Lecture VIII: Itô's lemma: stochastic calculus

Itô's lemma

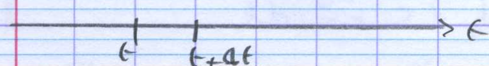
- Heath - Jarrow - Merton model.
  - ↳ Martingale
  - ↳ Arbitrage free models
  - ↳ relationship between Martingale & Arbitrage-free

$S(t)$ = price of asset at time t.

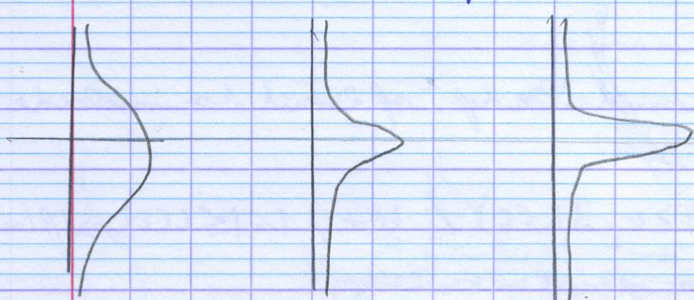
ex: price of Merrill Lynch stock.

$G(S(t), t)$: price of contract based on t and $S(t)$

$dW(t) = N(0, \Delta t)$, Δt = variance.



$\Delta S = S(t+\Delta t) - S(t) = \mu(t,S)\,dt + \sigma(t,S)\,dW(t)$



$\begin{cases} \mu(t,S) = \text{drift.} \\ \sigma(t,S) = \text{volatility.} \end{cases}$

$dS = \mu(t,S)\,dt + \sigma(t,S)\,dW(t)$

For ease we use, $\begin{cases} \mu(t,S) = \mu \\ \sigma(t,S) = \sigma \end{cases}$

What is $\Delta G = G(S(t+\Delta t), t+\Delta t) - G(S(t), t)$?

What is dG?

Recall Taylor's expansion:

$$\boxed{f(x+\Delta x) = f(x) + \Delta x\, f'(x) + \frac{\Delta x^2}{2} f''(x) + \dots}$$

$\Delta f = f(x+\Delta x) - f(x) = \Delta x\, f'(x) + \frac{\Delta x^2}{2} f''(x)$

$\Delta f \xrightarrow[\Delta t \to 0]{} df = f'(x)$

ex: $\begin{array}{|l} f(x) = x^2 \\ f(1) = 1 \\ f(1.01) = 1.2001 \end{array}$ $f(1.01) - f(1) = 0.2$

$$\boxed{df = (2 \cdot x)\,dx}$$

$G(x,y) = \ldots$

$G(x+\Delta x, y+\Delta y) = G(x,y) + \left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}\right) G + \frac{1}{2!}\left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}\right)^2 G$

$\Delta G = G(x+\Delta x, y+\Delta y) - G(x,y)$

$= \Delta x \frac{\partial G}{\partial x} + \Delta y \frac{\partial G}{\partial y} + \frac{1}{2}\Delta x^2 \frac{\partial^2 G}{\partial x^2} + \Delta x \Delta y \frac{\partial^2 G}{\partial x \partial y} + \frac{\Delta y^2}{2} \frac{\partial^2 G}{\partial y^2}$

$\Delta x \to 0,\ \Delta y \to 0.$

$dG = \frac{\partial G}{\partial t} dt + \frac{\partial G}{\partial S} dS + \frac{1}{2}\left[\frac{\partial^2 G}{\partial t^2} dt^2 + 2\, dS\, dt \frac{\partial^2 G}{\partial t \partial S} + \frac{\partial^2 G}{\partial S^2} dS^2\right]$

By avoid every term at an higher power than dt.

$\rightarrow dG = \frac{\partial G}{\partial t} dt + \frac{\partial G}{\partial S}(\mu dt + \sigma dW_t) + \frac{1}{2}\frac{\partial^2 G}{\partial S^2} dS^2$

because $dt^2 \ll dt$.

$dS \sim \sqrt{dt}. \Rightarrow \begin{cases} dS\, dt \simeq dt^{3/2} \ll dt. \\ dS \simeq dt. \end{cases}$

$dS^2 = (\mu dt + \sigma dW_t)^2$

$= \mu^2 dt^2 + 2\mu\sigma\, dt\, dW_t + \sigma^2 dW_t^2$

$\simeq \sigma^2 dW_t^2$

def! We have to extend the def$^{\circ}$ of limit to include random variables.

$\lim_{x\to 0} f(x) = A \iff \forall \varepsilon > 0, \exists L(\varepsilon) \,/\, \forall x, |x| \leq L(\varepsilon) \Rightarrow |f(x) - A| \leq \varepsilon$

- Convergence in probability:

(X_n): random variables,

then: $X_n \to X$ in probability

$\iff \forall \varepsilon > 0, \lim_{n\to\infty} P[\omega, |X_n(\omega) - X(\omega)| > \varepsilon] = 0.$

- Let's prove that $(dW_t)^2 \simeq dt$.

$W_t \sim \mathcal{N}(0, t)$

$\Delta W_t \sim \mathcal{N}(0, \Delta t)$

$dW_t \sim \sqrt{dt}\, \mathcal{N}(0,1)$

$(dW_t)^2 \sim dt\, (\mathcal{N}(0,1))^2$

$E[(dW_t)^2] = dt\, E[(\mathcal{N}(0,1))^2]$

$= dt\, Var[\mathcal{N}(0,1)] = dt.$

$$\text{Var}[(dW_t)^2] = \text{Var}[dt\,(N(0,1))^2]$$
$$= dt^2\,\text{Var}[N(0,1)^2]$$
$$= dt^2\left[E[N(0,1)^4] - (E[N(0,1)^2])^2\right].$$
$$= dt^2(3-1)$$
$$= 2\,dt^2$$

$$\text{Var}[(dW_t)^2] \xrightarrow[dt\to 0]{} 0$$
$$\Rightarrow (dW_t)^2 \xrightarrow[dt\to 0]{} E[(dW_t)^2]$$

$$\Rightarrow \boxed{(dW_t)^2 \xrightarrow[dt\to 0]{} dt.}$$

$\Rightarrow$ Itô's Lemma:

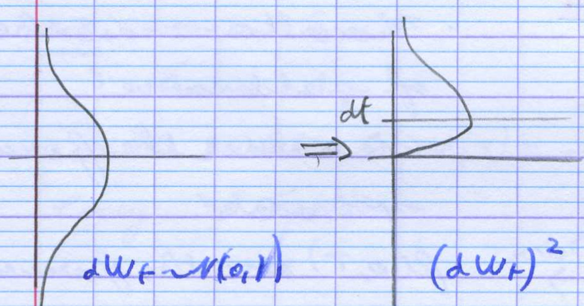
$$\boxed{dG(S(t),t) = \left[\frac{\partial G}{\partial t} + \mu\frac{\partial G}{\partial S} + \frac{\sigma^2}{2}\frac{\partial^2 G}{\partial S^2}\right]dt + \sigma\frac{\partial G}{\partial S}\,dW_t.}$$

- Recall if $X \sim N(0,1)$ then: $E[X] = 0$

$E[X^2] = \text{Var}(X) = 1$

skewness: $E[X^3] = 0$

kurtosis: $E[X^4] = 3$.



example:

① $G(S(t),t) = S(t)^2.$ with $dS(t) = \mu\,dt + \sigma\,dW(t)$

$G(x,t) = G(x) = x^2.$ using: $dG = \left[\frac{\partial G}{\partial t} + \mu S\frac{\partial G}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 G}{\partial S^2}\right]dt + \sigma\frac{\partial G}{\partial S}S\,dW_t.$

$$\frac{\partial G}{\partial t} = 0$$
$$\frac{\partial G}{\partial x} = 2x$$
$$\frac{\partial^2 G}{\partial x^2} = 2$$
$$Y(t) = S(t)^2$$
$$dY(t) = \left[\mu \times 2S^2(t) + \frac{\sigma^2}{2}\times 2S^2\right]dt + \sigma \times 2S^2(t)\,dW_t.$$
$$= [2\mu + \sigma^2]\,Y(t)\,dt + 2\sigma\,Y(t)\,dW_t.$$

def: Chebytchev's Inequality

. X is a random variable with mean μ, variance σ^2

$$\boxed{P\left[\,|X-\mu| \geq \varepsilon\,\right] \leq \frac{Var(X)}{\varepsilon^2}}$$

rem: if we prove that $E(X) \longrightarrow \mu$

and $Var(X) \xrightarrow[dt \to 0]{} 0$

we have $P\left[\,|X-\mu| > \varepsilon\,\right] \xrightarrow[dt\to 0]{} 0$

and we obtain the convergence in probability thanks to Bienaymé - Chebytchev's Inequality!

. for $X = dW(t)^2$

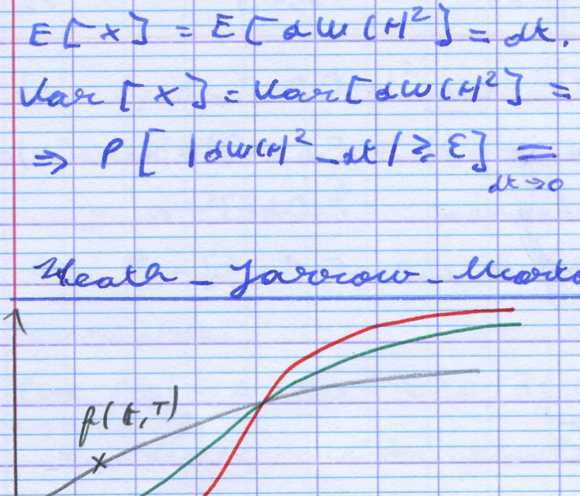
$E[X^2] = E[dW(t)^4] = 3\,dt^2$

$E[X] = E[dW(t)^2] = dt.$

$Var[X] = Var[dW(t)^2] = 3dt^2 - dt^2 = 2\,dt^2 \xrightarrow[dt\to 0.]{} 0$

$\Rightarrow P\left[\,|dW(t)^2 - dt| \geq \varepsilon\,\right] \underset{dt\to 0}{=} dt.$

Heath - Jarrow - Morton Model.

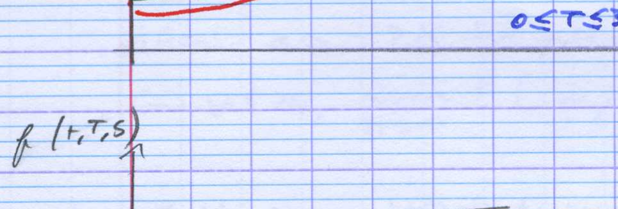


. How to model the movements of yield curve?

. We model the forward spot rate.

. $f(0,T)$ are the forward rates observed today.

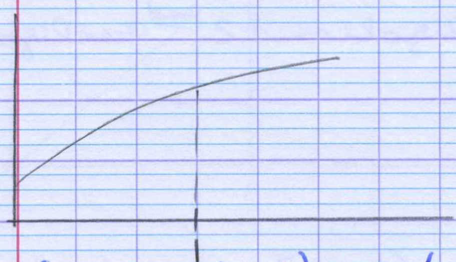
or equivalently we can model the movement of 0-coupon bond prices $P(t,T)$



recall $f(t,T) = -\dfrac{\partial \log P(t,T)}{\partial T}$

$$d\,f(t,T) = \mu(t,T)\,dt + \sum_{i=1}^{n} \sigma_i(t,T,f(t,T))\,dW_i(t)$$

and $dW_i(t)$ are independent wiener processes.



Can $\mu(t,T)$ and $\sigma_i(t,T,P(t,T))$ take any form?

<u>Arbitrage:</u>

- consider $t=0$ and suppose we have n assets $A_1, \ldots, A_n$, with $V_1(0), \ldots, V_n(0)$.

Let $\sum_{i=1}^{n} x_i V_i(0) = 0$ be a portfolio of these assets.

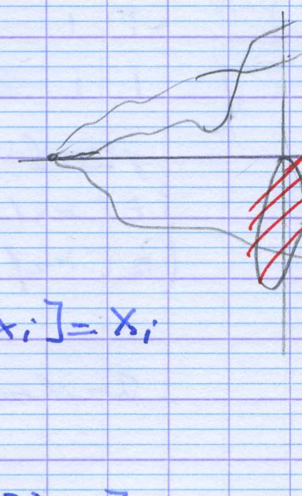
Let $V(t) = \sum_{i=1}^{n} x_i V_i(t)$ be the value of this portfolio at time t which is random.

There is an arbitrage $\Leftrightarrow V(t) \geq 0$ for all realization of prices at time t and $V(t) > 0$ for some realization

<u>ie:</u> $P(V(t) \geq 0) = 1$.
$P(V(t) > 0) > 0$.

<u>ex:</u> GM 36.
Ford 10

sell 10 GM, buy 36 Ford: $V(0) = -10\,GM + 36\,F = 0$



<u>def:</u> X_1, X_2 is a <u>martingale</u> $\Leftrightarrow E[X_{i+1}/X_i] = X_i$

<u>pty:</u> $\boxed{E[X_{i+1}] = E[X_i]}$

<u>proof:</u>

$E[X_{i+1}/X_i] = X_i \Rightarrow E[\,E[X_{i+1}/X_i]\,] = E[X_i]$
$\Rightarrow E[X_{i+1}] = E[X_i]$

- $V(0), V(1)$

$E[V(1)] = \sum$ prob.(prices) $\times$ prices.

- Suppose V is a martingale: $E[V(1)/V(0)] = V(0) = 0$.
$\forall$ values of $V(1) = 0$.

S_1	S_2	
0	0	0.5
1	0	0.3
0	1	0.1
1	1	0.1

$X_1 = 1^{st}$ realization $= S_1$
$X_2 = 1^{st}$ realization $+ 2^{nd}$ realization
$= S_1 + S_2$.

- If $X_1 = 0$, $E[X_2/X_1] = E[X_2/X_1=0]$
$$= 0 . \frac{0.5}{0.6} + 1 \times \frac{.1}{.6} = \frac{1}{6}.$$
- $E[X_2/X_1=1] = 1 . \frac{.3}{.4} + 2 \times \frac{.1}{.4}$

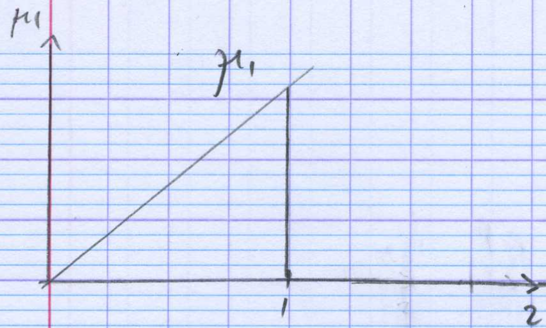
Under this probability the process is not a martingale.
Is it possible to change the probabilities (measure) such that, under this new probability the process is a martingale.
Are these probabilities equivalent?

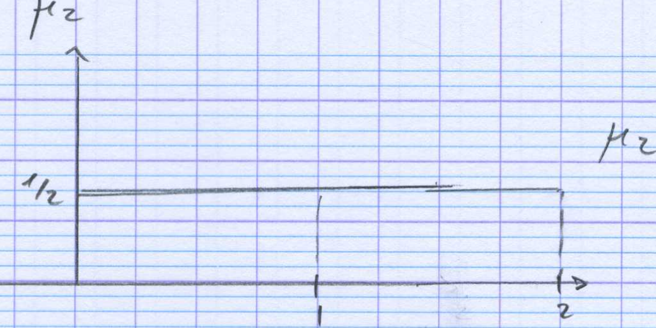
		old measure	New proba.
-1	-1	.5	.25
1	-1	.3	.25
-1	1	.1	.25
1	1	.1	.25

proof:
$$\begin{cases} E[X_2/X_1=-1] = -2 \times \frac{1}{2} + 0 \times \frac{1}{2} = -1 \\ E[X_2/X_1=1] = 0 \times \frac{1}{2} + 2 \times \frac{1}{2} = 1 \end{cases}$$

- Probability space $(\Omega, \mathcal{F}, \mu)$
$$\begin{pmatrix} -1 & -1 \\ 1 & -1 \\ -1 & 1 \\ 1 & 1 \end{pmatrix}$$

def: μ_1 and μ_2 are equivalent $\iff$ A is such that
$\mu_1(A) = 0 \iff \mu_2(A) = 0$.





$$f(x) = \begin{cases} 2x, & 0 \leq x \leq 1 \\ 0, & 1 \leq x \leq 2 \end{cases} \qquad f(x) = \frac{1}{2}, \quad 0 \leq x \leq 2.$$

$\mu_1, \rho_1 [1.5, 2] = 0.$

$\rightarrow$ μ_1 and μ_2 are not equivalent.

<u>Th:</u> <u>Radon Nykodym Theorem.</u>

If μ_1 and μ_2 are equivalent

<u>then</u>: there are functions that enable us to calculate the probability of one event using μ_1 that is the same as the other over μ_2.