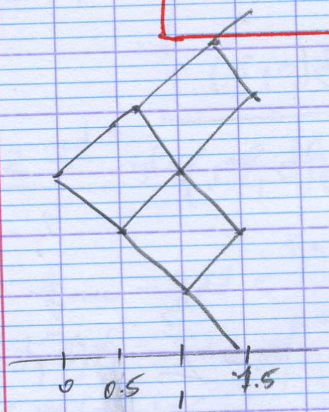
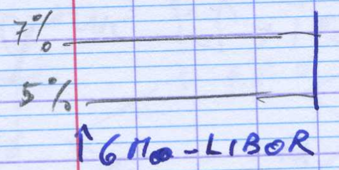
Lecture VI: Introduction to stochastic calculus



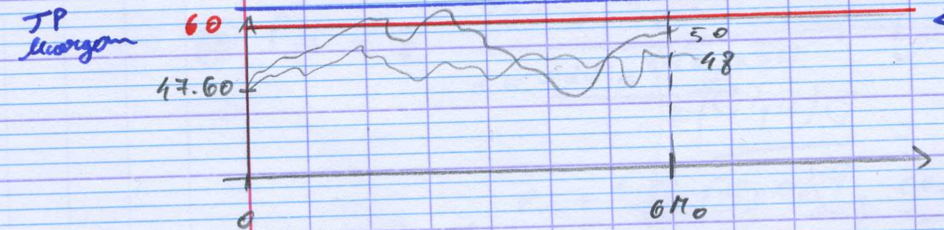
Discrete time : $\Delta t = 6$ Mo.
6 months interest rate.

. **Daily Accrual note:**



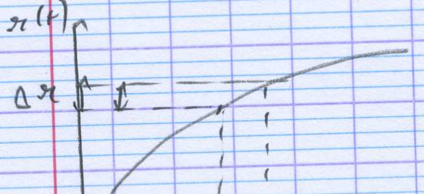
$$\text{Libor} \times \frac{\#\text{ of days LIBOR was at btw the bond}}{\#\text{ of days in the period}} \; (1+\alpha)$$

. **Knock in and knock out options:**



← barrier that should not be crossed.

. When $\Delta t \longrightarrow 0$ (Δt very small), we denote it by dt.
The rates that we will model are all future spot rates "short rates"
$r(t)$ rate is a random variable (up or down)
We model change in rate for a very small increment in time (dt)
$r(t+\Delta t) - r(t) = \Delta r$: change in rate.
$\rightarrow$ dr = evolution of interest rates.
or $dS(t)$ = evolution of an asset.



$f(s), f(s)$: asset or stock.

We want $df(S)$

<u>ex1</u>: $f(s) = s^2 + s$

$$df(s) = \frac{df}{ds} \cdot ds = (2s+1)\,ds.$$

<u>ex2</u>: if $S = 10 \Rightarrow f(10) = 110$.

$dS = 0.01$

$\Rightarrow df(s) = (2 \times 10 + 1) \times 0.01 = 0.21$

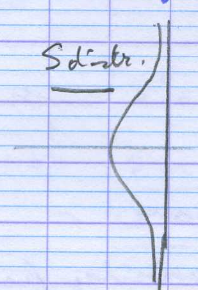
$$\begin{cases} f(10.01) = (10.01)^2 + 10.01 = 100.20 + 10.01 = 110.21 \\ f(10) = 110 \end{cases}$$

$\Delta f \simeq df = 0.21$.

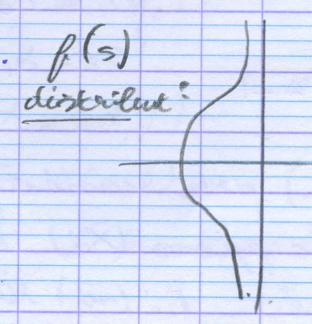
- However, S is a random variable,.
  $f(S)$ = option = is a random ''.

→ how do we determine $df(S)$? ⇒ ITO's LEMMA.

we look for the link:



between



<u>rem</u>: if S is deterministic: $df(S) = f'(S)\,dS$. (easy)

- to be able to describe the phenomenon, we have to extend the notion of limit and derivative. (integration).

⇝ STOCHASTIC CALCULUS.

<u>Extension of concept of convergence and limit</u>.

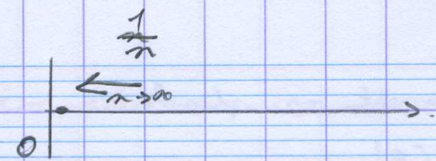
$$\lim_{n \to \infty} \frac{1}{n} = 0$$

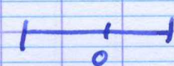
- <u>historical remark</u>: Achille paradox.

$\longrightarrow x$ — A. will never pass before the tortoise

Achille → tortoise

· Cauchy :

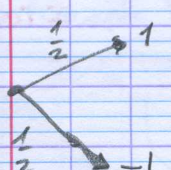


• $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ means :
$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall n \geq N,\ \left|\frac{1}{n}\right| \leq \varepsilon$.
$\Leftrightarrow$ given any interval 
we have, after a while, there is $N_0(\varepsilon)$ such that, for all $n > N_0(\varepsilon)$, $\left|\frac{1}{n} - 0\right| \leq \varepsilon$.

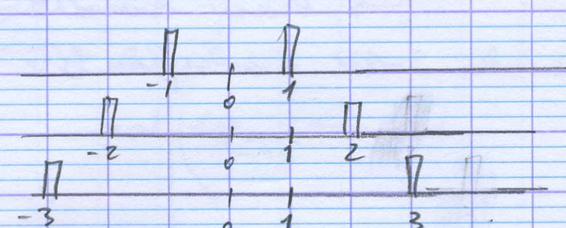
• Generally : $\lim\limits_{n\to\infty} f(n) = L$
$\Leftrightarrow \forall \varepsilon > 0,\ \exists N_0 \in \mathbb{N},\ \forall n \geq N_0,\ |f(n) - L| \leq \varepsilon$

• $f_n(x) = x^{\frac{1}{n}}$, $x \geq 0$.
$\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty} x^{\frac{1}{n}} = f(x)$
if given $\forall x$, $\lim\limits_{n\to\infty} f_n(x) = f(x)$.

ex ① :



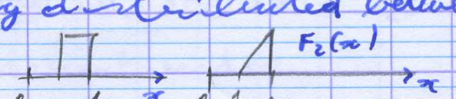
$$\begin{cases} S_1 = X_1 \\ S_2 = X_1 + X_2 \\ S_3 = X_1 + X_2 + X_3 \end{cases}$$

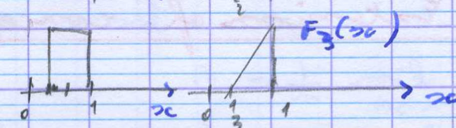


(S_n) are random variable.
. Suppose (X_i) are random var.
Let $F_i(x)$ the cumulative density funct° of X_i
$F_i(a) = P(X_i \leq a)$

déf : $X_n \xrightarrow[n\to\infty]{} X$ in distribution if $\lim\limits_{n\to\infty} F_n(x) = F(x)$
We say : convergence in distribution.

ex② : $(X_n)_{n \geq 2}$ are uniformly distributed between $[\frac{1}{n}, 1]$
$X_2 \sim \mathcal{U}([\frac{1}{2}, 1])$
$X_3 \sim \mathcal{U}([\frac{1}{3}, 1])$
$X_{100} \sim \mathcal{U}([\frac{1}{100}, 1])$





for X_n : $f_n(x) = \dfrac{1}{1 - \frac{1}{n}}$
$F_n(x) = \dfrac{x - \frac{1}{n}}{1 - \frac{1}{n}} \xrightarrow[n\to\infty]{} \dfrac{x}{1} = x$ for $x \in [0, 1]$.

Central limit theorem:

If (X_i) are iid (independent and identically distributed) random variables (RV) with

$$\begin{cases} E[X_i] = \mu \\ var[X_i] = \sigma^2 \end{cases}$$

$$S_n = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma \sqrt{n}}$$

then: $\boxed{S_n \underset{n\to\infty}{\Longrightarrow} \mathcal{N}(0, 1)}$: normal $(0, 1)$
mean variance.

ex① (following) $\left| \begin{array}{l} S_1 = X_1 \\ S_2 = X_1 + X_2 \\ S_n = \sum_{i=1}^{n} X_i \end{array} \right.$

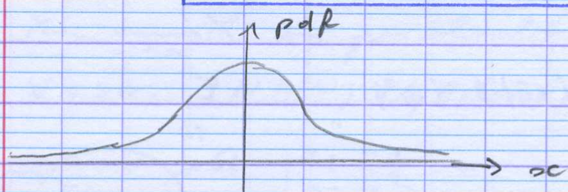
we suppose $\left| \begin{array}{l} E[X_i] = 0 \\ var[X_i] = 1 \end{array} \right.$ and (X_i) iid.

then: $S'_n = \frac{\sum_{i=1}^{n} X_i - 0}{\sqrt{n} \times 1} \Longrightarrow \mathcal{N}(0,1)$

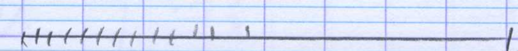
$\Rightarrow S'_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \sim \mathcal{N}(0,1)$

$\Rightarrow S_n = \sum X_i \sim \sqrt{n}\, \mathcal{N}(0,1)$

$\Rightarrow \boxed{S_n \sim \mathcal{N}(0, n)}$ when n large.



ex③:



Let $n = \left[\frac{T}{\Delta x}\right]$ where $\left[\frac{T}{x_n}\right]$ is the largest integer $\leq \frac{T}{x^n}$

$[x] = \max\{n \in \mathbb{N} \,/\, n \leq x\}$.

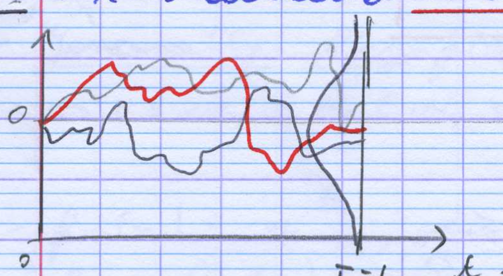
Note $W_n = \Delta x (X_1 + X_2 + \dots + X_n) = \Delta x \times \left(\sum_{i=1}^{n} X_i\right)$

. (X_i) are iid, $\left| \begin{array}{l} E[X_i] = 0. \\ var[X_i] = 1 \end{array} \right.$

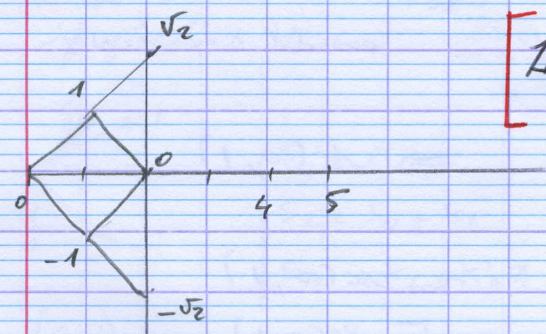
- $E[W_n] = E(\Delta x \times \sum X_i) = \Delta x \times (\sum_{i=1}^{n} E(X_i)) = 0$.
- $Var[W_n] = Var(\Delta x \times \sum X_i) = \Delta x^2 \times (\sum Var(X_i))$ by independence
$= \Delta x^2 \times n$
- If $\boxed{\Delta x = \frac{1}{\sqrt{n}}} \Rightarrow \begin{cases} E[W_n] = 0 \\ Var[W_n] = 1 \end{cases}$

$$W_n = \frac{\sum X_i - 0}{1 \times \sqrt{n}} \underset{n \to \infty}{\Rightarrow} N(0,1)$$

__def:__ W_n is called a __Wiener process__ or a __Brownian Motion__

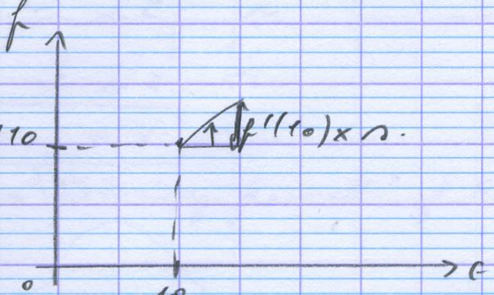


mean = 0
Var = $\sqrt{t}$.



__pty 1:__ all paths are continuous.
(no jumps)

__ex:__ $f(s) = s^2 + s$
$f(10) = 110$.
$f'(s) = 2s + 1 \Rightarrow f'(10) = 30$.



__pty 2:__ The path are almost nowhere differentiable.

__Convergence in probability!__

Almost equivalent to convergence in norm.

__Kolmogorov:__ $(\Omega, \mathcal{F}, \mu)$

Ω = sample space.

$\Omega = \{1, 2, 3, \dots\} = \{0, \to \infty\}$.

$\mathcal{F}$: is a collection of subsets of Ω.

def: · $\emptyset \in \mathcal{F}$

· $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$.

· $F_i \in \mathcal{F}$ and $F_j \in \mathcal{F} \Rightarrow F_i \cup F_j \in \mathcal{F}$.

· If $\forall i \in [\![1, \infty]\!]$, $F_i \in \mathcal{F} \Rightarrow \cup F_i \in \mathcal{F}$ $\Rightarrow$ σ-additivity.

$\mathcal{F}$ is called a σ-field.

ex: $\mathcal{F} = \{\emptyset, \Omega, \{2,4,6\}, \{1,3,5\}\}$.

$\mathcal{F} = \{\emptyset, \Omega\}$.

$\mathcal{F} = \{\emptyset\}$.

def: μ called probability measure is defined by:

$\mu : \mathcal{F} \longrightarrow [0,1]$

$\forall A \neq \emptyset, \mu(A) \geq 0$.

$\mu(\emptyset) = 0$

$\mu(\Omega) = 1$

$A \cap B = \emptyset \Rightarrow \mu(A \cup B) = \mu(A) + \mu(B)$

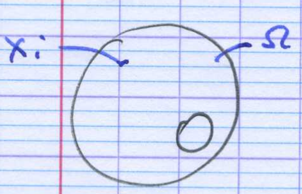
σ-additivity; if $\forall (i,j)$ $F_i \in \mathcal{F}$, $F_i \cap F_j = \emptyset$

then: $\mu\left(\bigcup_{i=1}^{\infty} F_i\right) = \sum_{i=1}^{\infty} \mu(F_i)$

def: A random variable is a function $X : \Omega \longrightarrow \mathbb{R}$.

def: $X_n \xrightarrow[n\to\infty]{} X$ in probability $\Leftrightarrow$

$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall \eta > 0, \forall n \geq N, P(|X_n - X| \leq \eta) \leq \varepsilon$.



$X_1 = \frac{1}{2}$.

$X_2 = \frac{1}{3}$.

$X_3 = \frac{1}{3}$.

$\vdots$

$X(i) = 0$.

$\forall \varepsilon > 0, \lim_{n\to\infty} P_n[\omega : \omega \in \Omega, |X_n(\omega) - X(\omega)| > \varepsilon] = 0$

$\Leftrightarrow \forall \varepsilon > 0, \lim_{n\to\infty} P[\omega, |X_n(\omega) - X(\omega)| < \varepsilon] = 1$

* Kolmogorov: $(\Omega, \mathcal{F}, \mu)$: we flip a coin H = head, T = tail

$\Omega = \{H, T\}$

$\mathcal{F} = \{\emptyset, \Omega, \{H\}, \{T\}\}$

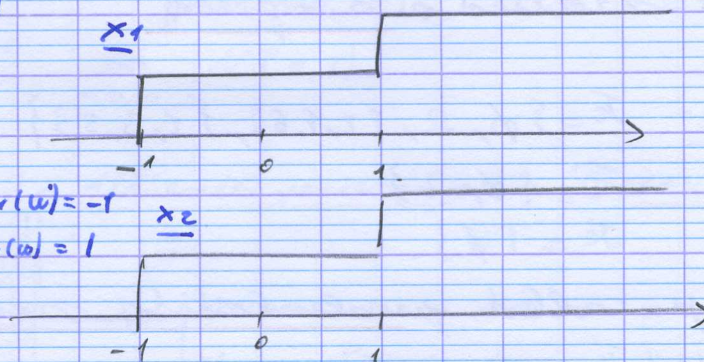
Convergence in distribution:

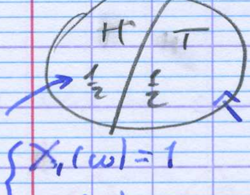
X_1: $H \mapsto 1$ $\quad\quad \mu(H) = \mu(T) = \frac{1}{2}$

$\quad\quad T \mapsto -1$

X_2: $H \mapsto -1$

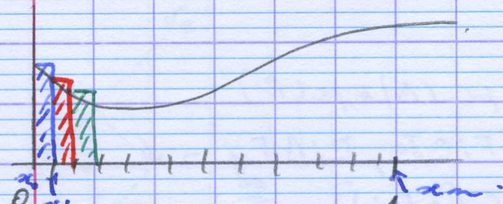
$\quad\quad T \mapsto 1$





$\begin{cases} X_1(\omega) = 1 \\ X_2(\omega) = -1 \end{cases}$ $\quad \begin{cases} X_1(\omega) = -1 \\ X_2(\omega) = 1 \end{cases}$

- $\int_0^1 f(x)\,dx$: definite integral.
- Riemain integral.

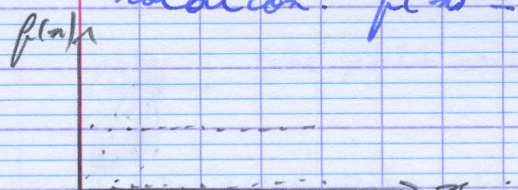


def: $\left[\lim_{n \to \infty} \sum_{i=0}^{n-1} f(x_i)(x_{i+1} - x_i) = \int_0^1 f(x)\,dx. \right.$

this is the Riemann integral.

counter-ex: $f(x) = \begin{cases} 1 & \text{if } x \text{ irrational} \\ 0 & \text{if } x \text{ is rational } \frac{p}{q}. \end{cases}$

notation: $f(x) = \chi_{\mathbb{IR}}(x)$



$\Rightarrow$ the Riemann integral does not exist.

$\begin{cases} \sum f(x_i)(x_{i+1} - x_i) = 0. \\ \sum f(x_i)(x_{i+1} - x_i) = 1. \end{cases}$

$\Rightarrow$ depending on the subdivision of $(0, x)$