

# ELEN 4720: Machine Learning for Signals, Information and Data

## Lecture 13

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# BAGGING CLASSIFIERS

## Algorithm: Bagging binary classifiers

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Given  $(x_1, y_1), \dots, (x_n, y_n), x \in \mathcal{X}, y \in \{-1, +1\}$

- ▶ For  $b = 1, \dots, B$ 
  - ▶ Sample a bootstrap dataset  $\mathcal{B}_b$  of size  $n$ . For each entry in  $\mathcal{B}_b$ , select  $(x_i, y_i)$  with probability  $\frac{1}{n}$ . Some  $(x_i, y_i)$  will repeat and some won't appear in  $\mathcal{B}_b$ .
  - ▶ Learn a classifier  $f_b$  using data in  $\mathcal{B}_b$ .
- ▶ Define the classification rule to be

$$f_{bag}(x_0) = \text{sign} \left( \sum_{b=1}^B f_b(x_0) \right).$$

- 
- ▶ With bagging, we observe that a *committee* of classifiers votes on a label.
  - ▶ Each classifier is learned on a *bootstrap sample* from the data set.
  - ▶ Learning a collection of classifiers is referred to as an *ensemble method*.

# BOOSTING

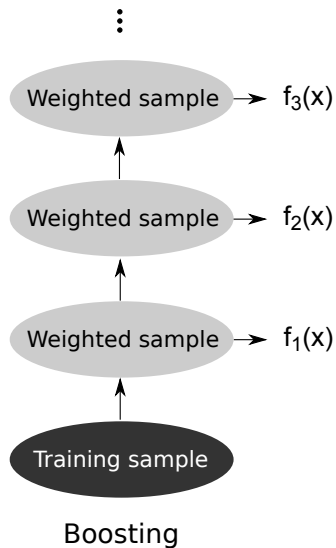
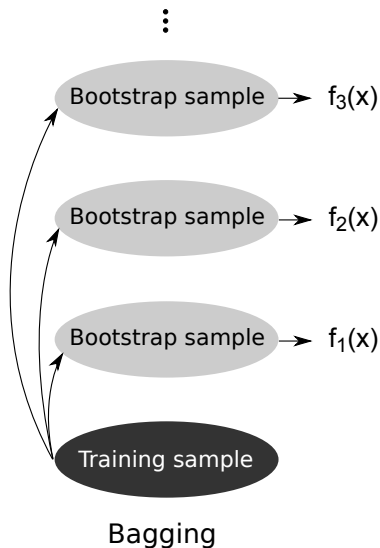
**Boosting** is another powerful method for ensemble learning. It is similar to bagging in that a set of classifiers are combined to make a better one.

It works for any classifier, but a “weak” one that is easy to learn is usually chosen. (weak = accuracy a little better than random guessing)

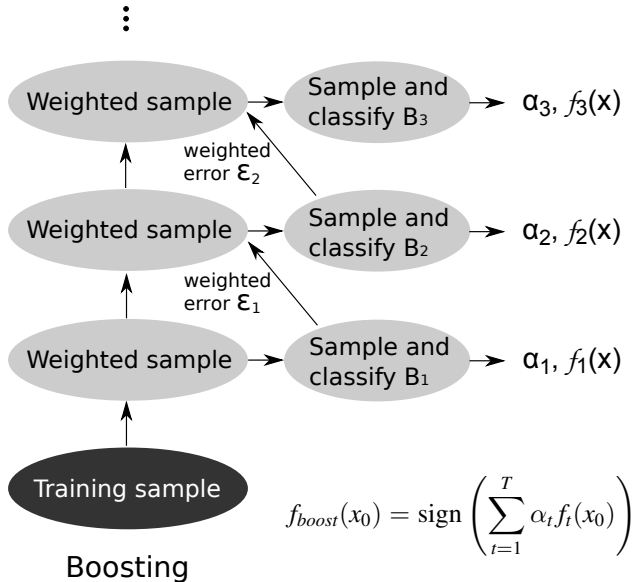
**Source:**

Robert E. Schapire and Yoav Freund, *Boosting: Foundations and Algorithms*, MIT Press, 2012. See this textbook for many more details. (I borrow figures from here.)

# BAGGING VS BOOSTING (OVERVIEW)



# THE ADABOOST ALGORITHM (SAMPLING VERSION)



# THE ADABOOST ALGORITHM (SAMPLING VERSION)

## Algorithm: Boosting a binary classifier

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Given  $(x_1, y_1), \dots, (x_n, y_n)$ ,  $x \in \mathcal{X}$ ,  $y \in \{-1, +1\}$ , set  $w_1(i) = \frac{1}{n}$  for  $i = 1 : n$

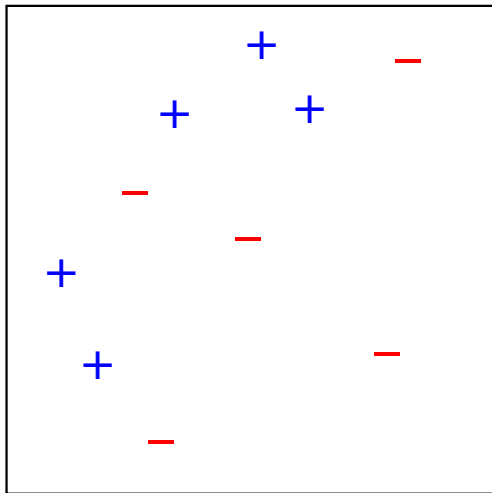
- ▶ For  $t = 1, \dots, T$ 
  1. Sample a new dataset  $\mathcal{B}_t$  of size  $n$  according to distribution  $w_t$ . Notice that we now pick  $(x_i, y_i)$  with probability  $w_t(i)$  and not  $\frac{1}{n}$ .
  2. Learn a classifier  $f_t$  using data in  $\mathcal{B}_t$ .
  3. Set  $\epsilon_t = \sum_{i=1}^n w_t(i) \mathbb{1}\{y_i \neq f_t(x_i)\}$  and  $\alpha_t = \frac{1}{2} \ln \left( \frac{1-\epsilon_t}{\epsilon_t} \right)$ .
  4. Scale  $\hat{w}_{t+1}(i) = w_t(i)e^{-\alpha_t y_i f_t(x_i)}$  and set  $w_{t+1}(i) = \frac{\hat{w}_{t+1}(i)}{\sum_j \hat{w}_{t+1}(j)}$ .
- ▶ Set the classification rule to be

$$f_{\text{boost}}(x_0) = \text{sign} \left( \sum_{t=1}^T \alpha_t f_t(x_0) \right).$$

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**Comment:** #1 & #2 usually simplified to “learn a classifier  $f_t$  using distribution  $w_t$ .”

# BOOSTING A DECISION STUMP (EXAMPLE 1)

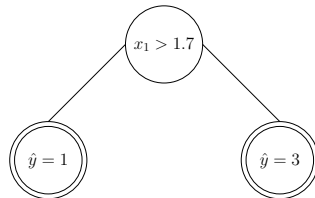


## Original data

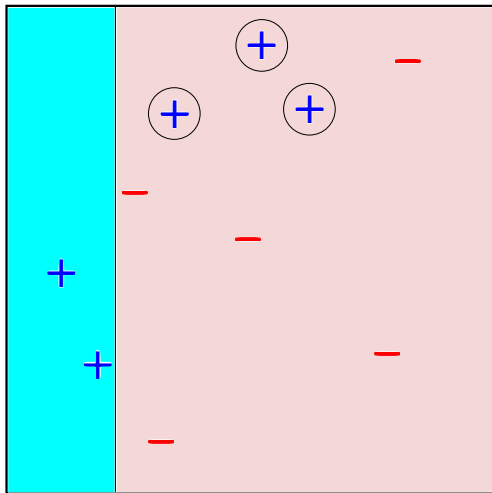
Uniform distribution,  $w_1$

Learn *weak classifier*

Here: Use a decision stump



# BOOSTING A DECISION STUMP (EXAMPLE 1)

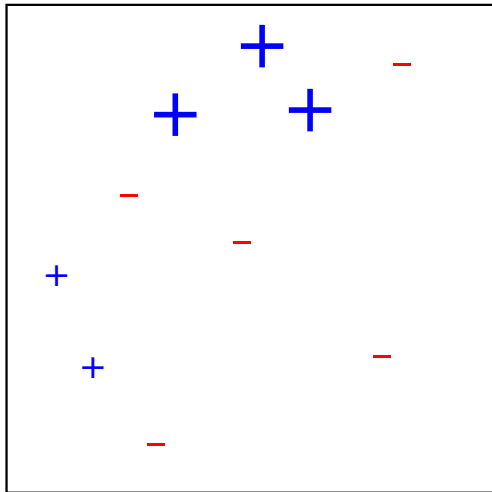


**Round 1 classifier**

Weighted error:  $\epsilon_1 = 0.3$

Weight update:  $\alpha_1 = 0.42$

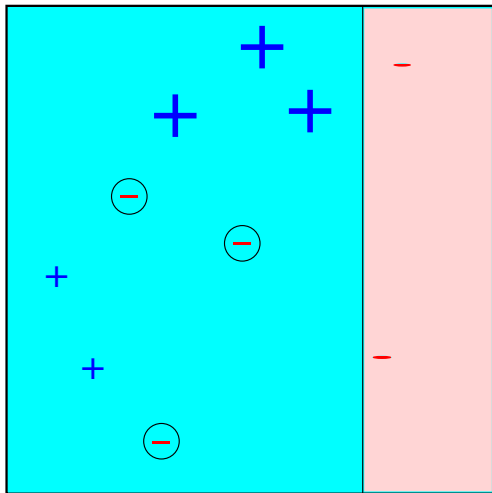
# BOOSTING A DECISION STUMP (EXAMPLE 1)



**Weighted data**

After round 1

## BOOSTING A DECISION STUMP (EXAMPLE 1)

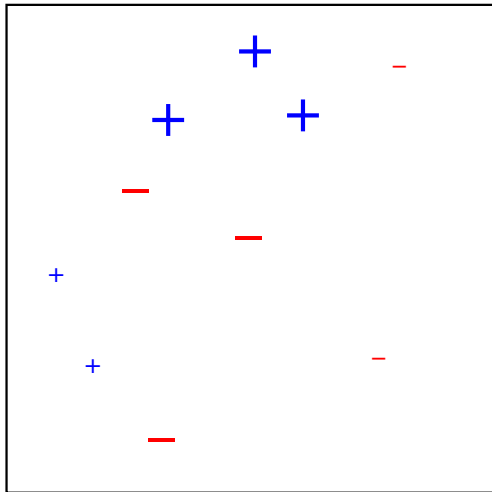


**Round 2 classifier**

Weighted error:  $\epsilon_2 = 0.21$

Weight update:  $\alpha_2 = 0.65$

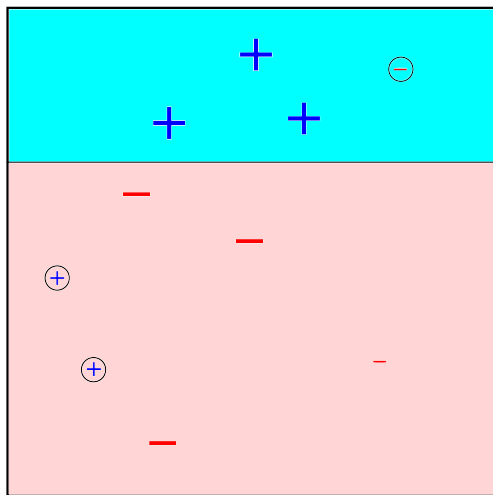
# BOOSTING A DECISION STUMP (EXAMPLE 1)



**Weighted data**

After round 2

# BOOSTING A DECISION STUMP (EXAMPLE 1)

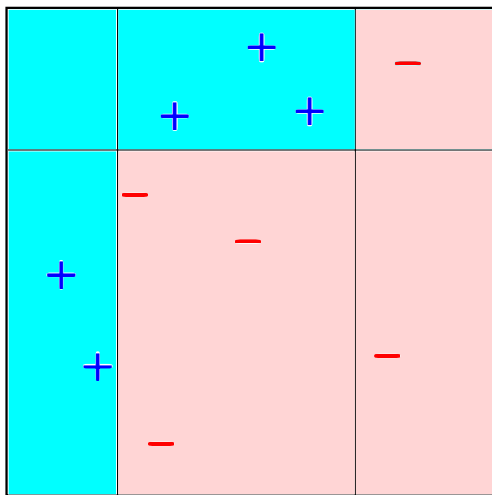


**Round 2 classifier**

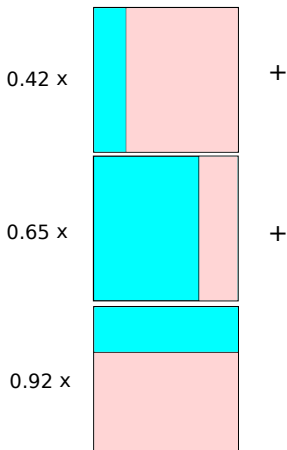
Weighted error:  $\epsilon_3 = 0.14$

Weight update:  $\alpha_3 = 0.92$

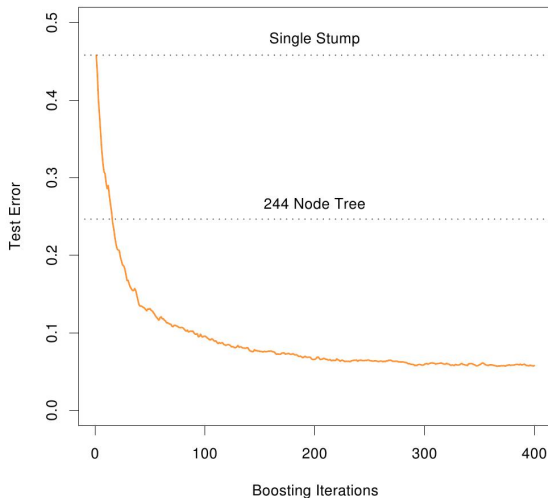
# BOOSTING A DECISION STUMP (EXAMPLE 1)



**Classifier after three rounds**



# BOOSTING A DECISION STUMP (EXAMPLE 2)



## Example problem

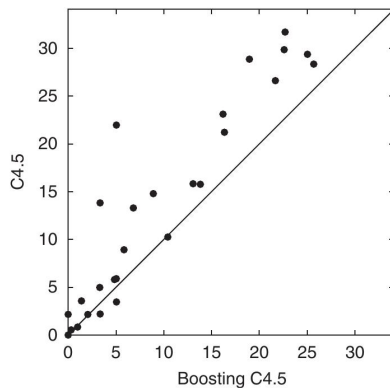
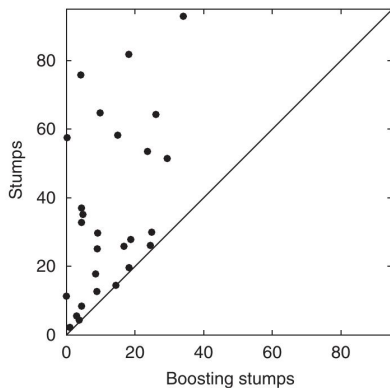
**Random guessing**  
50% error

**Decision stump**  
45.8% error

**Full decision tree**  
24.7% error

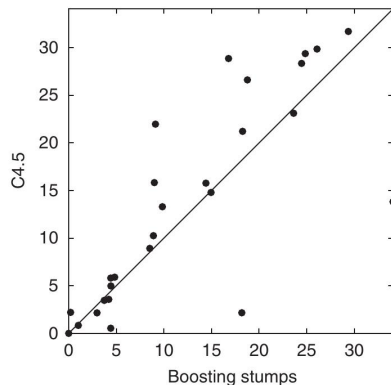
**Boosted stump**  
5.8% error

# BOOSTING

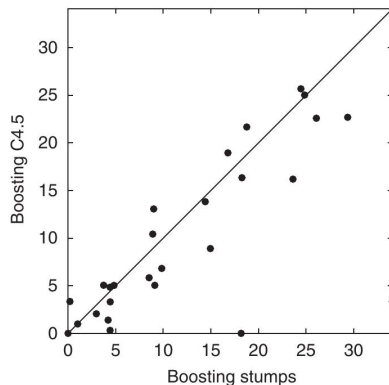


Point = one dataset. Location = error rate w/ and w/o boosting. The boosted version of the same classifier almost always produces better results.

# BOOSTING



(left) Boosting a bad classifier is often better than not boosting a good one.



(right) Boosting a good classifier is often better, but can take more time.

# BOOSTING AND FEATURE MAPS

**Q:** What makes boosting work so well?

**A:** This is a well-studied question. We will present one analysis later, but we can also give intuition by tying it in with what we've already learned.

The classification for a new  $x_0$  from boosting is

$$f_{boost}(x_0) = \text{sign} \left( \sum_{t=1}^T \alpha_t f_t(x_0) \right).$$

Define  $\phi(x) = [f_1(x), \dots, f_T(x)]^\top$ , where each  $f_t(x) \in \{-1, +1\}$ .

- ▶ We can think of  $\phi(x)$  as a high dimensional feature map of  $x$ .
- ▶ The vector  $\alpha = [\alpha_1, \dots, \alpha_T]^\top$  corresponds to a hyperplane.
- ▶ So the classifier can be written  $f_{boost}(x_0) = \text{sign}(\phi(x_0)^\top \alpha)$ .
- ▶ Boosting learns the feature mapping and hyperplane simultaneously.

# ANALYSIS OF BOOSTING

# ANALYSIS OF BOOSTING

## Training error theorem

We can use *analysis* to make a statement about the accuracy of boosting *on the training data*.

**Theorem:** Under the AdaBoost framework, if  $\epsilon_t$  is the weighted error of classifier  $f_t$ , then for the classifier  $f_{boost}(x_0) = \text{sign}(\sum_{t=1}^T \alpha_t f_t(x_0))$ ,

$$\text{training error} = \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{y_i \neq f_{boost}(x_i)\} \leq \exp\left(-2 \sum_{t=1}^T \left(\frac{1}{2} - \epsilon_t\right)^2\right).$$

Even if each  $\epsilon_t$  is only a little better than random guessing, the sum over  $T$  classifiers can lead to a large negative value in the exponent when  $T$  is large.

For example, if we set:

$$\epsilon_t = 0.45, T = 1000 \rightarrow \text{training error} \leq 0.0067.$$

# PROOF OF THEOREM

## Setup

We break the proof into three steps. It is an application of the fact that

$$\text{if } \underbrace{a < b}_{\text{Step 2}} \quad \text{and} \quad \underbrace{b < c}_{\text{Step 3}} \quad \text{then} \quad \underbrace{a < c}_{\text{conclusion}}$$

- ▶ Step 1 calculates the value of  $b$ .
- ▶ Steps 2 and 3 prove the two inequalities.

Also recall the following step from AdaBoost:

- ▶ Update  $\hat{w}_{t+1}(i) = w_t(i)e^{-\alpha_t y_i f_t(x_i)}$ .
- ▶ Normalize  $w_{t+1}(i) = \frac{\hat{w}_{t+1}(i)}{\sum_j \hat{w}_{t+1}(j)} \longrightarrow$  Define  $Z_t = \sum_j \hat{w}_{t+1}(j)$ .

# PROOF OF THEOREM ( $a \leq \mathbf{b} \leq c$ )

## Step 1

We first want to expand the equation of the weights to show that

$$w_{T+1}(i) = \frac{1}{n} \frac{e^{-y_i \sum_{t=1}^T \alpha_t f_t(x_i)}}{\prod_{t=1}^T Z_t} := \frac{1}{n} \frac{e^{-y_i h_T(x_i)}}{\prod_{t=1}^T Z_t} \rightarrow h_T(x_i) := \sum_{t=1}^T \alpha_t f_t(x_i)$$

### Derivation of Step 1:

Notice the update rule:  $w_{t+1}(i) = \frac{1}{Z_t} w_t(i) e^{-\alpha_t y_i f_t(x_i)}$

Do the same expansion for  $w_t(i)$  and continue until reaching  $w_1(i) = \frac{1}{n}$ ,

$$w_{T+1}(i) = w_1(i) \frac{e^{-\alpha_1 y_i f_1(x_i)}}{Z_1} \times \cdots \times \frac{e^{-\alpha_T y_i f_T(x_i)}}{Z_T}$$

**The product  $\prod_{t=1}^T Z_t$  is “ $\mathbf{b}$ ” above.** We use this form of  $w_{T+1}(i)$  in Step 2.

# PROOF OF THEOREM ( $\mathbf{a} \leq \mathbf{b} \leq \mathbf{c}$ )

## Step 2

Next show the training error of  $f_{boost}^{(T)}$  (boosting after  $T$  steps) is  $\leq \prod_{t=1}^T Z_t$ .  
Currently we know

$$w_{T+1}(i) = \frac{1}{n} \frac{e^{-y_i h_T(x_i)}}{\prod_{t=1}^T Z_t} \Rightarrow w_{T+1}(i) \prod_{t=1}^T Z_t = \frac{1}{n} e^{-y_i h_T(x_i)} \quad \& \quad f_{boost}^{(T)}(x) = \text{sign}(h_T(x))$$

## Derivation of Step 2:

Observe that  $0 < e^{z_1}$  and  $1 < e^{z_2}$  for any  $z_1 < 0 < z_2$ . Therefore

$$\underbrace{\frac{1}{n} \sum_{i=1}^n \mathbb{1}\{y_i \neq f_{boost}^{(T)}(x_i)\}}_a \leq \frac{1}{n} \sum_{i=1}^n e^{-y_i h_T(x_i)} \\ = \sum_{i=1}^n w_{T+1}(i) \prod_{t=1}^T Z_t = \underbrace{\prod_{t=1}^T Z_t}_b$$

*“a” is the training error – the quantity we care about.*

# PROOF OF THEOREM ( $a \leq \mathbf{b} \leq \mathbf{c}$ )

## Step 3

The final step is to calculate an upper bound on  $Z_t$ , and by extension  $\prod_{t=1}^T Z_t$ .

### Derivation of Step 3:

This step is slightly more involved. It also shows why  $\alpha_t := \frac{1}{2} \ln \left( \frac{1-\epsilon_t}{\epsilon_t} \right)$ .

$$\begin{aligned} Z_t &= \sum_{i=1}^n w_t(i) e^{-\alpha_t y_i f_t(x_i)} \\ &= \sum_{i: y_i = f_t(x_i)} e^{-\alpha_t} w_t(i) + \sum_{i: y_i \neq f_t(x_i)} e^{\alpha_t} w_t(i) \\ &= e^{-\alpha_t} (1 - \epsilon_t) + e^{\alpha_t} \epsilon_t \end{aligned}$$

Remember we defined  $\epsilon_t = \sum_{i: y_i \neq f_t(x_i)} w_t(i)$ , the probability of error for  $w_t$ .

## PROOF OF THEOREM ( $a \leq \mathbf{b} \leq \mathbf{c}$ )

### Derivation of Step 3 (continued):

Remember from Step 2 that

$$\text{training error} = \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{y_i \neq f_{\text{boost}}(x_i)\} \leq \prod_{t=1}^T Z_t.$$

and we just showed that  $Z_t = e^{-\alpha_t}(1 - \epsilon_t) + e^{\alpha_t}\epsilon_t$ .

We want the training error to be small, so we pick  $\alpha_t$  to *minimize*  $Z_t$ . Minimizing, we get the value of  $\alpha_t$  used by AdaBoost:

$$\alpha_t = \frac{1}{2} \ln \left( \frac{1 - \epsilon_t}{\epsilon_t} \right).$$

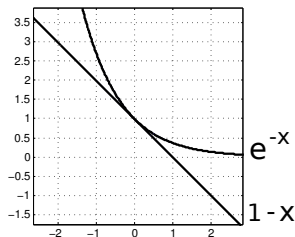
Plugging this value back in gives  $Z_t = 2\sqrt{\epsilon_t(1 - \epsilon_t)}$ .

# PROOF OF THEOREM ( $a \leq b \leq c$ )

**Derivation of Step 3** (continued):

Next, re-write  $Z_t$  as

$$\begin{aligned} Z_t &= 2\sqrt{\epsilon_t(1-\epsilon_t)} \\ &= \sqrt{1-4\left(\frac{1}{2}-\epsilon_t\right)^2} \end{aligned}$$



Then, use the inequality  $1 - x \leq e^{-x}$  to conclude that

$$Z_t = \left(1 - 4\left(\frac{1}{2} - \epsilon_t\right)^2\right)^{\frac{1}{2}} \leq \left(e^{-4\left(\frac{1}{2} - \epsilon_t\right)^2}\right)^{\frac{1}{2}} = e^{-2\left(\frac{1}{2} - \epsilon_t\right)^2}.$$

# PROOF OF THEOREM

## Concluding the right inequality ( $a \leq \mathbf{b} \leq c$ )

Because both sides of  $Z_t \leq e^{-2(\frac{1}{2}-\epsilon_t)^2}$  are positive, we can say that

$$\prod_{t=1}^T Z_t \leq \prod_{t=1}^T e^{-2(\frac{1}{2}-\epsilon_t)^2} = e^{-2 \sum_{t=1}^T (\frac{1}{2}-\epsilon_t)^2}.$$

This concludes the “ $b \leq c$ ” portion of the proof.

## Combining everything

$$\text{training error} = \overbrace{\frac{1}{n} \sum_{i=1}^n \mathbb{1}\{y_i \neq f_{\text{boost}}(x_i)\}}^a \leq \overbrace{\prod_{t=1}^T Z_t}^b \leq \overbrace{e^{-2 \sum_{t=1}^T (\frac{1}{2}-\epsilon_t)^2}}^c.$$

We set out to prove “ $a \leq c$ ” and we did so by using “ $b$ ” as a stepping-stone.

# TRAINING VS TESTING ERROR

**Q:** Driving the training error to zero leads one to ask, does boosting overfit?

**A:** Sometimes, but very often it doesn't!

