

# ELEN 4720: Machine Learning for Signals, Information and Data

## Lecture 19

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# PRINCIPAL COMPONENT ANALYSIS

# DIMENSIONALITY REDUCTION

We're given data  $x_1, \dots, x_n$ , where  $x \in \mathbb{R}^d$ . This data is often high-dimensional, but the “information” doesn't use the full  $d$  dimensions.



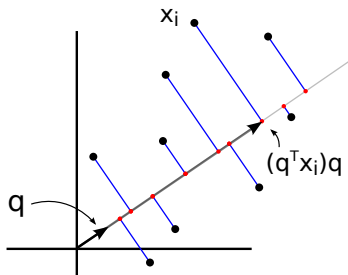
For example, we could represent the above images with three numbers since they have three degrees of freedom. Two for shifts and a third for rotation.

*Principal component analysis* can be thought of as a way of automatically mapping data  $x_i$  into some new low-dimensional coordinate system.

- ▶ It captures most of the information in the data in a few dimensions
- ▶ Extensions allow us to handle missing data, and “unwrap” the data

# PRINCIPAL COMPONENT ANALYSIS

**Example:** How can we approximate this data using a unit-length vector  $q$ ?



$q$  is a unit-length vector,  $q^T q = 1$ .

Red dot: The length,  $q^T x_i$ , to the axis after projecting  $x$  onto the line defined by  $q$ .

The vector  $(q^T x_i)q$  takes  $q$  and stretches it to the corresponding red dot.

So what's a good  $q$ ? How about minimizing the squared approximation error,

$$q = \arg \min_q \sum_{i=1}^n \|x_i - qq^T x_i\|^2 \quad \text{subject to} \quad q^T q = 1$$

$qq^T x_i = (q^T x_i)q$  : The approximation of  $x_i$  by stretching  $q$  to the “red dot.”

# PCA : THE FIRST PRINCIPAL COMPONENT

This is related to the problem of finding the largest eigenvalue,

$$\begin{aligned} q &= \arg \min_q \sum_{i=1}^n \|x_i - qq^T x_i\|^2 \quad \text{s.t.} \quad q^T q = 1 \\ &= \arg \min_q \sum_{i=1}^n x_i^T x_i - q^T \underbrace{\left( \sum_{i=1}^n x_i x_i^T \right)}_{= XX^T} q \end{aligned}$$

We've defined  $X = [x_1, \dots, x_n]$ . Since the first term doesn't depend on  $q$  and we have a negative sign in front of the second term, equivalently we solve

$$q = \arg \max_q q^T (XX^T) q \quad \text{subject to} \quad q^T q = 1$$

This is the eigendecomposition problem:

- ▶  $q$  is the first eigenvector of  $XX^T$
- ▶  $\lambda = q^T (XX^T) q$  is the first eigenvalue

# PCA: GENERAL

The general form of PCA considers  $K$  eigenvectors,

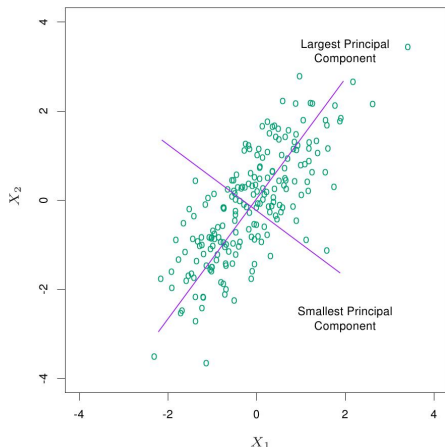
$$\begin{aligned} q &= \arg \min_q \sum_{i=1}^n \left\| x_i - \underbrace{\sum_{k=1}^K (x_i^T q_k) q_k}_{\text{approximates } x} \right\|^2 \quad \text{s.t. } q_k^T q_{k'} = \begin{cases} 1, & k = k' \\ 0, & k \neq k' \end{cases} \\ &= \arg \min_q \sum_{i=1}^n x_i^T x_i - \sum_{k=1}^K q_k^T \underbrace{\left( \sum_{i=1}^n x_i x_i^T \right)}_{= XX^T} q_k \end{aligned}$$

The vectors in  $Q = [q_1, \dots, q_K]$  give us a  $K$ -dimensional subspace with which to represent the data:

$$x_{\text{proj}} = \begin{bmatrix} q_1^T x \\ \vdots \\ q_K^T x \end{bmatrix}, \quad x \approx \sum_{k=1}^K (q_k^T x) q_k = Q x_{\text{proj}}$$

The eigenvectors of  $(XX^T)$  can be learned using built-in software.

# EIGENVALUES, EIGENVECTORS AND THE SVD



An equivalent formulation of the problem is to find  $(\lambda, q)$  such that

$$(XX^T)q = \lambda q$$

Since  $(XX^T)$  is a PSD matrix, there are  $r \leq \min\{d, n\}$  eigenpairs  $(\lambda_i, q_i)$  such that

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r \geq 0,$$

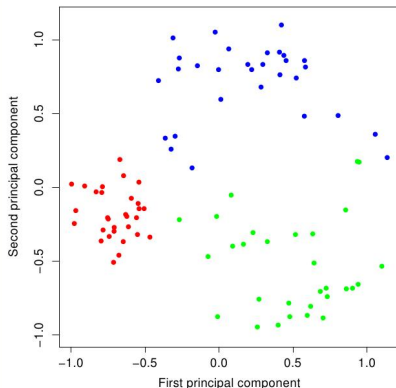
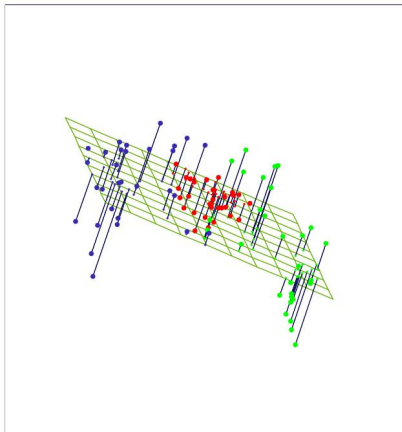
$$q_k^T q_k = 1, \quad q_k^T q_{k'} = 0$$

Why is  $(XX^T)$  PSD? Using the SVD,  $X = USV^T$ , we have that

$$(XX^T) = US^2U^T \Rightarrow Q = U, \quad \lambda_i = (S^2)_{ii} \geq 0$$

(Preprocessing: Usually first subtract off the mean of each dimension of  $x$ .)

# PCA: EXAMPLE OF PROJECTING FROM $\mathbb{R}^3$ TO $\mathbb{R}^2$



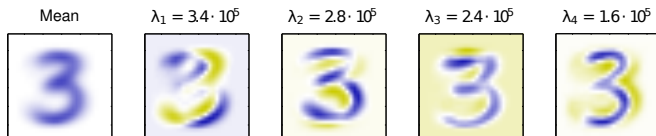
For this data, most information (structure in the data) can be captured in  $\mathbb{R}^2$ .

(left) The original data in  $\mathbb{R}^3$ . The hyperplane is defined by  $q_1$  and  $q_2$ .

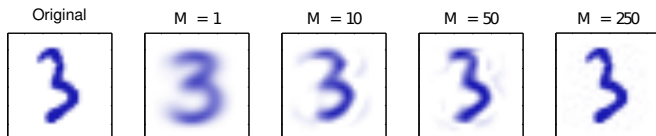
(right) The new coordinates for the data:  $x_i \rightarrow x_i^{proj} = \begin{bmatrix} x_i^T q_1 \\ x_i^T q_2 \end{bmatrix}$ .

# EXAMPLE: DIGITS

**Data:**  $16 \times 16$  images of handwritten 3's (treat as vectors in  $\mathbb{R}^{256}$ )



Above: The first four eigenvectors  $q$  and their eigenvalues  $\lambda$ .



Above: Reconstructing a 3 using the first  $M - 1$  eigenvectors plus the mean.

Let  $\bar{x} = x - \text{mean}$ , then

$$x \approx \text{mean} + \sum_{k=1}^{M-1} (\bar{x}^T q_k) q_k$$

# PROBABILISTIC PCA

# PCA AND THE SVD

We've discussed how any matrix  $X$  has a singular value decomposition,

$$X = USV^T, \quad U^T U = I, \quad V^T V = I$$

and  $S$  is a diagonal matrix with non-negative entries.

Therefore,

$$XX^T = US^2U^T \quad \Leftrightarrow \quad (XX^T)U = US^2$$

$U$  is a matrix of eigenvectors, and  $S^2$  is a diagonal matrix of eigenvalues.

# A MODELING APPROACH TO PCA

Using the SVD perspective of PCA, we can also derive a probabilistic model for the problem and use the EM algorithm to learn it.

This model will have the advantages of:

- ▶ Handling the problem of missing data
- ▶ Allowing us to learn additional parameters such as noise
- ▶ Provide a framework that could be extended to more complex models
- ▶ Gives distributions used to characterize uncertainty in predictions
- ▶ etc.

# PROBABILISTIC PCA

In effect, this is a new matrix factorization model.

- ▶ With the SVD, we had  $X = USV^T$ .
- ▶ We now approximate  $X \approx WZ$ , where
  - ▶  $W$  is a  $d \times K$  matrix. In different settings this is called a “factor loadings” matrix, or a “dictionary.” It’s like the eigenvectors, but no orthonormality.
  - ▶ The  $i$ th column of  $Z$  is called  $z_i \in \mathbb{R}^K$ . Think of it as a low-dimensional representation of  $x_i$ .

The generative process of Probabilistic PCA is

$$x_i \sim N(Wz_i, \sigma^2 I), \quad z_i \sim N(0, I).$$

In this case, we don’t know  $W$  or any of the  $z_i$ .

# THE LIKELIHOOD

## Maximum likelihood

Our goal is to find the maximum likelihood solution of the matrix  $W$  under the marginal distribution, i.e., with the  $z_i$  vectors integrated out,

$$W_{\text{ML}} = \arg \max_W \ln p(x_1, \dots, x_n | W) = \arg \max_W \sum_{i=1}^n \ln p(x_i | W).$$

This is intractable because  $p(x_i | W) = N(x_i | 0, \sigma^2 I + WW^T)$ ,

$$N(x_i | 0, \sigma^2 I + WW^T) = \frac{1}{(2\pi)^{\frac{d}{2}} |\sigma^2 I + WW^T|^{\frac{1}{2}}} e^{-\frac{1}{2} x^T (\sigma^2 I + WW^T)^{-1} x}$$

We can set up an EM algorithm that uses the vectors  $z_1, \dots, z_n$ .

# EM FOR PROBABILISTIC PCA

## Setup

The marginal log likelihood can be expressed using EM as

$$\begin{aligned} \sum_{i=1}^n \ln \int p(x_i, z_i | W) dz_i &= \sum_{i=1}^n \int q(z_i) \ln \frac{p(x_i, z_i | W)}{q(z_i)} dz_i && \leftarrow \mathcal{L}(W) \\ &+ \sum_{i=1}^n \int q(z_i) \ln \frac{q(z_i)}{p(z_i | x_i, W)} dz_i && \leftarrow \text{KL} \end{aligned}$$

**EM Algorithm:** Remember that EM has two steps. At iteration  $t$ ,

1. Set  $q(z_i) = p(z_i | x_i, W_{t-1})$  for each  $i$  and calculate  $\mathcal{L}(W)$
2. Maximize  $\mathcal{L}$  with respect to  $W$

Again, for this to work well we need that

- ▶ we can calculate the conditional posterior distribution  $p(z_i | x_i, W)$ , and
- ▶ maximizing  $\mathcal{L}$  is easy, i.e., we update  $W$  using a simple equation

# THE ALGORITHM

## EM for Probabilistic PCA

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**Given:** Data  $x_{1:n}$ ,  $x_i \in \mathbb{R}^d$  and model  $x_i \sim N(Wz_i, \sigma^2)$ ,  $z_i \sim N(0, I)$ ,  $z \in \mathbb{R}^K$

**Output:** Point estimate of  $W$  and posterior distribution on each  $z_i$

**E-Step:** Set each  $q(z_i) = p(z_i|x_i, W) = N(z_i|\mu_i, \Sigma_i)$  where

$$\Sigma_i = (I + W^T W / \sigma^2)^{-1}, \quad \mu_i = \Sigma_i W^T x_i / \sigma^2$$

**M-Step:** Update  $W$  by maximizing the objective  $\mathcal{L}$  from the E-step

$$W = \left[ \sum_{i=1}^n x_i \mu_i^T \right] \left[ \sum_{i=1}^n (\mu_i \mu_i^T + \Sigma_i) \right]^{-1}$$

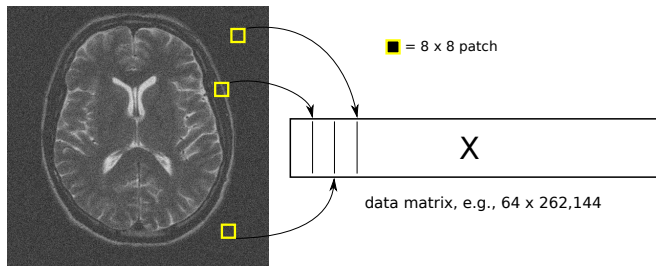
**Iterate** E and M steps until increase in  $\sum_{i=1}^n \ln p(x_i|W)$  is “small.”

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Comment:

- The probabilistic framework gives a way to learn  $K$  and  $\sigma^2$  as well.

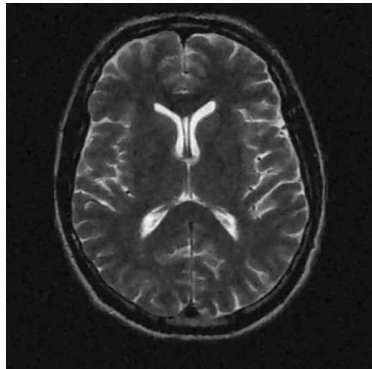
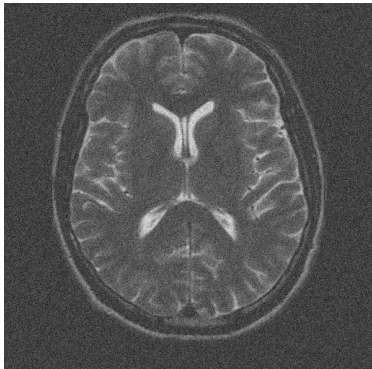
# EXAMPLE: IMAGE PROCESSING



For image problems such as denoising or inpainting (missing data)

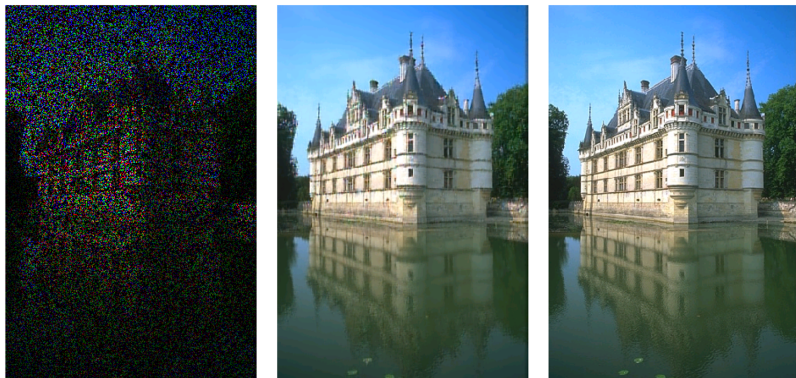
- ▶ Extract overlapping patches (e.g.,  $8 \times 8$ ) and vectorize to construct  $X$
- ▶ Model with a factor model such as Probabilistic PCA
- ▶ Approximate  $x_i \approx W\mu_i$ , where  $\mu_i$  is the posterior mean of  $z_i$
- ▶ Reconstruct the image by replacing  $x_i$  with  $W\mu_i$  (and averaging)

## EXAMPLE: DENOISING



Noisy image on left, denoised image on right. The noise variance parameter  $\sigma^2$  was learned for this example.

# EXAMPLE: MISSING DATA



Another somewhat extreme example:

- ▶ Image is  $480 \times 320 \times 3$  (RGB dimension)
- ▶ Throw away 80% at random
- ▶ (left) Missing data, (middle) reconstruction, (right) original image

# KERNEL PCA

# KERNEL PCA

We've seen how we can take an algorithm that uses dot products,  $x^T x$ , and generalize with a nonlinear kernel. This generalization can be made to PCA.

Recall: With PCA we find the eigenvectors of the matrix  $\sum_{i=1}^n x_i x_i^T = XX^T$ .

- ▶ Let  $\phi(x)$  be a feature mapping from  $\mathbb{R}^d$  to  $\mathbb{R}^D$ , where  $D \gg d$
- ▶ We want to solve the eigendecomposition

$$\left[ \sum_{i=1}^n \phi(x_i) \phi(x_i)^T \right] q_k = \lambda_k q_k$$

without having to work in the higher dimensional space.

- ▶ That is, how can we do PCA without explicitly using  $\phi(\cdot)$  and  $q$ ?

Notice that we can reorganize the operations of the eigendecomposition

$$\sum_{i=1}^n \phi(x_i) \underbrace{(\phi(x_i)^T q_k)}_{= a_{ki}} / \lambda_k = q_k$$

That is, the eigenvector  $q_k = \sum_{i=1}^n a_{ki} \phi(x_i)$  for some vector  $\mathbf{a}_k \in \mathbb{R}^n$ .

The trick is that instead of learning  $q_k$ , we'll learn  $\mathbf{a}_k$ . Plug the above equation for  $q_k$  back into the equation on the previous slide:

$$\sum_{i=1}^n \phi(x_i) \sum_{j=1}^n a_{kj} \underbrace{\phi(x_i)^T \phi(x_j)}_{= K(x_i, x_j)} = \lambda_k \sum_{i=1}^n a_{ki} \phi(x_i)$$

Let's try to write this in matrix-vector form. . .

# KERNEL PCA

Let  $\Phi = [\phi(x_1), \dots, \phi(x_n)]$ . Then the previous equation:

$$\sum_{i=1}^n \phi(x_i) \sum_{j=1}^n a_{kj} \underbrace{\phi(x_i)^T \phi(x_j)}_{= K(x_i, x_j)} = \lambda_k \sum_{i=1}^n a_{ki} \phi(x_i)$$

is equal to  $\Phi K \mathbf{a}_k = \lambda_k \Phi \mathbf{a}_k$ , where  $K$  is the  $n \times n$  kernel matrix of the data.

Assuming  $\phi$  is high-dimensional, we often can (and will) assume that

$$\Phi K \mathbf{a}_k = \lambda_k \Phi \mathbf{a}_k \quad \Longleftrightarrow \quad K \mathbf{a}_k = \lambda_k \mathbf{a}_k$$

We now perform “regular” PCA, but on the kernel matrix  $K$  instead of the data matrix  $XX^T$ . We summarize the algorithm on the following slide.

# KERNEL PCA ALGORITHM

## Kernel PCA

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**Given:** Data  $x_1, \dots, x_n, x \in \mathbb{R}^d$ , and a kernel function  $K(x_i, x_j)$ .

**Construct:** The kernel matrix on the data, e.g.,  $K_{ij} = b \exp \left\{ -\frac{\|x_i - x_j\|^2}{c} \right\}$ .

**Solve:** The eigendecomposition

$$K\mathbf{a}_k = \lambda_k \mathbf{a}_k$$

for the first  $r \ll n$  eigenvector/eigenvalue pairs  $(\lambda_1, \mathbf{a}_1), \dots, (\lambda_r, \mathbf{a}_r)$ .

**Output:** A new coordinate system for  $x_i$  by (implicitly) mapping  $\phi(x_i)$  and then projecting  $q_k^T \phi(x_i)$

$$x_i \xrightarrow{\text{projection}} \begin{bmatrix} \lambda_1 a_{1i} \\ \vdots \\ \lambda_r a_{ri} \end{bmatrix}$$

where  $a_{ki}$  is the  $i$ th dimension of the  $k$ th eigenvector  $\mathbf{a}_k$ .

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# KERNEL PCA AND NEW DATA

**Q:** How do we handle new data,  $x_0$ ? Before, we projected  $x_0^T q_k$ , but  $\mathbf{a}_k \neq q_k$ .

**A:** Recall the relationship of  $\mathbf{a}_k$  to  $q_k$  in kernel PCA is

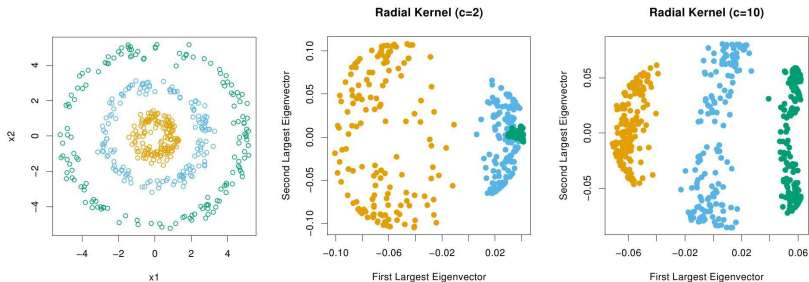
$$q_k = \sum_{i=1}^n a_{ki} \phi(x_i).$$

We used the “kernel trick” to avoid working with or even defining  $\phi(x_i)$ . As with regular PCA, after mapping  $x_0$  we want to project onto eigenvectors

$$x_0 \xrightarrow{\text{projection}} \begin{bmatrix} \phi(x_0)^T q_1 \\ \vdots \\ \phi(x_0)^T q_r \end{bmatrix}$$

Plugging in for  $q_k$  gives  $\phi(x_0)^T q_k = \sum_{i=1}^n a_{ki} \phi(x_0)^T \phi(x_i) = \sum_{i=1}^n a_{ki} K(x_0, x_i)$ .

# EXAMPLE RESULTS



An example of kernel PCA using the Gaussian kernel.

(left) Original data, colored for reference (but could be class labels)

(middle) New coordinates using kernel width  $c = 2$

(right) New coordinates using kernel width  $c = 10$

Terminology: What we are doing is closely related to “spectral clustering” and can be considered an instance of “manifold learning.”