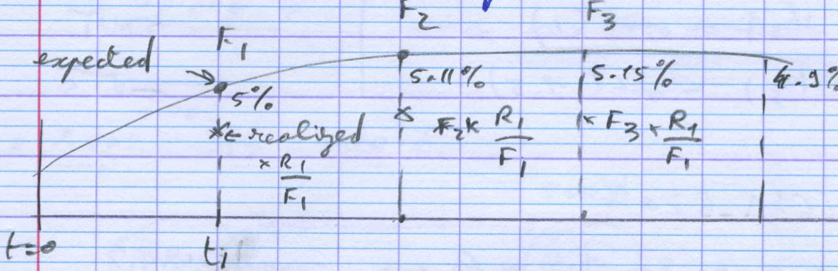
Lecture VII: Stochastic integrals.

* Movement of the yield curve:

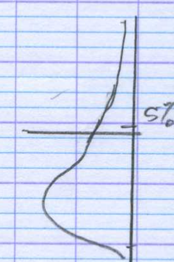
one factor parallel shift model:

- lognormal $\Rightarrow \ln f =$ Normal $\Rightarrow$ Insures no $\ominus$ rates
- Normal $\longrightarrow f \sim$ Normal $\Rightarrow$ $\ominus$ quantities.



today $\Rightarrow$ at t_1 the realized 6 mo is random.

This random var. has lognormal distr.



$\ln R \sim N(\mu, \sigma^2)$

$\sigma =$ volatility annualized $\times \sqrt{\text{time}}$

F_1

$R_1 \Rightarrow \log R_1 \sim N(\mu, \sigma\sqrt{0.5})$

F_2

$R_2 \Rightarrow \log R_2 \sim N(\mu, \sigma\sqrt{1})$ with $\sigma = 25\%$

. Forward rates are unbiased estimate of future spot rates.

$E[R_1] = F_1$

$E[R_t] = F_t \Rightarrow \mu_t = \ln F_t - \frac{1}{2}(\sigma\sqrt{t})^2$

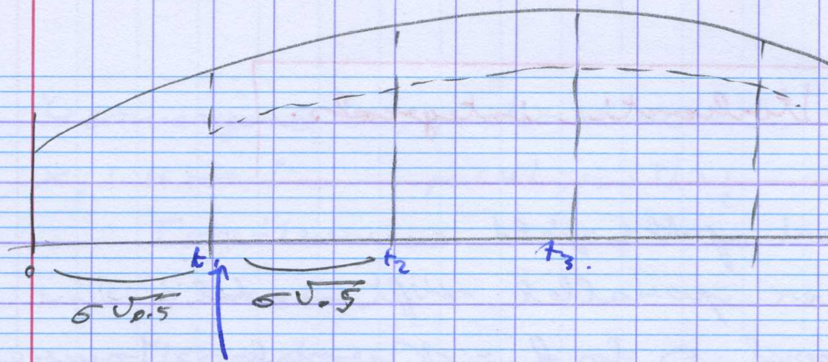
with $\frac{1}{2}\sigma^2 t$ called the drift.

Follows from the following fact:

ppty: [if X is a RV. and g is a function
then: $E[g(X)] \neq g(E(X))$.

. We adjust $\mu_2 \to \mu'_2$

$\mu_3 \to \mu'_3 \ldots$



$$\begin{cases} Dn(\text{realizat}^{\circ}\ t_1) = \sigma^2(.5) \\ Dn(\text{———}\ t_2) = \sigma^2(1) \\ Dn(\text{———}\ t_3) = \sigma^2(1.5) \end{cases} \Rightarrow \begin{cases} \text{vol}^{\circ} \text{ of } \sigma\sqrt{.5} \\ \sigma\sqrt{1} \\ \sigma\sqrt{1.5} \end{cases}$$

<u>Caplet:</u> (619-620)

you give a rate 2Y from now: 5%

payoff = (R - 5%, 0) = Caplet.

<u>ex1</u>: if infact R = 7%.

⇒ payoff = 2%.

<u>ex2</u>: Max (R - $R_{\text{previous setting}}$ - 50bps; 0)

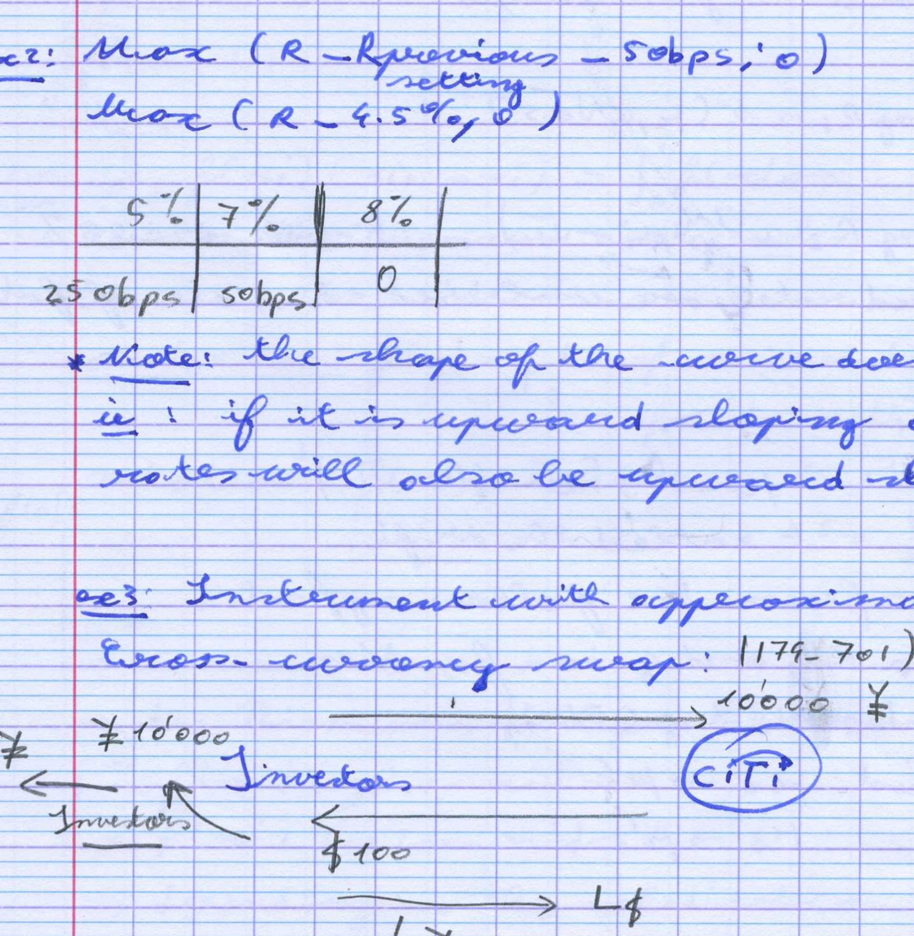
Max (R - 4.5%, 0)

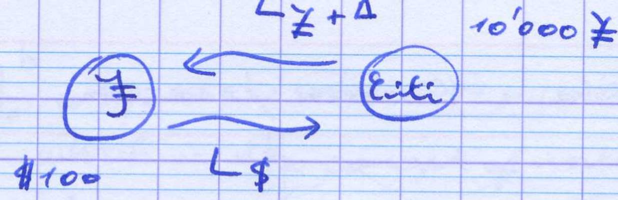
5%	7%	8%
250bps	50bps	0

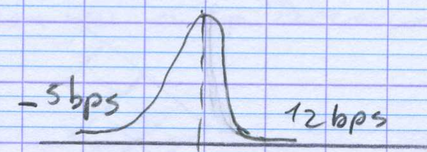
* <u>Note</u>: the shape of the curve does not change <u>ie</u>: if it is upward sloping all forward rates will also be upward sloping.

<u>ex3</u>: Instrument with approximately normal distr.

Cross-currency swap: (179-701)



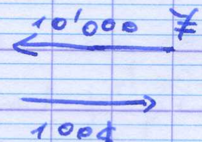




Δ: the spread only affects the ¥

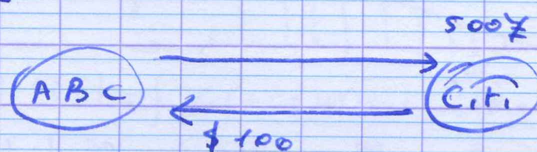
1 $ = 100 ¥.

J borrows 10000 ¥ ⟺ 100 $

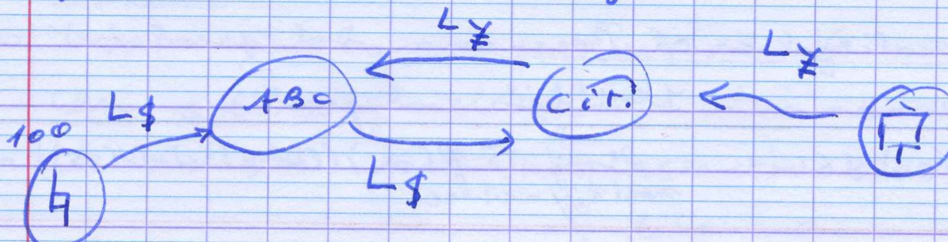


no matter what the exchange rate is.

* t=0: 1$ = 100 ¥



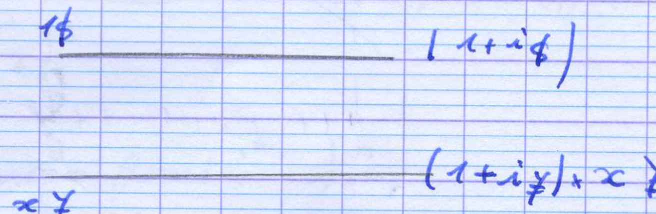
If there is initial exchange



* carry trade:

2$ 2¥

450bps = 4.5% 50bps

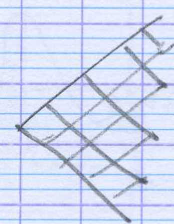


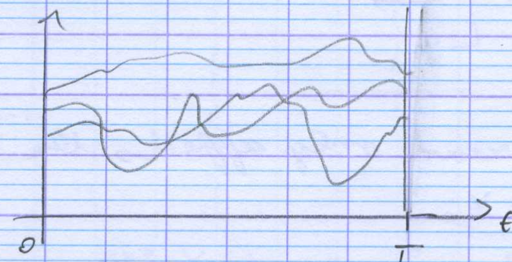
$$\Rightarrow FX = \frac{x ¥ (1+i_{¥})}{1+i_{\$}} \quad \text{: foreign exchange.}$$

$$= \frac{116 \times (1.005)^{1/2}}{1.045} \quad \Leftrightarrow \quad 98 ¥ = 1 \$$$

borrow ¥ and pay low yen interest rates by taking FX risk.

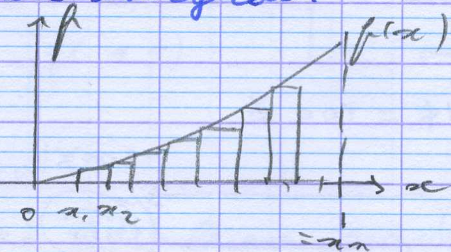
• Brownian Motion - Wiener process.





Recall Riemann Integral:

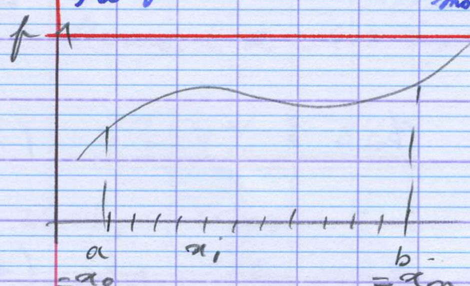
$\int_0^1 f(x)\,dx$



def: If: $\sum\limits_{\max|x_{i+1}-x_i|\to 0} f(x_i)(x_{i+1}-x_i) = A$. then A is called the Riemann integral of f.

* Generalization of Riemann integral is called Riemann - Stieltjes integral.

$$\boxed{\int_a^b f(x)\,dG(x) = \lim_{\max|x_{i+1}-x_i|\to 0} \sum f(x_i)\left(G(x_{i+1})-G(x_i)\right) = B}$$

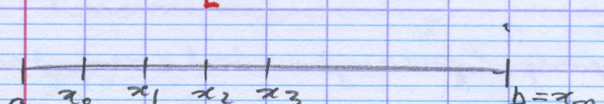


Consider the Brownian Motion (Wiener Process): $W(t)$

$W(t)$, $t\in[a,1]$ be one realization of this process.

Let $G(\cdot)$ be a continuous f°.

Define: $\left[\int_a^b w(t)\,dG(t) = \sum\limits_{i=0}^{n-1} w(x_i)\left[G(x_{i+1})-G(x_i)\right]\right.$



Let us rewrite this:

$\int_a^b w(t)\,dG(t) = -W(a)G(a) + G(x_1)\left(w(x_1)-W(a)\right) + G(x_2)\left(w(x_2)-w(x_1)\right)$
$+ \ldots$

$$\boxed{\int_a^b w(t)\,dG(t) = w(b)G(b) - w(a)G(a) - \int_a^b G(t)\,dW(t)}$$

$\Rightarrow$ which is the concept of integration w.r.t a Brownian Motion.

$W(t)$ is a stochastic process:

$\int_a^b G(t)\,dW(t)$: takes a stoc. process and transform into a random variable

We can view this random variable as $S(t)$.

$F(x) = \int_0^x f(u)\,du$.

$F'(x) = f(x) \iff dF(x) = f(x)\,dx$.

ppty: If $S(t) = \int G(u)\,dW(u)$ then $dS(t) = G(t)\,dW(t)$

Tangential:

df: Suppose you have a Vector Space V.

Then: a linear funct° L is:

$L : V \longrightarrow \mathbb{R}$ such that
$$\begin{cases} L(av) = aL(v) & , \forall a \in \mathbb{R}, \forall v \in V \\ L(0) = 0 \\ L(v_1 + v_2) = L(v_1) + L(v_2) & , \forall (v_1, v_2) \in V^2 \end{cases}$$

ex: ① $V = \mathbb{R}^2$

$L(x, y) = x + y$. (is a linear function)

$L(0, 1) = 0 + 1 = 1$

$L(7, 4) = 7 + 4 = 11$

② $\forall (a, b) \in \mathbb{R}^2$, $L(x, y) = ax + by$.

Let's take the basis $((1,0), (0,1))$

$L(1, 0) = a$

$L(0, 1) = b$.

$L(b_1 x + b_2 y) = L(b_1 x) + L(b_2 y) = b_1 L(x) + b_2 L(y)$

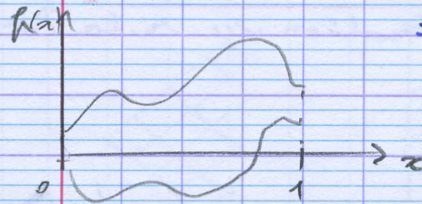
i.e: L is a linear function.

df: V is a Vector Space if:

$\begin{vmatrix} V_1 \in V \\ V_2 \in V \end{vmatrix} \Rightarrow \begin{cases} aV_1 \in V \\ V_1 + V_2 \in V. \end{cases}$

ex③: Consider $V = \mathcal{C}^0([0,1])$

$= \{ \text{continuous } f^\circ \text{ over } [0,1] \}$.



Indeed $0 : x \mapsto 0$ belongs to V.

$[0,1] \to \mathbb{R}$

. $\forall (f,g) \in V^2, \forall (a,b) \in \mathbb{R}^2, af + bg \in V$.

th: If L is a linear function on $\mathcal{C}^0[0,1]$
then there is a G such that
$L(f(x)) = \int_0^1 f(x)\, dG(x)$ Riemann Stieltjes Integral

Suppose we have a product or structure whose value depends on t ie: $(t, S(t))$ with: t = time

$S(t)$ = price of the asset.

ex: call, put.

Suppose: $dS(t) = \mu(t,S)\, dt + \sigma(t,S)\, dW(t)$

$dS(t) = \mu(t)\, dt + \sigma(t)\, dW(t)$ with: $\mu(t)$: drift,

$\sigma(t)$: volatility.

What is $df(t, S(t))$?

- Recall $df(x) = f(x+\Delta x) - f(x)$

Taylor's expansion implies:

$f(x+\Delta x) = f(x) + \Delta x f'(x) + \frac{\Delta x^2}{2} f''(x) + \frac{\Delta x^3}{3!} f^{(3)}(x)$

- Now suppose: $f(x,y)$.

$f(x+\Delta x, y+\Delta y) = f(x,y) + \left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}\right) f(x,y)$
$+ \frac{1}{2!}\left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}\right)^2 f + \frac{1}{3!}\left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}\right) \dots$

$\downarrow$

$\frac{1}{2}\left(\Delta x^2 \frac{\partial^2}{\partial x^2} + \Delta y^2 \frac{\partial^2}{\partial y^2} + \Delta x \Delta y \frac{\partial^2}{\partial x \partial y}\right) f(x,y)$

$$\Rightarrow \boxed{\begin{aligned} df(t,S) &= f(t+\Delta t, S(t+\Delta t)) - f(t, S(t)) \\ &= \Delta S \frac{\partial f}{\partial S} + \Delta t \frac{\partial f}{\partial t} + \frac{1}{2}\left[\Delta S^2 \frac{\partial^2 f}{\partial S^2} + \Delta t^2 \frac{\partial^2 f}{\partial t^2} + 2 \Delta t \Delta S \frac{\partial^2 f}{\partial S \partial t}\right] \end{aligned}}$$