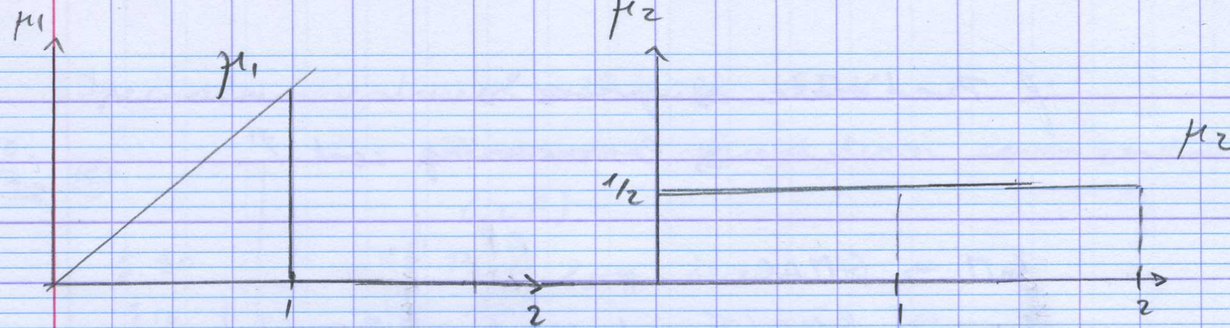


$f(x) = \begin{cases} 2x, & 0 \leq x \leq 1 \\ 0, & 1 \leq x \leq 2 \end{cases}$ $\quad f(x) = \frac{1}{2}, \ 0 \leq x \leq 2.$

$\mu_1\ p_1 [1.5, 2] = 0.$

$\rightarrow \mu_1$ and μ_2 are not equivalent.

th: <u>Radon Nykodym theorem.</u>

If μ_1 and μ_2 are equivalent

<u>then:</u> there are functions that enable us to calculate the probability of one event using μ_1 that is the same as the other over μ_2.

<u>Lecture IX:</u> $\boxed{\text{Credit derivatives}}$

rem: <u>bonds and company health:</u>

$B_1, \ldots, B_{100}.$

if the company is closed to bankruptcy

F = 100 \$

P = 50 ds / bonds is the total amount we hope to recover.

<u>Credit rating companies:</u>

Moody's

S&P (Standard & Poors)

Fitch

<u>rating:</u>			
	AAA	A	CCC$^+$
	AA$^+$	A$^-$	|
	AA	BBB$^+$	D.
	AA$^-$	BBB	
	A$^+$	BBB$^-$	

if Bank rate ⇒ price ↘

⇒ borrowing cost ↗

GM → GMAC

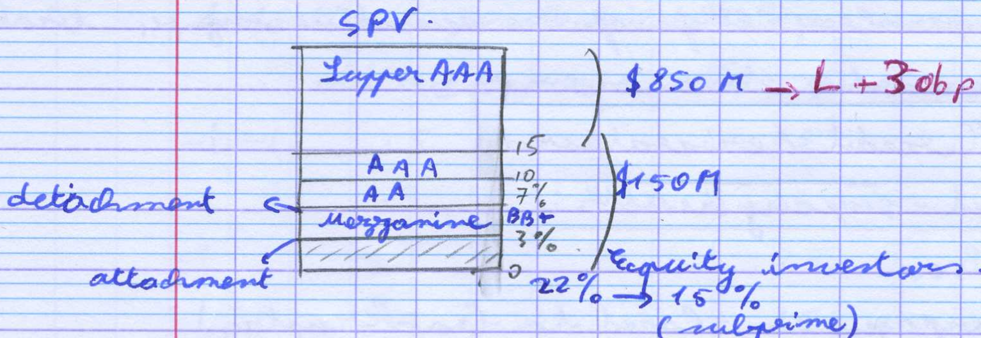
F → FMC (Ford Mutual Credit)

borrow at 6% → GMAC → lend at 8%

<u>1 Company has:</u> → It creates a Special purpose vehicle SPV (Special Investment vehicle)

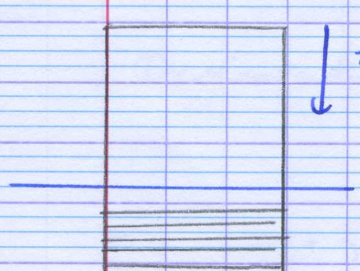
bonds B_1	A	10M	
B_2	BBB		
B_3	BB+		
⋮	⋮		
B_{100}	B		$= \boxed{1 \text{ bn } \$}$

SPV.



<u>What is the purpose?</u> leverage.

- if investment directly in BB+ → 5%.
- if investment in BB+ tranche → 10%



payment 1st tranche is paid, then 2, ...

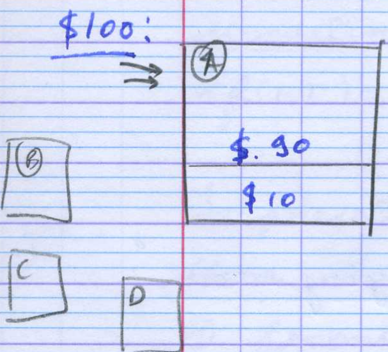
paid

in case of default

not paid

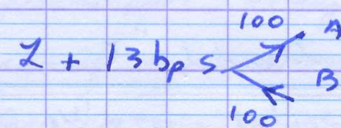
CDO: collateralized debt obligation.

Special investment vehicle : SIV



. Borrow short term commercial paper (CP)

• Buy long-term.



if A has asset A (99 \$) obliged to sell → 95 \$

then B with the same asset mark it to market → 94 \$.

then C ... → 93 \$ ....

Citi, B of America and LJ (CITIC) make

super SIV 100 bn \$

II - Interest rates! Term structure of interest rates

Modeling evolution of interest rates.

- There are several models
- following models describe the evolution of "short rate"

1°) Vasicek model

$\boxed{dr_t = \alpha(b - r_t)\,dt + \sigma\, dW_t}$ "Mean reversion"

. b : is a typical rate (5%)

. if $r_t \nearrow$ <u>then</u> drift < 0

$\Rightarrow r_t \searrow$ to the "b".

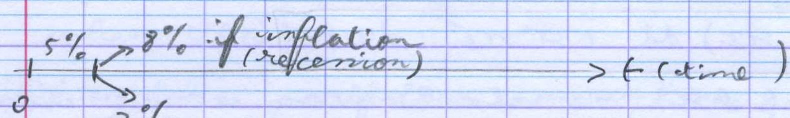
. if $r_t \searrow$ <u>then</u> drift > 0

$\Rightarrow r_t \nearrow$ to the "b"

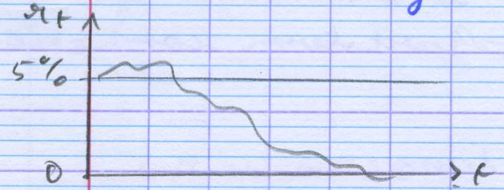
. Points to be careful about:

The mean reversion level "b" is difficult to estimate.

Now $\overline{r_t} \simeq 5\%$

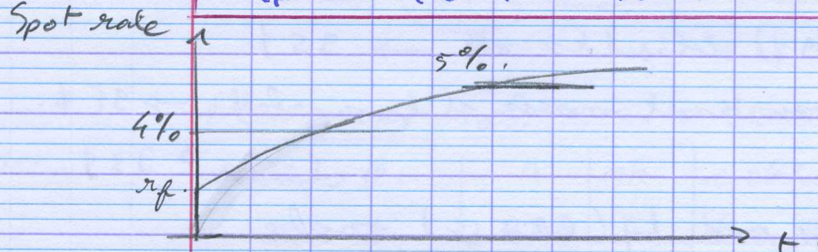


- Theoretically rates could be negative.



2°) Hull_White Model "Time varying mean"

$$\boxed{dr_t = \alpha(\theta(t) - r_t)dt + \sigma\, dW_t}$$



$\theta(t)$ = mean_reversion level.

$\theta(t)$ is varying with t.

- This makes it easier to fit the model observed values in the market,

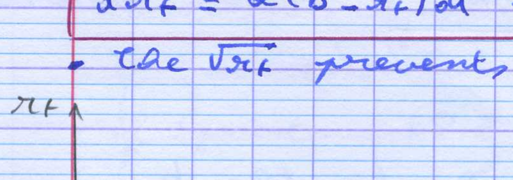
meth: simulation different paths → take the "average" one.

- However, r_t can be negative

3°) Cox - INGERSOLL - ROSS (CIR)

$$\boxed{dr_t = \alpha(b - r_t)dt + \sigma\sqrt{r_t}\, dW_t}$$ "Square root process".

- The $\sqrt{r_t}$ prevents the rates to become < 0,



if $r_t \sim 0^+$, volatility $= \sigma\sqrt{r_t}$

vol $\sim 0^+$

$\Rightarrow$ only the term $\alpha(b - r_t)dt$ is relevant

$\Rightarrow$ $\underline{dr_t > 0}$

4°) Longstaff and Sanders model

$$\boxed{dr_t = \alpha(\mu - r_t)\, dt + \sigma r_t^{\gamma}\, dW_t}$$

This model has constant mean_reversion level μ but has a slightly more flexible volatility function

5°) Heath - Jarrow - Morton model

Modeling forward:

$$\boxed{df(t,T) = \mu(t,T)\,dt + \sum_{i=1}^{n} \sigma_i(t,T)\, f(t,T)\, dW_i(t)}$$

In previous expressions: only 1 source of randomness dW_t

Here: n sources of randomness: $dW_i(t)$

① front of the curve ⇒ usually $\boxed{n=3}$

② end ————

③ middle ————

⇒ we have more flexibility to change the shape of the curve.

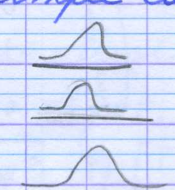
- Can $\mu(t,T)$ & $\sigma(t,T)$ be arbitrary?

No, if we want to exclude possibility of arbitrage.

6°) Arbitrage and martingales

n assets	price	T
A_1	$P_1(0)$	$P_1(T)$
|	|	|
A_n	$P_n(0)$	$P_n(T)$

Simple case of 2-period economy.



$\begin{cases} x_i > 0 : \text{we are long} \\ x_i < 0 : \text{we are short.} \end{cases}$

$t=0 \longrightarrow T \quad t\,(\text{time})$

Consider a portfolio of x_i units of asset i ($x_i \in \mathbb{R}$)

$\Pi = \sum_{i=1}^{n} x_i\, p_i(0) = V(0)$ = value of portfolio at $t=0$.

$V(T)$ = value of portfolio at T (RANDOM)

def 1: There is an arbitrage opportunity if:

① $V(0)=0$ ⟺ it doesn't cost anything to build the port.

② $P[V(T) \geq 0] = 1$ ⟺ there is no outcome where $V(T) < 0$.

③ for some outcomes $P[V(T) > 0] > 0$ ⟺ there are outcomes where $V(T) > 0$.

def 2: ① A riskless portfolio cannot have a return > riskless rate

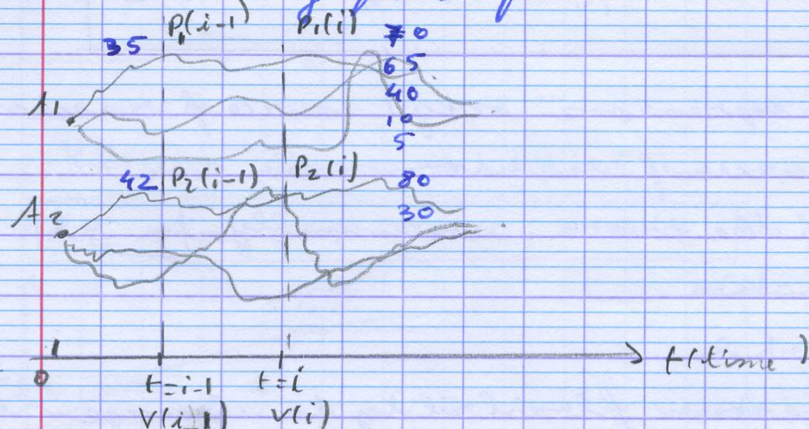
② If portfolios A & B have the same value (random) in future

They must have the same value today.

def: $X = \{X_1, x_2, \dots, x_n\}$.

X_i are martingale $\Leftrightarrow E[x_i / x_{i-1}] = x_{i-1}$

→ extremely powerful.



$E[V(i) / V(i-1) = 35+42 = 77] = 77.$

$E[V(T) / V(0) = 0] = 0$ ⇒ on average, I am not going to make money on this portfolio.

_ We show that martingale is not consistent with arbitrage.

rem: prob values.

$V(T) \geq 0 \Big\} = 0$

$\vdots$

$V(T) > 0 \Big\} > 0.$

$\sum p_i V_i(T) > 0$ (over scenarios)

 arbitrage-free.

Y_1	Y_2	P_1	P_2	f_1	f_2	P_3
1	1	1/8	1/4	2	1/2	0
1	-1	1/2	1/4	1/2	2	1/3
-1	1	1/4	1/4	1	1	1/3
-1	-1	1/8	1/4	1/2	2	1/3

rem: Equivalent measure

P_1 is equivalent to P_2 ($P_1 \sim P_2$) because there is a function f_1 & f_2 / $\left| \begin{array}{l} P_1 f_1 = P_2 \\ P_2 f_2 = P_1 \end{array} \right.$

there is no f_3 / $P_3 f_3 = P_1$

tricks: Indeed P_1 and P_2 are equivalent
$\iff \left(P_1(A) = 0 \iff P_2(A) = 0\right)$.
That means that 2 prob. measures are equivalent
$\iff$ we have the same zeros on the same lines.

P_1	P_2
0 $\leftrightarrow$	0
$1/2$	$3/4$
$1/2$	$1/4$

th: Martingale $\Rightarrow$ Arbitrage free
Arbitrage-free $\not\Rightarrow$ Martingale

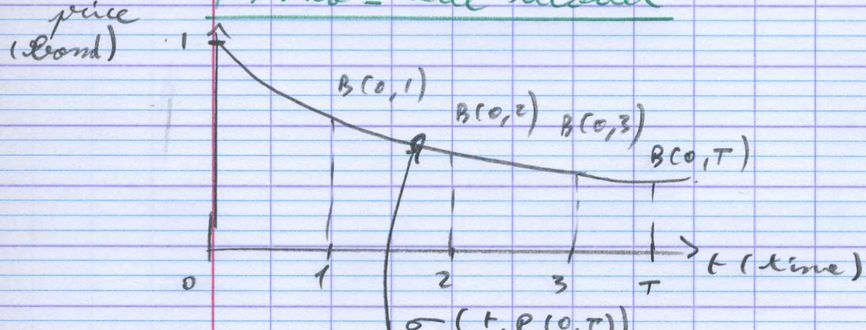
rem: $V(T) = \begin{pmatrix} \\ \end{pmatrix}$ w.p. : Under P_1, V is not a Martingale
can we find P_2, such that V is a MG.

ex: recall objective prob. $\Rightarrow$ risk-neutral prob.
$\mathbb{P} \Rightarrow \mathbb{Q}$.

And under <u>$\mathbb{Q}$, the price is a Martingale</u>.

rem: <u>HJM model</u>
The drift is determined by the volatility
$df(t,T) = \mu(t,T)\,dt + \sum_{i=1}^{n} \sigma_i(t,T) f(t,T)\, dW_i(t)$
If $\mu(t,T) = \sum_{i=1}^{n} \sigma_i(t,T,f(t,T)) \int_t^T \sigma_i(t,u,f(t,T))\,du$
we have a Martingale for $f(t,T)$.

7°) Ho – Lee Model



Is this process arbitrage-free?