$  \hspace{1cm} (1.11)

with $R, D$ polynomials and $\deg(R) < \deg(Q)$.

**Proof.** To show that such a representation exists, we give an algorithm for constructing it. For an arbitrary polynomial $H(z^{-1}) = \sum_{\ell=0}^{d} h_\ell z^{-\ell}$, let $\text{lead}(H) = a_d z^{-d}$ denote the leading term. The algorithm is as follows:

1. Set $D = 0, R = P$.
2. while $\deg(R) \geq \deg(Q)$,
3. \hspace{0.5cm} Set $D = D + \frac{\text{lead}(R)}{\text{lead}(Q)}$.

\(^1\)This can be deduced from the fact that the Vandermonde matrix constructed from $N$ distinct points has full rank $N$.\]
Set \( R = R - \frac{\text{lead}(R)}{\text{lead}(Q)} Q \).

end while

To show that this algorithm produces a representation of the desired form, note that because at each iteration \( \deg(R) \geq \deg(Q) \), the term \( \frac{\text{lead}(R)}{\text{lead}(Q)} Q \) is always a polynomial in \( z^{-1} \). Hence, \( D \) remains a polynomial in \( z^{-1} \), as does \( R \). Moreover, the degree of \( R \) decreases by at least one in each iteration. Hence, after a certain number of iterations, the algorithm terminates with \( D \) and \( R \) polynomials in \( z^{-1} \) and \( \deg(R) < \deg(Q) \).

To show that this representation is unique, note that if \( X(z) = D'(z^{-1}) + R'(z^{-1})/Q(z^{-1}) \) is another factorization of the same form, then

\[
Q(z^{-1})(D - D'(z^{-1})) = (R' - R)(z^{-1}).
\]

(1.12)
The left hand side is either zero (if \( D' = D \)) or has degree at least \( \deg(Q) \). The right hand side has degree strictly smaller than \( \deg(Q) \), and so the only possibility is that \( D' = D \), which then implies that \( R' = R \).

The proof of the above proposition gives a very practical way for computing \( D \) and \( R \), which you may recognize as polynomial long division. If we are confronted with a function \( X(z) \) of the form

\[
X(z) = \frac{b_0 \prod_{i=1}^{M} (1 - \zeta_i z^{-1})}{a_0 \prod_{\ell=1}^{N} (1 - \rho_{\ell} z^{-1})},
\]

(1.13)

with \( \rho_1, \ldots, \rho_N \) distinct, but \( M \geq N \), we can use Proposition 1.2 and Proposition 1.1 to express \( X(z) \) as

\[
X(z) = D(z^{-1}) + \frac{R(z^{-1})}{\prod_{\ell=1}^{N} (1 - \rho_{\ell} z^{-1})}
\]

(1.14)
\[
= D(z^{-1}) + \sum_{\ell=1}^{N} \frac{A_{\ell}}{1 - \rho_{\ell} z^{-1}},
\]

(1.15)
which can be inverted piece-by-piece.

The final generalization needed is to consider rational \( X(z) \) in which there may be repeated poles. In this case, the form of the partial fraction expansion becomes slightly more complicated:

**Proposition 1.3.** Suppose that

\[
X(z) = \frac{P(z^{-1})}{Q(z^{-1})} = \frac{P(z^{-1})}{\prod_{\ell=1}^{U} (1 - \rho_{\ell} z^{-1})^{d_{\ell}}},
\]

(1.16)

where \( \deg(P) < \deg(Q) \), \( U \) is the number of unique (distinct) roots of \( Q \), and \( \rho_1, \ldots, \rho_U \) are distinct. Then

\[
X(z) = \sum_{\ell=1}^{U} \sum_{k=1}^{d_{\ell}} \frac{A_{\ell,k}}{(1 - \rho_{\ell} z^{-1})^k},
\]

(1.17)
with

\[
A_{\ell,k} = \frac{1}{(d_{\ell} - k)!(-\rho_{\ell})^{d_{\ell} - k}} \left[ \frac{d^{(d_{\ell} - k)}(1 - \xi_{\ell} w)^d X(w^{-1})}{dw^{d_{\ell} - k}} \right]_w=\rho_{\ell}^{-1}
\]

(1.18)
An example. The following example is taken from the text. We consider

$$X(z) = \frac{z^{-2} + 2z^{-1} + 1}{\frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1}, \quad |z| > 1. \quad (1.19)$$

Factoring the numerator and denominator, we obtain

$$X(z) = \frac{(1 + z^{-1})^2}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})}. \quad (1.20)$$

Thus, there are no repeated poles. However, the numerator and denominator have the same degree, and so we cannot directly apply partial fraction expansion. Using polynomial long division, we obtain that

$$X(z) = 2 + \frac{5z^{-1} - 1}{(1 - \frac{1}{2}z^{-1})(1 - z^{-1})} \quad (1.21)$$

We then express the second term as

$$\frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}} \quad (1.22)$$

where

$$A_1 = \left. \frac{5z^{-1} - 1}{1 - z^{-1}} \right|_{z=1/2} = -9 \quad (1.23)$$

$$A_2 = \left. \frac{5z^{-1} - 1}{1 - \frac{1}{2}z^{-1}} \right|_{z=1} = 8, \quad (1.24)$$

and so

$$X(z) = 2 - \frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}}. \quad (1.25)$$

Since

$$2\delta[n] \leftrightarrow 2 \quad (1.26)$$

$$\left(\frac{1}{2}\right)^n u[n] \leftrightarrow \frac{1}{1 - \frac{1}{2}z^{-1}}, \quad |z| > 1/2 \quad (1.27)$$

$$u[n] \leftrightarrow \frac{1}{1 - z^{-1}}, \quad |z| > 1 \quad (1.28)$$

we conclude that

$$x[n] = 2\delta[n] - 9 \left(\frac{1}{2}\right)^n u[n] + 8u[n]. \quad (1.29)$$

Power series expansion. The last trick for avoiding heavy calculation in the inverse \(Z\)-transform is to use power series expansion. Namely, we seek to express \(X(z)\) as a sum \(X(z) = \sum_{n=-\infty}^{\infty} \alpha_n z^{-n}\), and then read of the values \(x[n] = \alpha_n\). This approach is perhaps best illustrated through an example.
Let us suppose that \( X(z) = \log(1 + az^{-1}) \), with ROC \(|z| > |a|\). We know that \( \log(1 + s) \) has a convergent Taylor series for \(|s| \leq 1\), namely,

\[
\log(1 + s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} s^n}{n}.
\]

(1.30)

Setting \( s = az^{-1} \), we obtain

\[
\log(1 + az^{-1}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n},
\]

(1.31)

from which we read off

\[
x[n] = \begin{cases} 
\frac{(-1)^{n+1} a^n}{n} & n \geq 1 \\
0 & \text{else}
\end{cases}.
\]

(1.32)