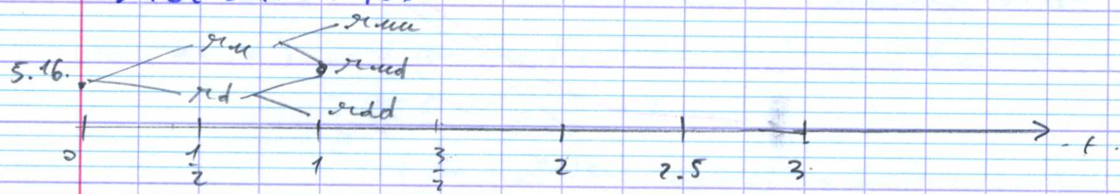
Lecture V:

6Mo: 5.16 %   } deposit rates = LIBOR.
1Yr: 4.96 %. }

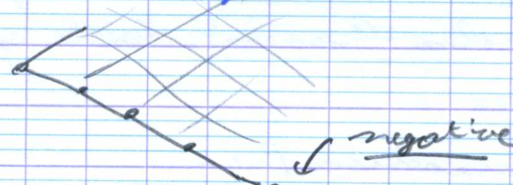
2Yr: 4.70 %.

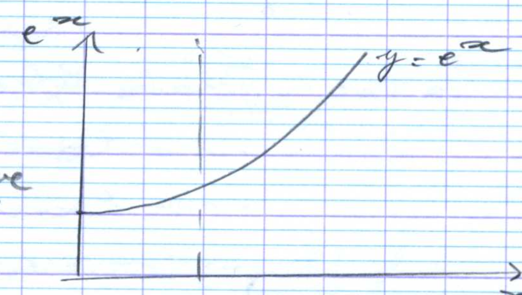
3Yr: 4.74 %.



{ non recombinant = 2^n possibilities after n times.
{ recombinant = αn ——— after n —.

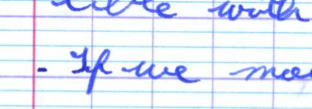
① What should the elements of the tree be ...?
How should the tree be constructed so it is compatible with market observed rates & prices?
- If we model rates after sufficiently large numbers of steps, some values of the tree become negative.
Something not acceptable!
indeed:





. If $\underbrace{R = e^x}_{>0} \Leftrightarrow \ln R = x$

$\Rightarrow$ Consequently, we model $\ln R$.



In this case $\Delta t = 6\,Mo = \frac{1}{2}\,Yr.$

$\Delta \ln r_t = \ln r_{t+\Delta t} - \ln r_t = \ln\left(\frac{r_{t+\Delta t}}{r_t}\right)$

$= \ln\left(\frac{r_{t+\Delta t}}{r_t} - 1 + 1\right)$

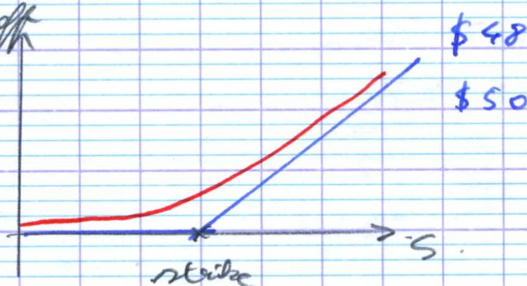
$= \ln\left(\frac{r_{t+\Delta t} - r_t}{r_t} + 1\right) \simeq \boxed{\frac{r_{t+\Delta t} - r_t}{r_t} = \Delta \ln r_t}$

= rate of return that can be negative.

ex: 100 ↘ 80 $\frac{80 - 100}{100}$ — 2

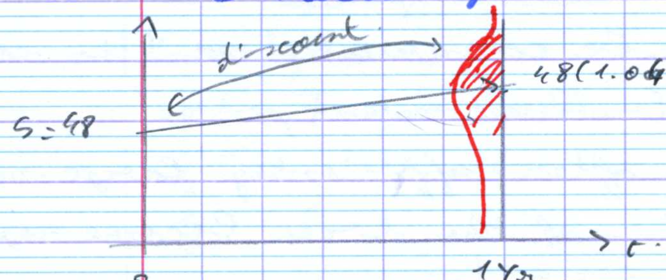
. σ in table 2 is annualized ⇒ that's why in Black-Scholes model ⇒ $\sigma\sqrt{T}$.

rem: Call option.



S = stock price
K = strike
T = maturity.

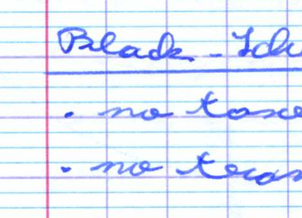
R = interest rate
σ_T = volatility.



Black-Scholes Models: lack of reality
- no taxes
- no transaction fees ...

⇒ σ = compensator for imperfection.

× ln r_t has a normal distribution (μ, σ^2) with volatility = σ (annual basis).



what volatility?
↳ vol = $\sigma\sqrt{\Delta t}$ with | σ: annualized
| Δt: in years.

⇒ Let us determine the relationship between r_u & r_d such that it is "compatible" with observed volatility in the market.

$\frac{1}{2}$ $\ln r_u$

$\frac{1}{2}$ $\ln r_d$

this random variable should have a $SD = \sigma\sqrt{t}$.

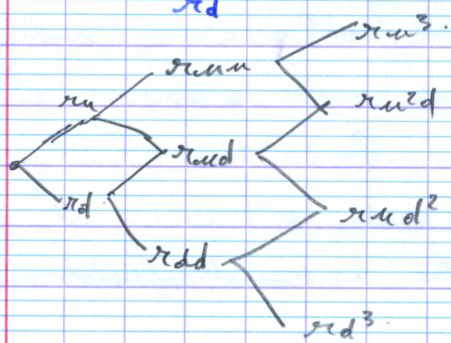
RV	prob
$\ln r_u$	$1/2$
$\ln r_d$	$1/2$

$$E[\ln r] = \frac{1}{2}(\ln r_u + \ln r_d)$$

$$\begin{aligned}
Var[\ln r] &= \frac{1}{2}(\ln r_u)^2 + \frac{1}{2}(\ln r_d)^2 - \left(\frac{1}{2}(\ln r_u + \ln r_d)\right)^2 \\
&= (E[\ln^2 r]) - (E[\ln r])^2. \\
&= \frac{1}{2}(\ln r_u)^2 + \frac{1}{2}(\ln r_d)^2 - \frac{1}{4}(\ln r_u)^2 - \frac{1}{4}(\ln r_d)^2 - \frac{1}{2}\ln r_u \times \ln r_d \\
&= \frac{1}{4}(\ln r_u)^2 + \frac{1}{4}(\ln r_d)^2 - \frac{1}{2}\ln u \ln r_d. \\
&= \frac{1}{4}(\ln r_u - \ln r_d)^2. \\
&= \frac{1}{4}\left(\ln \frac{r_u}{r_d}\right)^2 = \sigma^2 \Delta t.
\end{aligned}$$

$$\Rightarrow \ln \frac{r_u}{r_d} = 2\sigma\sqrt{\Delta t} \Rightarrow \frac{r_u}{r_d} = e^{2\sigma\sqrt{\Delta t}}$$

$$\Rightarrow \boxed{r_u = r_d\, e^{2\sigma\sqrt{\Delta t}}}$$



$$\Rightarrow r_{u^3} = r_{u^2d}\, e^{2\sigma\sqrt{\Delta t}}$$
$$= r_{ud^2}\, e^{2\times 2\sigma\sqrt{\Delta t}} = r_{ud^2}\, e^{4\sigma\sqrt{\Delta t}}$$
$$= r_{ddd}\, e^{8\sigma\sqrt{\Delta t}}.$$

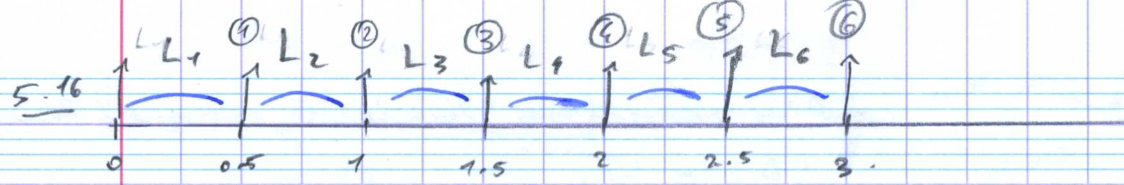
We should determine the price, hence the yield of a zero-coupon bond maturing at the nodes of interest to us (0.5 Yr, 1, 1.5, 2, 2.5, 3) Yr.

<u>1st step:</u>

0.5	$\rightarrow$	5.16%
1	$\rightarrow$	4.95%
1.5	$\rightarrow$	4.825%
2	$\rightarrow$	4.7%
2.5	$\rightarrow$	4.72%
3	$\rightarrow$	4.74%.

interpolat°

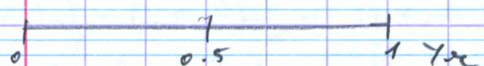
$$\boxed{yield(1.5) = \frac{yield(1) + yield(2)}{2}}$$



$P_1 = \dfrac{100}{\left(1+\frac{L_1}{2}\right)}$: for 1 period. $\boxed{L_1 = 5.16\%}$

$P_2 = \dfrac{100}{\left(1+\frac{L_1}{2}\right)\left(1+\frac{L_2}{2}\right)}$

$P_3 = \dfrac{100}{\left(1+\frac{L_1}{2}\right)\left(1+\frac{L_2}{2}\right)\left(1+\frac{L_3}{2}\right)}$



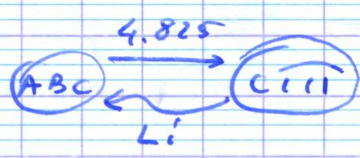
$(1+0.0496) = \left(1+\frac{0.051663}{2}\right)\left(1+\frac{L_2}{2}\right)$

$\Rightarrow \boxed{L_2 = 4.74\%}$

⇒ After we have to use swaps rates:

<u>Calculating L_3:</u>

- the 1.5 yr swap rate = 4.825 %



⇒ we are indifferent between the following cash flows.

<u>i.e.</u> the present value of the 2 cash-flows should be the same.

	L_i	PV
L_1	5.16	.9748
L_2	4.74	$.9523 = \frac{1}{1+0.0474/2} \times 0.9748$
$L_3 =$	5 %	$\left[\frac{1}{\left(1+\frac{0.05}{2}\right)}\right](.9523)$ ⇐ if

$(E_1) \Rightarrow \left(\frac{5.16}{2}\right)(.9748) + \left(\frac{4.74}{2}\right)(.9523) + \left(\frac{5}{2}\right)\left[\frac{1}{1+\frac{0.05}{2}}\right](.9523)$

$(E_2) = \left(\frac{4.825}{2}\right)(.9748) + \left(\frac{4.825}{2}\right)(0.9523) + \left(\frac{4.825}{2}\right)\left(\frac{1}{1+\frac{0.05}{2}}\right)(.9523)$

E_1
E_2
E_3

$\Rightarrow \boxed{L_3 = 4.57\%}$

→ After: bootstrapping

zero-coupon price:

0.5	97.4849	5.16%
1	95.2303	4.95%
1.5	93.1046	4.82%
2	91.1115	4.69%
2.5	89.0032	4.71%
3	86.8972.	4.74%.

* 2 riskless bonds (no risk of default, US gov, highly rated) with same maturity have same yield.

yield:

$$93.1046 = \sum_{i=1}^{n} \frac{c_i}{(1+\frac{y}{2})^i} + \frac{100}{(1+\frac{y}{2})^n} = \frac{100}{(1+\frac{y}{2})^3} \Rightarrow y \text{ yield}$$

⇒ A coupon bond that pays semi-annual with coupon of 4.69 will be par (i.e 100) with mat. 2 yr.

* Consider a par bond with coupon of 4.95% that matures in 1 yr.

today 5.16%

$$r_u = r_d e^{2\times(0.1537)} \rightarrow \frac{\left(\frac{4.95}{2}+100\right)}{(1+r_u/2)}\cdot\left(\frac{1}{2}\right)$$

$$= r_d e^{2\sigma\sqrt{\Delta t}}$$

r_d

$$\rightarrow \frac{\left(\frac{4.95}{2}+100\right)}{\left(1+\frac{r_d}{2}\right)}\left(\frac{1}{2}\right)$$

price(0)

$$\text{price}(0.5) = \frac{1}{2}\left(\frac{4.95}{2}+100\right)\times\left[\frac{1}{1+\frac{r_d e^{2\times0.1537}}{2}}+\frac{1}{1+\frac{r_d}{2}}\right].$$

$\frac{4.95}{2}$ $\frac{4.95}{2}+100$

0   $\frac{1}{2}$ 1

price of this bond $(t=0) = 100$.

$$\Rightarrow \text{price}(0) = \left[\frac{1}{2}\left(\frac{4.95}{2}+100\right)\left(\frac{1}{1+\frac{r_d e^{2\times0.1537}}{2}}+\frac{1}{1+\frac{r_d}{2}}\right)+\frac{4.95}{2}\right]\times\underbrace{\frac{1}{\left(1+\frac{0.0516}{2}\right)}}_{=0.974849}$$

$$= \underline{\underline{100}} \Rightarrow r_d$$

price:

t (years)	0	$\frac{1}{2}$	1	$\frac{3}{2}$
price	100	$\frac{4.82}{2}$	$\frac{4.82}{2}$	$100+\frac{4.82}{2}$

2nd step of the binomial tree:

- $r_{uu} = r_{dd}\, e^{4\sigma\sqrt{\Delta t}} = r_{dd}\, e^{4\times(0.1537)}$
- $r_{ud} = r_{dd}\, e^{2\sigma\sqrt{\Delta t}} = r_{dd}\, e^{2\times 0.1537}$
- r_{dd}

5.45 → $\dfrac{100+\frac{4.82}{2}}{\left(1+\frac{r_{dd}}{2}e^{4\times 0.1537}\right)}$, $\dfrac{100+4.82/2}{\left(1+\frac{r_{dd}}{2}e^{2\times 0.1537}\right)}$

4.01 → $\dfrac{100+4.82/2}{\left(1+\frac{r_{dd}}{2}e^{2\times 0.1537}\right)}$, $\dfrac{100+4.82/2}{(1+r_{dd}/2)}$

5.45 $\xrightarrow{P_u}$ P_{uu} $\frac{1}{2}$

P_{ud} $\frac{1}{2}+\frac{1}{2}$

4.01 $\xrightarrow{P_d}$ P_{dd} $\frac{1}{2}$

$\frac{4.82}{2}$

$$P_u = \frac{1}{2}\,\frac{100+\frac{4.82}{2}}{1+\frac{r_{dd}\,e^{4\times(0.1537)}}{2}} + \frac{1}{2}\,\frac{100+\frac{4.82}{2}}{1+\frac{r_{dd}\,e^{2\times(0.1537)}}{2}}$$

$P_u + \frac{4.82}{2}$.

$$P_d = \frac{1}{2}\,\frac{100+\frac{4.82}{2}}{1+\frac{r_{dd}}{2}e^{2\times(0.1537)}} + \frac{1}{2}\,\frac{100+\frac{4.82}{2}}{1+\frac{r_{dd}}{2}}$$

$\frac{1}{2}\left(P_u+\frac{4.82}{2}\right)\times\left(\frac{1}{1+\frac{0.0545}{2}}\right)$

$\frac{1}{2}\left(P_d+\frac{4.82}{2}\right)\left(\frac{1}{1+\frac{0.0401}{2}}\right)$

→ discount by $\dfrac{1}{\left(1+\frac{0.0516}{2}\right)}$ $= 10$.

$\frac{4.82}{2}$

- 5.16 → 5.45, 4.01
- 5.45 → 6.06, 4.96
- 4.01 → 4.96, 3.28
- 6.06 → $r_{uuu} = r_d^3\, e^{8\times(.1537)}$, $r_{u^2d} = r_d^3 e^{4\times(.1537)}$
- 4.96 → r_{u^2d}, $r_{ud^2} = r_d^3\, e^{2(.1537)}$
- 3.28 → r_{ud^2}, r_d^3.