

ELEN 4720: Machine Learning for Signals, Information and Data

Lecture 10

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FEATURE EXPANSIONS

FEATURE EXPANSIONS

Feature expansions (also called **basis expansions**) are names given to a technique we've already discussed and made use of.

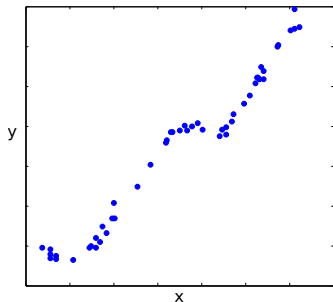
Problem: A linear model on the original feature space $x \in \mathbb{R}^d$ doesn't work.

Solution: Map the features to another space $\phi(x) \in \mathbb{R}^D$, often where $D > d$, and do linear modeling there.

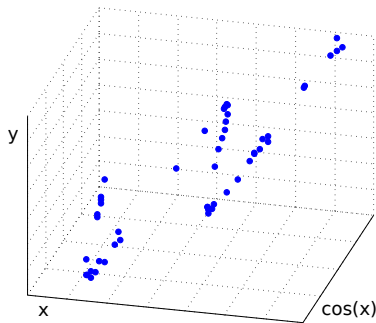
Examples

- ▶ For polynomial regression on $\mathbb{R}$, we let $\phi(x) = (x, x^2, \dots, x^p)^T$.
- ▶ For one jump discontinuity, we could set $\phi(x) = (x, \mathbb{1}\{x < a\})^T$.

MAPPING EXAMPLE FOR REGRESSION



(a) Data for linear regression



(b) Same data mapped to higher dimension

High-dimensional maps can transform the data so output is linear in inputs.

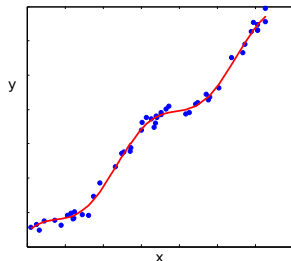
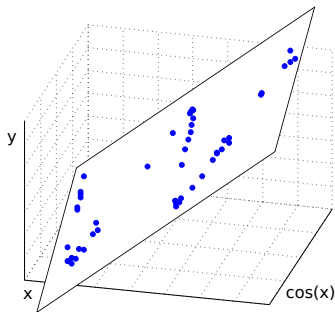
Left: Original $x \in \mathbb{R}$ and response y .

Right: x mapped to $\mathbb{R}^2$ using $\phi(x) = (x, \cos x)^T$.

MAPPING EXAMPLE FOR REGRESSION

Using the mapping $\phi(x) = (x, \cos x)^T$, learn the linear regression model

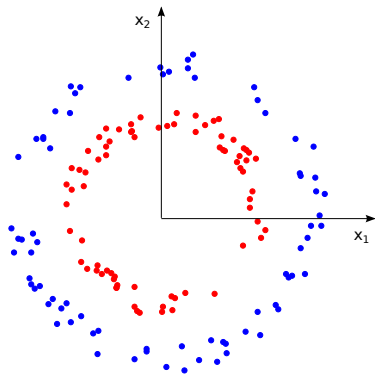
$$\begin{aligned}y &\approx w_0 + \phi(x)^T w \\ &\approx w_0 + w_1 x + w_2 \cos x.\end{aligned}$$



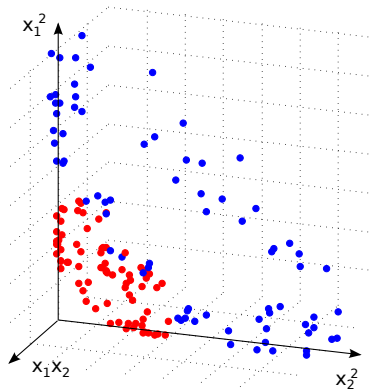
Left: Learn (w_0, w_1, w_2) to approximate data on the left with a plane.

Right: For each point x , map to $\phi(x)$ and predict y . Plot as a function of x .

MAPPING EXAMPLE FOR CLASSIFICATION



(e) Data for binary classification



(f) Same data mapped to higher dimension

High-dimensional maps can transform data so it becomes linearly separable.

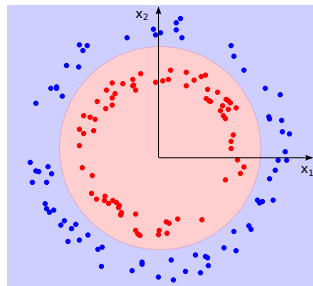
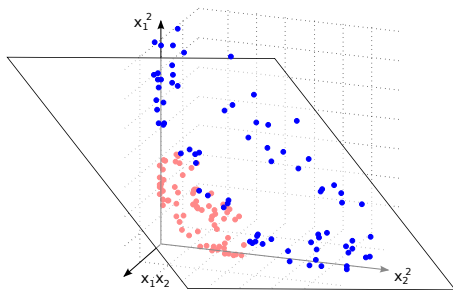
Left: Original data in $\mathbb{R}^2$.

Right: Data mapped to $\mathbb{R}^3$ using $\phi(x) = (x_1^2, x_1x_2, x_2^2)^T$.

MAPPING EXAMPLE FOR CLASSIFICATION

Using the mapping $\phi(x) = (x_1^2, x_1x_2, x_2^2)^T$, learn a linear classifier

$$\begin{aligned}y &= \text{sign}(w_0 + \phi(x)^T w) \\ &= \text{sign}(w_0 + w_1x_1^2 + w_2x_1x_2 + w_3x_2^2).\end{aligned}$$



Left: Learn (w_0, w_1, w_2, w_3) to linearly separate classes with hyperplane.

Right: For each point x , map to $\phi(x)$ and classify. Color decision regions in $\mathbb{R}^2$.

FEATURE EXPANSIONS AND DOT PRODUCTS

What expansion should I use?

This is not obvious. The illustrations required knowledge about the data that we likely won't have (especially if it's in high dimensions).

One approach is to use the “kitchen sink”: If you can think of it, then use it. Select the useful features with an ℓ_1 penalty

$$w_{\ell_1} = \arg \min_w \sum_{i=1}^n f(y_i, \phi(x_i), w) + \lambda \|w\|_1.$$

We know that this will find a sparse subset of the dimensions of $\phi(x)$ to use.

Often we only need to work with dot products $\phi(x_i)^T \phi(x_j) \equiv K(x_i, x_j)$. This is called a **kernel** and can produce some interesting results.

KERNELS

PERCEPTRON (SOME MOTIVATION)

Perceptron classifier

Let $x_i \in \mathbb{R}^{d+1}$ and $y_i \in \{-1, +1\}$ for $i = 1, \dots, n$ observations. We saw that the Perceptron constructs the hyperplane from data,

$$w = \sum_{i \in \mathcal{S}} y_i x_i, \quad (\text{assume } \eta = 1 \text{ and } \mathcal{S} \text{ has no duplicates})$$

where $\mathcal{S}$ is the sequentially constructed set of misclassified examples.

Predicting new data

We also discussed how we can predict the label y_0 for a new observation x_0 :

$$y_0 = \text{sign}(x_0^T w) = \text{sign} \left(\sum_{i \in \mathcal{S}} y_i x_0^T x_i \right)$$

We've taken feature expansions for granted, but we can explicitly write it as

$$y_0 = \text{sign}(\phi(x_0)^T w) = \text{sign} \left(\sum_{i \in \mathcal{S}} y_i \phi(x_0)^T \phi(x_i) \right)$$

We can represent the decision using dot products between data points.

KERNELS

Kernel definition

A kernel $K(\cdot, \cdot) : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ is a symmetric function defined as follows:

Definition: If for any n points $x_1, \dots, x_n \in \mathbb{R}^d$, the $n \times n$ matrix K , where $K_{ij} = K(x_i, x_j)$, is *positive semidefinite*, then $K(\cdot, \cdot)$ is a “kernel.”

Intuitively, this means K satisfies the properties of a covariance matrix.

Mercer's theorem

If the function $K(\cdot, \cdot)$ satisfies the above properties, then there exists a mapping $\phi : \mathbb{R}^d \rightarrow \mathbb{R}^D$ (D can equal ∞) such that

$$K(x_i, x_j) = \phi(x_i)^T \phi(x_j).$$

If we first define $\phi(\cdot)$ and then K , this is obvious. However, sometimes we first define $K(\cdot, \cdot)$ and avoid ever using $\phi(\cdot)$.

GAUSSIAN KERNEL (RADIAL BASIS FUNCTION)

The most popular kernel is the Gaussian kernel, also called the radial basis function (RBF),

$$K(x, x') = a \exp \left\{ -\frac{1}{b} \|x - x'\|^2 \right\}.$$

- ▶ This is a good, general-purpose kernel that usually works well.
- ▶ It takes into account proximity in $\mathbb{R}^d$. Things close together in space have larger value (as defined by kernel width b).

In this case, the the mapping $\phi(x)$ that produces the RBF kernel is *infinite dimensional* (it's a continuous function instead of a vector). Therefore

$$K(x, x') = \int \phi_t(x) \phi_t(x') dt.$$

- ▶ $\phi_t(x)$ can be thought of as a function of t with parameter x that also has a Gaussian-looking form.

Another kernel

Map : $\phi(x) = (1, \sqrt{2}x_1, \dots, \sqrt{2}x_d, x_1^2, \dots, x_d^2, \dots, \sqrt{2}x_i x_j, \dots)$

Kernel : $\phi(x)^T \phi(x') = K(x, x') = (1 + x^T x')^2$

In fact, we can show $K(x, x') = (1 + x^T x')^c$, for $c > 0$ is a kernel as well.

Kernel arithmetic

Certain functions of kernels can produce new kernels.

Let K_1 and K_2 be any two kernels, then constructing K in the following ways produces a new kernel (among many other ways):

$$K(x, x') = K_1(x, x')K_2(x, x')$$

$$K(x, x') = K_1(x, x') + K_2(x, x')$$

$$K(x, x') = \exp\{K_1(x, x')\}$$

KERNELIZED PERCEPTRON

Returning to the Perceptron

We write the feature-expanded decision as

$$\begin{aligned}y_0 &= \text{sign} \left(\sum_{i \in \mathcal{S}} y_i \phi(x_0)^T \phi(x_i) \right) \\ &= \text{sign} \left(\sum_{i \in \mathcal{S}} y_i K(x_0, x_i) \right)\end{aligned}$$

We can pick the kernel we want to use. Let's pick the RBF (set $a = 1$). Then

$$y_0 = \text{sign} \left(\sum_{i \in \mathcal{S}} y_i e^{-\frac{1}{b} \|x_0 - x_i\|^2} \right)$$

Notice that we never actually need to calculate $\phi(x)$.

What is this doing?

- ▶ Notice $0 < K(x_0, x_i) \leq 1$, with bigger values when x_0 is closer to x_i .
- ▶ This is like a “soft voting” among the data picked by Perceptron.

KERNELIZED PERCEPTRON (OPTIONAL)

Learning the kernelized Perceptron

Recall: Given a current vector $w^{(t)} = \sum_{i \in \mathcal{S}_t} y_i x_i$, we update it as follows,

1. Find a new x' such that $y' \neq \text{sign}(x'^T w^{(t)})$
2. Add the index of x' to $\mathcal{S}$ and set $w^{(t+1)} = \sum_{i \in \mathcal{S}_{t+1}} y_i x_i$

Again we only need dot products, meaning these steps are equivalent to

1. Find a new x' such that $y' \neq \text{sign}(\sum_{i \in \mathcal{S}_t} y_i K(x', x_i))$
2. Add the index of x' to $\mathcal{S}$ *but don't bother calculating* $w^{(t+1)}$

The trick is to realize that we never need to work with $\phi(x)$.

- ▶ We don't need $\phi(x)$ to do Step 1 above.
- ▶ We don't need $\phi(x)$ to classify new data (previous slide).
- ▶ We only ever need to calculate $K(x, x')$ between two points.

KERNEL k -NN (OPTIONAL)

An extension

We can generalize kernelized Perceptron to *soft* k -NN with a simple change. Instead of summing over misclassified data $\mathcal{S}$, sum over *all* the data:

$$y_0 = \text{sign} \left(\sum_{i=1}^n y_i e^{-\frac{1}{b} \|x_0 - x_i\|^2} \right).$$

Next, notice the *decision* doesn't change if we divide by a positive constant.

Let : $Z = \sum_{j=1}^n e^{-\frac{1}{b} \|x_0 - x_j\|^2}$

Construct : Vector $p(x_0)$, where $p_i(x_0) = \frac{1}{Z} e^{-\frac{1}{b} \|x_0 - x_i\|^2}$

Declare : $y_0 = \text{sign} \left(\sum_{i=1}^n y_i p_i(x_0) \right)$

- ▶ We let all data vote for the label based on a “confidence score” $p(x_0)$.
- ▶ Set b so that most $p_i(x_0) \approx 0$ to only focus on neighborhood around x_0 .

KERNEL REGRESSION (OPTIONAL)

Nadaraya-Watson model

The developments are almost limitless.

Here's a regression example almost identical to the kernelized k -NN:

Before: $y \in \{-1, +1\}$

Now: $y \in \mathbb{R}$

Using the RBF kernel, for a new (x_0, y_0) predict

$$y_0 = \sum_{i=1}^n y_i \frac{K(x_0, x_i)}{\sum_{j=1}^n K(x_0, x_j)}.$$

What is this doing?

We're taking a locally weighted average of all y_i for which x_i is close to x_0 (as decided by the kernel width). *Gaussian processes* are another option. . .

GAUSSIAN PROCESSES (FOR REGRESSION)

KERNELIZED BAYESIAN LINEAR REGRESSION

Regression setup: For n observations, with response vector $y \in \mathbb{R}^n$ and their feature matrix X , we define the likelihood and prior

$$y \sim N(Xw, \sigma^2 I), \quad w \sim N(0, \lambda^{-1} I).$$

Marginalizing: What if we integrate out w ? We can solve this,

$$p(y|X) = \int p(y|X, w)p(w)dw = N(0, \sigma^2 I + \lambda^{-1} XX^T).$$

Kernelization: Notice that $(XX^T)_{ij} = x_i^T x_j$. Replace each x with $\phi(x)$ after which we can say $[\phi(X)\phi(X)^T]_{ij} = K(x_i, x_j)$. We can define K directly, so

$$p(y|X) = \int p(y|X, w)p(w)dw = N(0, \sigma^2 I + \lambda^{-1} K).$$

This is called a *Gaussian process*. We never use w or $\phi(x)$, but just $K(x_i, x_j)$.

GAUSSIAN PROCESSES

Definition

- Let $f(x) \in \mathbb{R}$ and $x \in \mathcal{X}$.
- Define the *kernel* $K(x, x')$ between two points x and x' .
- Then $f(x)$ is a *Gaussian process* and $y(x)$ the noise-added process if for n observed pairs $(x_1, y_1), \dots, (x_n, y_n)$, where $x \in \mathcal{X}$ and $y \in \mathbb{R}$,

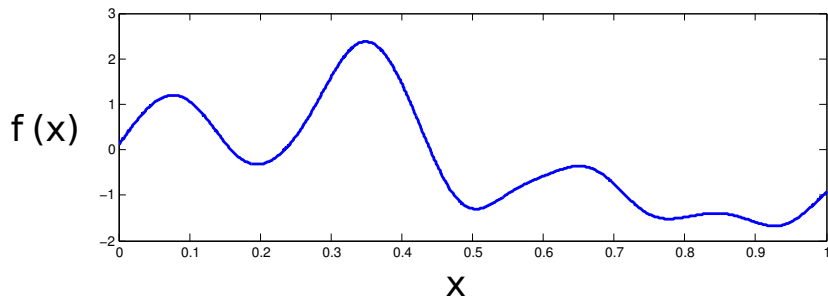
$$y | f \sim N(f, \sigma^2 I), \quad f \sim N(0, K) \quad \Longleftrightarrow \quad y \sim N(0, \sigma^2 I + K)$$

where $y = (y_1, \dots, y_n)^T$ and K is $n \times n$ with $K_{ij} = K(x_i, x_j)$.

Comments:

- ▶ We assume $\lambda = 1$ to reduce notation.
- ▶ Typical breakdown: $f(x)$ is the GP and $y(x)$ equals $f(x)$ plus i.i.d. noise.
- ▶ The kernel is what keeps this from being “just a Gaussian.”

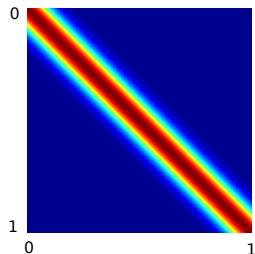
GAUSSIAN PROCESSES



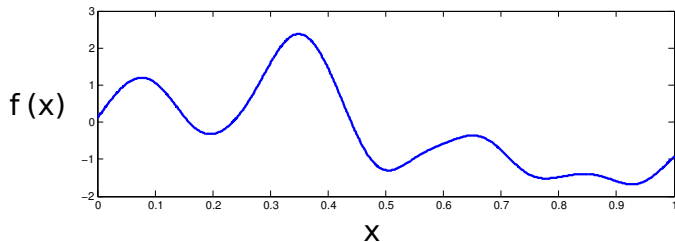
Above: A Gaussian process $f(x)$ ($x \in \mathbb{R}$) using

$$K(x_i, x_j) = \exp \left\{ -\frac{\|x_i - x_j\|^2}{b} \right\}.$$

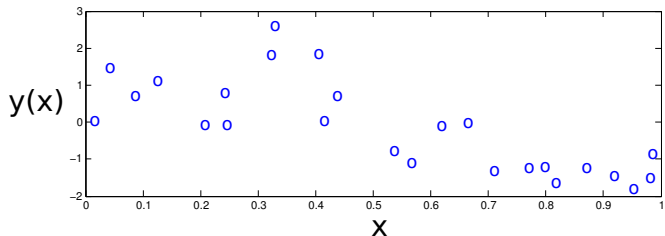
Right: The covariance of $f(x)$ defined by K .



GAUSSIAN PROCESSES



Top: Unobserved underlying function,
Bottom: Noisy observed data sampled from this function



PREDICTIONS WITH GAUSSIAN VECTORS

Bayesian linear regression

Imagine we have n observation pairs $\mathcal{D} = \{(x_i, y_i)\}_{i=1}^N$ and want to predict y_0 given x_0 . Integrating out w and setting $\lambda = 1$, the joint distribution is

$$\begin{bmatrix} y_0 \\ y \end{bmatrix} \sim \text{Normal} \left(\mathbf{0}, \sigma^2 I + \begin{bmatrix} x_0^T x_0 & (Xx_0)^T \\ Xx_0 & XX^T \end{bmatrix} \right)$$

We want to predict y_0 given $\mathcal{D}$ and x_0 . Calculations can show that

$$\begin{aligned} y_0 | \mathcal{D}, x_0 &\sim \text{Normal}(\mu_0, \sigma_0^2) \\ \mu_0 &= (Xx_0)^T (\sigma^2 I + XX^T)^{-1} y \\ \sigma_0^2 &= \sigma^2 + x_0^T x_0 - (Xx_0)^T (\sigma^2 I + XX^T)^{-1} (Xx_0) \end{aligned}$$

Since the Gaussian process is only ever evaluated at a finite set of points, this is useful. Notice all the dot products between x_i . These can become kernels.

PREDICTIONS WITH GAUSSIAN PROCESSES

Predictive distribution of $y(x)$

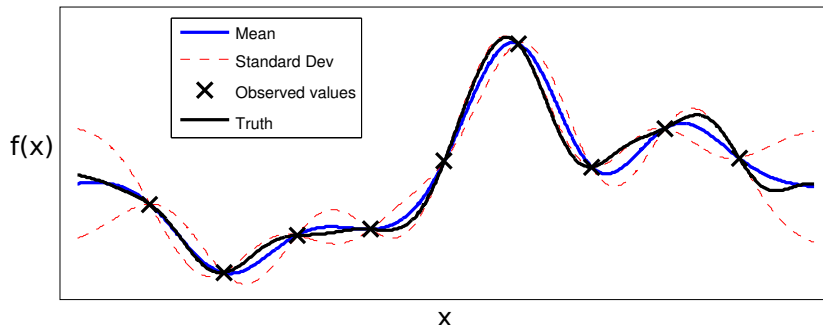
Given measured data $\mathcal{D}_n = \{(x_1, y_1), \dots, (x_n, y_n)\}$, the distribution of $y(x)$ can be calculated at any *new* x to make predictions.

Let $K(x, \mathcal{D}_n) = [K(x, x_1), \dots, K(x, x_n)]$ and K_n be the $n \times n$ kernel matrix restricted to points in $\mathcal{D}_n$. Then kernelizing the previous slide, we see that

$$\begin{aligned}y(x)|\mathcal{D}_n &\sim N(\mu(x), \Sigma(x)), \\ \mu(x) &= K(x, \mathcal{D}_n)(\sigma^2 I + K_n)^{-1}y, \\ \Sigma(x) &= \sigma^2 + K(x, x) - K(x, \mathcal{D}_n)(\sigma^2 I + K_n)^{-1}K(x, \mathcal{D}_n)^T\end{aligned}$$

For the posterior of $f(x)$ instead of $y(x)$, just remove σ^2 .

GAUSSIAN PROCESSES POSTERIOR



What does the posterior distribution of $f(x)$ look like?

- ▶ We have data marked by an $\times$.
- ▶ These values pin down the function $f(x)$ nearby
- ▶ We get a mean and variance for every possible x from previous slide.
- ▶ The distribution on $y(x)$ adds variance σ^2 (*very small above*) point-wise.