Lecture II: $\boxed{\text{Mode of CV - Wiener process}}$

risk-neutral probability.
price of risk
Excess return (portable a)
Relationship between risk-neutral prob. & obj. prob
Interest rate
Term structure of interest-rate.
Why Q is called risk-neutral?

$$S \begin{cases} \xrightarrow{p} S(H) \\ \xrightarrow{1-p} S(L) \end{cases}$$

- p = objective prob.
- i = interest rate.

$$G \begin{cases} G(H) \\ G(L) \end{cases}$$

ex: G = option we price (call opt°)

$$46.5 \begin{cases} 47 \\ 46. \end{cases} \qquad G(K=46.5) \begin{cases} 0.5 \\ 0 \end{cases}$$

def: $$Q = \frac{S/B - S(L)}{S(H) - S(L)}.$$

$$1 - Q = \frac{S(H) - S/B}{S(H) - S(L)}.$$

ex: $$\begin{cases} \xrightarrow{\frac{1}{2}} 4 \\ \xrightarrow{\frac{1}{2}} 2 \end{cases} \quad E(X) = 3. \quad SD(X) = \sqrt{Var(X)} = \sqrt{10 - 9} = 1$$

$\begin{pmatrix} 5 \\ 1 \end{pmatrix}$ $E = 3$, $SD = 2$ $\qquad SD_1 < SD_2 < SD_3.$

$\begin{pmatrix} 6 \\ 0 \end{pmatrix}$ $E = 3$, $SD = 3$ $\qquad \rightarrow$ risk-neutral

$\Rightarrow$ <u>neutral towards risk</u>.

because $E(X) = 3 = cte$, $\forall$ risk.

- Recall under prob. measure Q (= prob.)

(*) $G = B E_Q[G(T)]$ with $B = \frac{1}{1+i}$ $\quad \left(\frac{1}{B} = 1+i\right)$

return = $\dfrac{\text{final price}}{\text{present price}}$

Expected return = $E_Q[\text{return}] = E_Q\left[\dfrac{G(T)}{G}\right]$

From(*) $\Rightarrow E_Q\left[\frac{G(T)}{G}\right] = \frac{1}{B} = 1+i$

$$\Rightarrow \boxed{E_Q\left[\frac{G(T)}{G}\right] - 1 = i}$$

It is left to show that Expected return of S is also:
Expected return of $S = E_Q\left[\frac{S(T)}{S}\right] - 1$.

$S \xrightarrow{q} S(H)$

$S \xrightarrow{1-q} S(L)$

$$E_Q\left[\frac{S(T)}{S}\right] = \frac{1}{S} E_Q[S(T)]$$

$$= \frac{1}{S}\left[\frac{S/B - S(L)}{S(H) - S(L)} S(H) + \frac{S(H) - S/B}{S(H) - S(L)} S(L)\right].$$

$$= \frac{1}{S}\left[\frac{S/B\, S(H) - S(L)S(H) + S(H)S(L) - S/B\, S(L)}{S(H) - S(L)}\right].$$

$$= \frac{1}{B} \cdot \frac{S(H) - S(L)}{S(H) - S(L)} = \frac{1}{B}.$$

$$\Rightarrow \boxed{E_Q\left[\frac{S(T)}{S}\right] - 1 = i}$$

- All the assets under prob. Q have the same return as the riskless asset (i)
$\Rightarrow$ investors are neutral towards risk.

(Martingale) • $\frac{G}{B} = E_Q\left[\frac{G(T)}{B(T)}\right]$ recall $B(T) = 1$

$$G = B\, E_Q\left[\frac{G(T)}{1}\right] \Rightarrow \frac{G}{B} = E_Q\left[\frac{G(T)}{B(T)}\right].$$

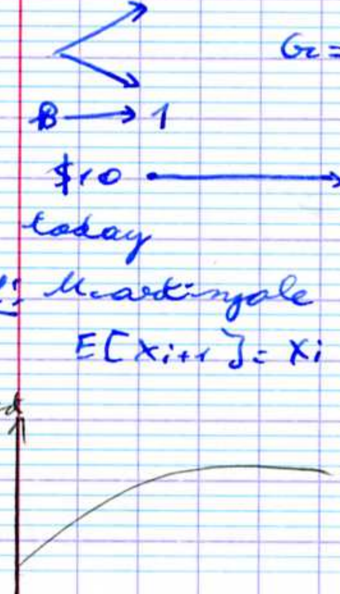
$B \longrightarrow 1$

\$10 $\longrightarrow$ expected outcome at time T
today

def: Martingale

$E[X_{i+1}] = X_i \quad \Leftrightarrow \begin{cases} E[X_{i+1}] \geq X_i \\ E[X_{i+1}] \leq X_i \end{cases}$

interest rate



term structure of I.R.

rem: the shape has a prob. structure.
Is it possible to construct a port. / ∀ shape of π, $R(\text{port.}) > 0$?
→ impossible.

ex ?: $46.5 < \begin{matrix} 47.5 \\ 45.5 \end{matrix}$

$5\$ <$

$$S \begin{cases} \xrightarrow{p} S(H) \\ \xrightarrow{1-p} S(L) \end{cases} \qquad S \begin{cases} \xrightarrow{Q} S(H) \\ \xrightarrow{1-Q} S(L) \end{cases}$$

$$G \begin{cases} \xrightarrow{p} G(H) \\ \xrightarrow{1-p} G(L) \end{cases} \qquad G \begin{cases} \xrightarrow{Q} G(H) \\ \xrightarrow{1-Q} G(L) \end{cases}$$

- Under the prob. obj.: ie (real world), investors are not risk neutral and

$E[\text{return } S] = E_Q\left[\frac{S(T)}{S}\right] - 1 = i$

$E[\text{—— } G] = E_Q\left[\frac{S(T)}{S}\right] - 1 = i$

$E[\text{—— } B] = E_Q\left[\frac{B(T)}{B}\right] - 1 = i$

they ask for extra return for the risk that they are taking.
Let i_G be the return of asset G under obj. prob.
— i_S ———— S ————.

$$\begin{matrix} & \frac{1}{2} \nearrow & 4 & 5 & 6 \\ & \frac{1}{2} \searrow & 2 & 1 & 0 \end{matrix}$$

$SD(x) = 1 \quad 2 \quad 3$

$i = 5\%$ $\quad i+\lambda_1 \mid i+\lambda_2 \mid i+\lambda_3$.

$i_G = E_p\left(\frac{G(T)}{G}\right) - 1$

real-world: $i_S = E_p\left[\frac{S(T)}{S}\right] - 1 = E[\text{return of asset S under obj. prob.}]$

real world = where markets thinks prices would be.

$i_S = \frac{1}{S}\left[\, p\,S(H) + (1-p)S(L)\right] - 1$

$$i_S \begin{cases} \xrightarrow{p} \frac{S(H)}{S} - 1 \\ \xrightarrow{1-p} \frac{S(L)}{S} - 1 \end{cases} \qquad i_G \begin{cases} \xrightarrow{p} \frac{G(H)}{G} - 1 \\ \xrightarrow{1-p} \frac{G(L)}{G} - 1 \end{cases}$$

The returns are random variables take can take values with prob. p and $(1-p)$.
Consider a new RV called "market".

$$M \begin{cases} \xrightarrow{p} \sqrt{\frac{1-p}{p}} \\ \xrightarrow{1-p} -\sqrt{\frac{p}{1-p}} \end{cases}$$

$$E[M] = p\sqrt{\frac{1-p}{p}} - (1-p)\sqrt{\frac{p}{1-p}} = \sqrt{p(1-p)} - \sqrt{p(1-p)} = \underline{\underline{0}}.$$

$$\text{var}[M] = E(M^2) - E(M)^2$$
$$= p\,\frac{1-p}{p} + (1-p)\,\frac{p}{1-p} = 1-p+p = \underline{\underline{1}}.$$

→ We want to find links between p, Q, risk.

rem! Martingale: no money on it.

- $\text{corr}(M,S) = \text{corr}(S,G) = \text{corr}(M,G) = 1$

because | when $M\nearrow$, $S\nearrow$, $G\nearrow$.
| — $M\searrow$, $S\searrow$, $G\searrow$.

The same holds for $\text{corr}(M, i_S) = \text{corr}(M, i_G) = 1$

similar to CAPM (α, β)

- $\boxed{\text{if } \text{corr}(X,Y) = 1 \Rightarrow y = a + bx}$, for (a,b)
- $\Rightarrow \begin{cases} i_G = a_G + b_G M \\ i_S = a_S + b_S M \end{cases}$, with (a_G, a_S, b_G, b_S)

ex: M = S&P 500

$\boxed{\text{odds} = \frac{1}{\text{odds ratio}}}$

<u>odds</u>	<u>odds Ratio</u>:
Betting concept	Statistical concept.
$\frac{\text{prob. of unfavorable event}}{\text{prob. of fav. event.}}$	$\frac{\text{prob. of fav. event.}}{\text{prob. of unfav. event.}}$

ex: $p = 25\% = p(H)$ flip a coin.
$1-p = 75\% = p(T)$
bet 1 \$.
if $T \to$ lose 1 \$
$H \to$ win x \$
what shall be x? $x = \text{odds} = \frac{.75}{.25} = 3$.

$(.75)(-1) + (.25)3 = 0$.

therefore,

$$\boxed{\begin{cases} i_G = a_G + b_G M \\ i_S = a_S + b_S M \end{cases}}$$

$\Rightarrow E[i_G] = E[a_G + b_G M] = a_G + b_G E[M] = a_G$.

$E[i_S] = E[a_S + b_S M] = a_S + b_S E[M] = a_S$.

* consider the portfolio $\boxed{b_S G - b_G S}$

the return is:

$b_S(a_G + b_G M) - b_G(a_S + b_S M) = b_S a_G - b_G a_S$.

ie: does not depend on $M \Rightarrow$ riskless.

therefore, the return should be i:

$b_S a_G - b_G a_S = i(b_S - b_G)$

$$\Rightarrow \boxed{\frac{a_G - i}{b_G} = \frac{a_S - i}{b_S} = \lambda} = \text{market price of risk.}$$

risk of asset G ↳ risk of asset S.

$i_G = a_G + b_G M$

risk ↗.

uncertainty of return due to $b_S M$.

ie: $Var(b_S M) = b_S^2 Var(M) = b_S^2 \Rightarrow \boxed{SD(i_S) = b_S}$

This guarantees lack of arbitrage under obj. prob.

• λ is market price of risk that can vary in time.

som: \$ weak, rates ↑, commodities ↑, inflation ↑.

$$S \begin{cases} \xrightarrow{Q} S(H) \\ \xrightarrow{1-Q} S(L) \end{cases} \quad S = \frac{S(H)Q + S(L)(1-Q)}{1+i} \qquad S \begin{cases} \xrightarrow{p} S(H) \\ \xrightarrow{1-p} S(L) \end{cases} \quad \frac{pS(H) + (1-p)S(L)}{1+i+\lambda b_S}.$$

$$G \begin{cases} \xrightarrow{Q} G(H) \\ \xrightarrow{1-Q} G(L) \end{cases} \quad G_1 = \frac{G(H)Q + (1-Q)G(L)}{1+i} \qquad G \begin{cases} \xrightarrow{p} G(H) \\ \xrightarrow{1-p} G(L) \end{cases} \quad \frac{pG(H) + (1-p)G(L)}{1+i+\lambda b_G}.$$

proof: $\dfrac{pS(H) + (1-p)S(L)}{1+i+\lambda b_S} = \dfrac{E_p(S) = (1+i_S)S}{1+i+\lambda b_S} \Rightarrow \dfrac{E_p[a_S + b_S M + 1]S}{1+i+\lambda b_S} = \dfrac{(a_S+1)S}{1+i+\lambda b_S}$

$E_p[G(T)] = E[a_G + b_G M + 1]G$

$G_1 = \dfrac{(a_G+1)G}{1+i+\lambda b_G}$

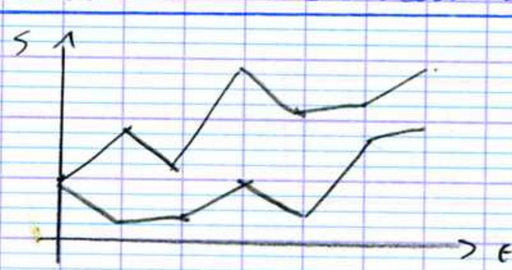
$$\begin{cases} a_G = i + \lambda b_G \\ a_S = i + \lambda b_S \end{cases}$$

$\frac{a_G - i}{\dots} = \lambda$ - substituted

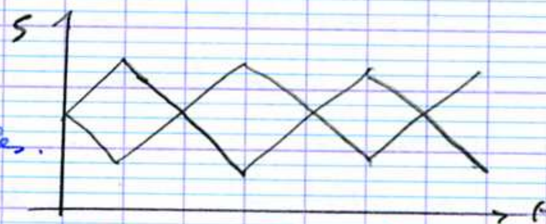
- Relationship between risk-neutral prob. & obj. prob.

$$S \xrightarrow{p} S(H), \quad S \xrightarrow{1-p} S(L)$$

$$G \xrightarrow{p} G(H), \quad G \xrightarrow{1-p} G(L)$$



here $\frac{S}{B}$, $\frac{G}{B}$ are not Martingales.



cf! only with Q, $\frac{S}{B}$, $\frac{G}{B}$ are MG.

$$\frac{G}{B} = E_Q\left[\frac{G(T)}{B(T)}\right] \qquad \frac{G}{B} \neq E_P\left[\frac{G(T)}{B(T)}\right]$$

$$\frac{S}{B} = E_Q\left[\frac{S(T)}{B(T)}\right] \qquad \frac{S}{B} \neq E_P\left[\frac{G(T)}{B(T)}\right]$$