Disclaimer: These notes are intended to be an accessible introduction to the subject, with no pretense at completeness. In general, you can find more thorough discussions in Oppenheim’s book. Please let me know if you find any typos.

Reading suggestions: Oppenheim and Schafer Sections 2.1-2.4

Homework: HW0 is due Wednesday September 16.

Optional (nongraded) homework: Spend some time pondering Euler’s formula. Make a picture in Matlab to convince yourself that Euler’s formula is true. Make sure you have a crisp understanding of the meaning of infinite summations. This is a technical point, but a clear understanding will be very helpful when we get to the Z-transform.

In this lecture, we will begin by finishing the material on complex math from the last lecture. We will focus our attention in particular on Euler’s formula and the technicalities associated with the existence of infinite sums. The remainder of this lecture serves to introduce some basic notation. We will talk about basic classes of discrete time signals, and some important considerations that arise when talking about discrete time sinusoids and periodic signals.

1 Classes of Complex Signals

Symmetric, and conjugate symmetric signals. We say that a signal \( x[n] \) is symmetric (or even) if

\[
x[n] = x[-n]. \tag{1.1}
\]

That is to say, the signal is unchanged if we flip it about zero. We say that a signal is antisymmetric, or odd, if

\[
x[n] = -x[-n]. \tag{1.2}
\]

That is to say, flipping the signal about zero produces its negative.
When we deal with complex signals \( x \in \mathbb{C}^Z \), it is often useful to consider the effect of both flipping the signal about \( n = 0 \), and taking the complex conjugate. We say that a signal \( x[n] \) is \textit{conjugate symmetric} if

\[
x[n] = x^*[−n],
\]

(1.3)

Check for yourself that \( x[n] \) is conjugate symmetric if and only if \( \text{Re}[x] \) is an even sequence and \( \text{Im}[x] \) is an odd sequence.

We say that \( x[n] \) is \textit{conjugate antisymmetric} if

\[
x[n] = −x^*[−n].
\]

(1.4)

A sequence \( x[n] \) is conjugate antisymmetric if and only if \( \text{Re}[x] \) is an odd sequence and \( \text{Im}[x] \) is an even sequence.

It is not particularly common to encounter (conjugate) symmetric signals in nature. However, we will see that the definition is still very useful for talking about properties of the Fourier and \( Z \) transforms.\(^1\) We can express any given input signal as a sum of a conjugate symmetric and conjugate antisymmetric signal:

**Proposition 1.1.** Every signal \( x[n] \) can be written as a sum

\[
x[n] = x_{cs}[n] + x_{ca}[n]
\]

(1.5)

of a conjugate symmetric signal \( x_{cs}[n] \) and a conjugate anti-symmetric signal \( x_{ca}[n] \).

**Proof.** Write

\[
x_{cs}[n] = \frac{x[n] + x^*[−n]}{2}
\]

(1.6)

and

\[
x_{ca}[n] = \frac{x[n] − x^*[−n]}{2}
\]

(1.7)

Notice that \( x_{cs}[−n] = x_{cs}[n] \), so \( x_{cs} \) is indeed conjugate-symmetric, while \( x_{ca}[−n] = −x_{ca}[n] \), so \( x_{ca} \) is indeed conjugate anti-symmetric. Finally, observe that

\[
x_{cs}[n] + x_{ca}[n] = \frac{1}{2}\left(x[n] + x^*[−n] + x[n] − x^*[−n]\right) = x[n],
\]

(1.8)

as desired. \( \square \)

## 2 Basic Signals

In this section, we list a few basic sequences which show up frequently in our analysis of discrete-time systems.

\(^1\)For example, you may recall that a (continuous-time) signal \( f(t) \) is real valued if and only if its (continuous-time) Fourier transform is conjugate symmetric.
The unit impulse. The unit impulse $\delta[n]$ is defined as

$$\delta[n] = \begin{cases} 1 & n = 0, \\ 0 & \text{else}. \end{cases}$$  \hspace{1cm} (2.1)$$

In contrast to the continuous-time Dirac delta, there are no serious technical complications in the definition of the unit impulse $\delta[n]$. To get an impulse at location $k$, we simply apply a shift:

$$\delta[n-k] = \begin{cases} 1 & n = k, \\ 0 & \text{else}. \end{cases}$$  \hspace{1cm} (2.2)$$

We can express any input signal $x[n]$ as a superposition of shifted impulses:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k].$$  \hspace{1cm} (2.3)$$

This basic formula will be very useful next lecture when we study linear time invariant (LTI) systems and their impulse response – because any signal $x[n]$ can be constructed as a superposition of shifted unit impulses, we will be able to completely characterize an LTI system by studying how it responds to a unit impulse.

The unit step. The unit step function is

$$u[n] = \begin{cases} 0 & n < 0, \\ 1 & n \geq 0. \end{cases}$$  \hspace{1cm} (2.4)$$

The unit step can be viewed as the “integral” of a unit impulse $\delta[\cdot]$:

$$u[n] = \sum_{k=-\infty}^{n} \delta[k].$$  \hspace{1cm} (2.5)$$

Please stop to think carefully about this formula! To verify that it is true, just check that the quantities on both sides agree for every choice of $n \in \mathbb{Z}$.

Please also take a moment to verify the following formula, which goes in the other direction and expresses $\delta[n]$ as a difference of unit steps:

$$\delta[n] = u[n] - u[n-1].$$  \hspace{1cm} (2.6)$$

Exponential sequences. An exponential sequence has the general form

$$x[n] = A \alpha^n.$$  \hspace{1cm} (2.7)$$

When $A$ and $\alpha$ are real, $x[n]$ either grows very rapidly in magnitude (if $|\alpha| > 1$) or decays very rapidly in magnitude (if $|\alpha| < 1$). If $A$ and $\alpha$ are complex, the situation is more complicated. If their polar representations are

$$A = |A| \exp(j\phi)$$  \hspace{1cm} (2.8)$$
and
\[ \alpha = |\alpha| \exp(j\omega) \] (2.9)
then we can write
\[ x[n] = A\alpha^n = |A||\alpha|^n \exp(j(\omega n + \phi)). \] (2.10)

Using Euler’s formula, we can further express this signal as
\[ x[n] = |A||\alpha|^n \cos(\omega n + \phi) + j|A||\alpha|^n \sin(\omega n + \phi). \] (2.11)

That is to say, the real and complex parts of \( x[n] \) are sinusoids, with frequency \( \omega \), and phase shift \( \phi \). They are modulated by an exponentially increasing (or decreasing) sequence \( |A||\alpha|^n \).

A discrete-time signal \( x[n] \) is only defined for integers \( n \). This means that the signals
\[ x[n] = A \exp(j\omega n) \] (2.12)
and
\[ y[n] = A \exp(j(\omega + 2\pi) n) \]
\[ = A \exp(j\omega n) \exp(2\pi n) \]
\[ = A \exp(j\omega n) \quad \forall n \in \mathbb{Z} \] (2.13)
\[ = x[n] \] (2.14)
are indistinguishable. So, for discrete time complex exponentials and sinusoids, the frequency \( \omega \) is only defined up addition by a multiple of \( 2\pi \). We usually restrict our attention to frequencies \( \omega \) in an interval of length \( 2\pi \): for example, \( -\pi < \omega \leq \pi \), or \( 0 \leq \omega < 2\pi \).

Because of this ambiguity, we need to be careful about the interpretation of the frequency \( \omega \). In continuous time, higher frequency signals oscillate more quickly. In discrete time, if we consider a sinusoidal or complex exponential signal with frequency \( \omega \), say,
\[ x[n] = \cos(\omega n), \] (2.15)
then as \( \omega \) increases from 0 to \( \pi \), the signal will oscillate more and more quickly. However, if we continue to increase \( \omega \) beyond \( \pi \), the signal \( x[n] \) oscillates less quickly. When \( \omega \) reaches \( 2\pi \), the signal is actually constant! So, in general, bigger \( \omega \) does not imply faster oscillation – this only happens when \( \omega \) belongs to certain intervals.

**Discrete-time periodic signals.** A signal \( x[n] \) is periodic with period \( N \), if
\[ x[n] = x[n + N], \quad \forall n \in \mathbb{Z}. \] (2.16)

That is to say, every \( N \) samples, the signal repeats itself. Because \( n \) is restricted to be an integer, discrete-time complex exponentials and sinusoids are not necessarily periodic. A necessary and sufficient condition for a discrete time sinusoidal sequence to have period \( N \) is that the frequency \( \omega \) is an integer multiple of \( 2\pi k/N \):
Proposition 2.1. The signals

\begin{align*}
x_1[n] &= \exp\{j(\omega n + \phi)\} & (2.17) \\
x_2[n] &= \cos(\omega n + \phi) & (2.18) \\
x_3[n] &= \sin(\omega n + \phi) & (2.19)
\end{align*}

are periodic with period \(N\) if and only if \(\omega = 2\pi k/N\) for some integer \(k \in \mathbb{Z}\).

Try to convince yourself of this fact. For completeness, I’ve sketched a proof in Appendix A.

Consider a complex exponential sequence \(x[n] = \exp\{j(\omega n + \phi)\}\), and suppose that this sequence is periodic with period \(N\). From the previous proposition, we know that \(\omega = 2\pi k/N\) for some integer \(k\). Moreover, for every integer \(L\), the frequencies \(\omega\) and \(\omega + 2\pi L\) produce exactly the same sequence \(x[n]\). So, we can assume that \(\omega\) belongs to the set

\[
\left\{ 0, \frac{2\pi}{N}, \frac{2\pi}{N} \times 2, \ldots, \frac{2\pi}{N} \times (N-1) \right\}.
\]

That is to say, there are exactly \(N\) distinct frequencies \(\omega\) of \(N\)-periodic complex exponentials. More formally:

**Proposition 2.2.** Suppose that \(x[n] = \exp\{j(\omega n + \phi)\}\) is periodic, with period \(N\). Then

\[
x[n] = \exp\{j(\omega_0 n + \phi)\}
\]

for some \(\omega_0\) in the set

\[
\Gamma = \left\{ 0, \frac{2\pi}{N}, \frac{2\pi}{N} \times 2, \ldots, \frac{2\pi}{N} \times (N-1) \right\}.
\]

This fact will be very important when we start working with finite length signals and the Discrete Fourier Transform (DFT).
Appendix: Proof of Proposition 2.1

Proof. The “if” portion of the claim is straightforward. The “only if” requires a bit of calculation. Consider first the function \( \sin(\omega n + \phi) \). Note that

\[
\sin(\theta) = \sin(\varphi)
\] (A.1)

implies that either

\[
\theta = \varphi + 2\pi M
\] (A.2)

or

\[
\pi - \theta = \varphi + 2\pi M,
\] (A.3)

for some integer \( M \). Periodicity implies that for all \( n \),

\[
\sin(\omega n + \phi) = \sin(\omega(n + N) + \phi).
\] (A.4)

If, for some \( n \),

\[
\omega n + \phi = \omega(n + N) + \phi + 2\pi M,
\] (A.5)

this implies that \( \omega = -2\pi M/N \) is of the desired form. Suppose instead that for every \( n \),

\[
\pi - (\omega n + \phi) = \omega(n + N) + \phi + 2\pi M_n,
\] (A.6)

for some integer \( M_n \). Evaluating at \( n = 0 \), we obtain that

\[
2\phi = \pi - \omega N - 2\pi M_0.
\] (A.7)

Simplifying (A.6), we obtain

\[
-\omega(2n + N) = -\pi + 2\phi + 2\pi M_n,
\] (A.8)

and plugging in gives

\[
-\omega(2n + N) = -\omega N + 2\pi(M_n - M_0),
\] (A.9)

Evaluating at \( n = 1 \), we obtain

\[
\omega = -\pi(M_n - M_0)
\] (A.10)

so \( \omega \) is an integer multiple of \( \pi \). Hence, there are essentially two possible frequencies \( \omega \): \( \omega = 0 \), and \( \omega = \pi \). \( \omega = 0 \) has the requisite form. Notice that if \( N \) is odd, \( \omega = \pi \) does not yield an \( N \)-periodic signal, and hence is impossible, while if \( N \) is even, \( \omega = \pi \) has the requisite form. This completes the proof for \( \sin \). Using that \( \cos \theta = \sin(\pi/2 - \theta) \), this implies the claim for \( \cos \). The results for \( \cos \) and \( \sin \) imply the claim for general complex exponentials, via Euler’s formula. \( \Box \)