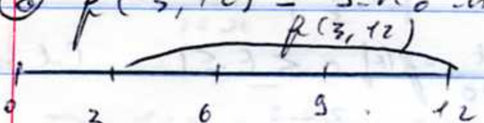
# Pricing Models IEOR 4630
## Midterm, fall 2007

**ex1**

| | |
|---|---|
| 3 Mo | 5% |
| 6 Mo | 5.5% |
| 1 Yr | 6.25% |

continuously compounded.

(a) $f(3,12)$ = 9-Mo interest rate in 3 Mo.




$$f(3,12) = \frac{s_2 t_2 - s_1 t_1}{t_2 - t_1} = \frac{12R(12) - 3R(3)}{9} = \frac{12 \times 6.25 - 3 \times 5}{9}$$

$$\boxed{f(3,12) = 6.67\%}$$

(b) zero-coupon bond, maturing in 1 Yr, 6 Mo from now.
face value: 100

assume 6 Mo-compounding.

$$P = \cancel{\frac{F}{\left(1+\frac{f(6,12)}{2}\right)}} = \boxed{F e^{-f(6,12)\times\frac{1}{2}} = P} \quad \text{continuous!}$$

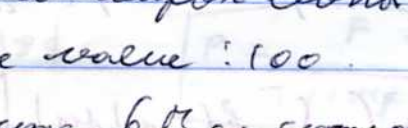
$$f(6,12) = \frac{12R(12) - 6R(6)}{6} = \frac{12 \times 6.25 - 6 \times 5.5}{6} = \boxed{7\% = f(6,12)}$$

$$P = \cancel{\frac{100}{1+\frac{0.07}{2}}} \Rightarrow P = 100\, e^{-\frac{1}{2}0.07}$$

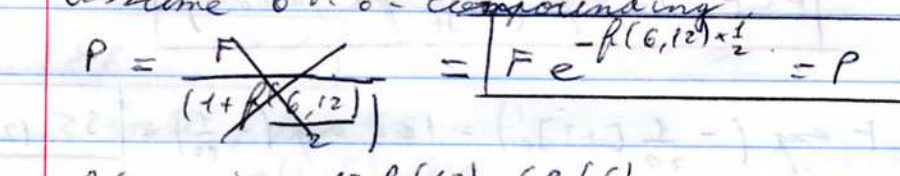
$$\boxed{P = 96.56\$}$$

rem: if $P = \dfrac{100}{1+\frac{0.07}{2}} \Rightarrow \underline{P = 96.62\$}$ close but different

(c)




1-Yr swap rate $L_1$



$$\Rightarrow \left(\frac{L_1}{2} e^{-R(6,12)\cdot\frac{1}{2}} + \frac{L_1}{2}\right)\cancel{e^{-\frac{1}{2}R(6)}} = \left(\frac{R(6,12)}{2} e^{-R(6,12)\cdot\frac{1}{2}} + \frac{R(6)}{2}\right)\cancel{e^{-\frac{1}{2}R(6)}}$$

$\Rightarrow L_1 \times 0.982802708 = 0.06129619$

$\Rightarrow L_1 = 6.2369\ \%$

$\boxed{L_1 = 6.24\ \%}$

rem: very close to $R(12M_0) = 6.25\%$

$\boxed{ex2}$ short rate at time $t = \frac{t}{10} = f(t)$, $0 \le t \le 1$ ($t$ in years)

(a) face value $= F = 100\$$

assume 6 Mo compounding

$$\boxed{P(0,0.5) = F \exp\left(-\int_0^{\frac{1}{2}} f(t)dt\right)}\ !$$

$$= 100 \exp\left(-\int_0^{\frac{1}{2}} \frac{t}{10} dt\right)$$

$$= 100 \exp\left(-\frac{1}{20}\left[t^2\right]_0^{\frac{1}{2}}\right) = 100 \exp\left(-\frac{1}{20} \times \frac{1}{4}\right) = 100 e^{-\frac{1}{80}}$$

$\boxed{P(0,0.5) = 98.76\$}$

Lecture 3 : $\boxed{P(t,T) = \exp\left(-\int_t^T f(t,u)du\right)}$

(b) $P(0,1) = F \exp\left(-\frac{1}{20}\left[t^2\right]_0^1\right) = 100 \exp\left(-\frac{1}{20}\right) = \boxed{95.12\ \$ = P(0,1)}$

(c) $\boxed{P(0.5,1) = F \exp\left(-\int_{\frac{1}{2}}^1 f(t)dt\right)} = 100 \exp\left(-\frac{1}{20}\left[t^2\right]_{\frac{1}{2}}^1\right) = \boxed{96.32\$ = P(\tfrac{1}{2},1)}$

(d) $P(0,0.5)P(0.5,1) = 98.76 \times 96.32 = 9512.28$ (4 decimals)

$F \times P(0,1)^* = 95.12^* \times 100 = 9512.29$

$$P(0,0.5)P(0.5,1) = F^2 \exp\left(-\int_0^{\frac{1}{2}} f(t)dt\right)\cdot \exp\left(-\int_{\frac{1}{2}}^1 f(t)dt\right)$$

$$= F^2 \exp\left(-\int_0^1 f(t)dt\right)$$

$$= F^* \times P(0,1)$$

otherwise take $\boxed{F = 1\$}$

Ex 3

$$S_0 \begin{cases} \xrightarrow{1/2} 1.15\, S_0 \\ \xrightarrow{1/2} 0.90\, S_0 \end{cases}$$

$t=0$ $\qquad$ $t=1$

$u = 1.15$, $d = 0.90$ $\qquad$ $d \neq \frac{1}{u}$

(a) if $Q = 1-q = \frac{1}{2}$ (risk-neutral prob)

no continuous compounding

$$S_0 = \frac{1}{1+r} E^{Q}[S(1)]$$

$$= \frac{1}{1+r}\left(\frac{1}{2} \times (1.15 S_0 + 0.90 S_0)\right) = \frac{2.05}{2(1+r)} \cancel{S_0} = \cancel{S_0}$$

$$\Rightarrow 1+r = \frac{2.05}{2} \Rightarrow \boxed{r = \frac{2.05}{2} - 1} \Rightarrow \boxed{r = 2.5\%}$$

(b) $P = \frac{F}{1+r} = \frac{1}{1+r} \Rightarrow \boxed{P = 0.9756\ \$}$

(c) $F = 100\ \$$

$P = 98\ \$$ $\qquad$ $\boxed{P = \frac{F}{1+r_f}} \Rightarrow \boxed{r_f = \frac{F}{P} - 1} = \frac{100}{98} - 1$

$\boxed{r_f = 2.04\%}$

- call, $K = 98$

$$C_0 \begin{cases} (1.15 S_0 - K)^+ \\ (0.90 S_0 - K)^+ \end{cases} \qquad S_0 \begin{cases} 1.15 S_0 \\ 0.90 S_0 \end{cases}$$

$t=0$ $\quad$ $t=1$ $\qquad$ $t=0$ $\quad$ $t=1$

$$q = \frac{S_0(1+r_f) - S_d}{S_u - S_d} = \frac{\cancel{S_0}(1+r_f) - 0.90 \cancel{S_0}}{(1.15 - 0.90)\cancel{S_0}} = \frac{0.10 + r_f}{0.25}$$

$\boxed{q = 0.4816 \quad 1-q = 0.5184}$ RN-prob

$$C_0 = \frac{1}{1+r_f} E^q[C(1)]$$

$$\boxed{C_0 = \frac{1}{1+r_f}\left[q(1.15 S_0 - K) + (1-q)(0.90 S_0 - K)^+\right]}$$

with $K = S_0$ $\quad$ $\boxed{C_0 = \frac{1}{1+r_f}\left[q \cdot 0.15 S_0\right]}$

$$\boxed{C_0 = \frac{0.15q}{1+r_f} S_0}$$

$$\frac{0.15q}{1+r_f} = \frac{0.15 \times 0.4816}{1+0.0204} = 0.0708 \Rightarrow \boxed{7.08\%}$$

$$\boxed{C_0 = 7.08\% \text{ of } S_0}$$

2)
$$\begin{cases} C_0 = \dfrac{1}{1+r_f} E^Q[C(1)] \\ C_0 = \dfrac{1}{1+r_f+\text{spread}} E^P[C(1)] \end{cases}$$

$$\frac{0.15q}{1+r_f} S_0 = \frac{1}{1+r_f+\text{spread}} \left( p \cdot 0.15 S_0 + \cancel{(1-p) \times 0} \right)$$

$$\Leftrightarrow \frac{q}{1+r_f} = \frac{p}{1+r_f+\text{spread}} \Leftrightarrow \boxed{\text{spread} = \frac{p(1+r_f)}{q} - 1 - r_f}$$

$$\text{spread} = \frac{0.5(1+0.0204)}{0.4816} - 1 - 0.0204.$$

$$\text{spread} = 0.03899 \Rightarrow \boxed{\text{spread} = 3.90\%}$$

$\boxed{ex4}$ non-dividend paying stock.

$S_0 = 50\$$ $\qquad r = 10\%$

$$\begin{tikzcd}[row sep=small] & 1.20\,S_0 \\ S_0 \arrow[ur, "0.6"] \arrow[dr, "0.4"'] & \\ & 0.9\,S_0 \end{tikzcd}$$

put: $K = 49\$$

$T = 1\,Yr$

payoff(put) $= \max(K - S_T, 0)$

$\boxed{t=0}$ $\quad \boxed{t=1}$ $\qquad \boxed{t=0}$ $\quad \boxed{t=1}$

$$\begin{tikzcd}[row sep=small] & 60 \\ 50 \arrow[ur, "0.6"] \arrow[dr, "0.4"'] & \\ & 45 \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] & 0 \\ P_0 \arrow[ur, "0.6"] \arrow[dr, "0.4"'] & \\ & 4 \end{tikzcd}$$

stock: $\qquad$ put:

(a) 1st step: compute the risk-neutral probabilities.

$$q = \frac{S_0(1+r) - S_d}{S_u - S_d} = \frac{S_0(1+r) - 0.90\,S_0}{1.2\,S_0 - 0.9\,S_0}$$

$$\boxed{q = \frac{(1+r) - 0.9}{1.2 - 0.9}} \qquad \boxed{\begin{array}{l} q = 0.666 = \frac{2}{3} \\ 1 - q = \frac{1}{3} \end{array}}$$

2nd step: $\boxed{P_0 = \frac{1}{1+r} \cdot E^{Q}[P(1)]}$

$$P_0 = \frac{1}{1+r} \cdot (q \times 0 + (1-q) \times 4) = \frac{4}{3(1.1)} \Rightarrow \boxed{P_0 = 1.21\ \$}$$

(b) if prob. are reversed, $q$ is unchanged

indeed, risk-neutral probabilities and objective probabilities are independent!

$\Rightarrow \boxed{P_0 = 1.21\$}$

Ex 5

R : spot short rate

$\Delta t = 1$

$\sigma = 0.25 \Rightarrow \boxed{\sigma\sqrt{\Delta t} = 0.25}$

| | price (0) | T |
|---|---|---|
| Bond 1 | 95.24 | 1 |
| Bond 2 | 89 | 2 |

$$R \begin{cases} \xrightarrow{\frac{1}{2}} R_u \\ \xrightarrow{\frac{1}{2}} R_d \end{cases}$$

$\boxed{q = 1-q = \frac{1}{2}}$ $\quad F = 100\,\$$

(a) Let's assume yearly compounding

$$B(0,1) = E^Q\left(\frac{F}{1+R}\right) = \frac{F}{1+R}$$

$$\Rightarrow \boxed{R = \frac{F}{B(0,1)} - 1} \Rightarrow R = \frac{100}{95.24} - 1 \Rightarrow \boxed{R = 5.00\,\%}$$

We have zero-coupon bonds:

$$B(0,2) = E^Q\left[\frac{1}{1+R} \cdot \frac{F}{1+R_1}\right]$$

$R_1$ : random variable

$R_1 = R_u$ w p $\frac{1}{2}$

$R_1 = R_d$ w p $\frac{1}{2}$

$$B(0,2) = \frac{100}{1+R}\left(\frac{1/2}{1+R_u} + \frac{1/2}{1+R_d}\right)$$

but $R_u = R_d\, e^{2\sigma\sqrt{\Delta t}} = \boxed{R_d\, e^{0.5} = R_u}$

$$\Rightarrow \boxed{B(0,2) = \frac{1/2\,F}{1+R}\left(\frac{1}{1+R_d} + \frac{1}{1+R_d e^{0.5}}\right)} = 89$$

$$\Rightarrow \boxed{\begin{aligned} R_d &= 5.31 \\ R_u &= 8.76 \end{aligned}}$$

(b) Expected rate for the 2nd period

$R_2 = \frac{1}{2}(R_u + R_d)$

$R_2 = 0.0704$

$\boxed{R_2 = 7.04\,\%}$

With the forward rates:

$$(1+s(1))(1+f(1,2)) = (1+s(2))^2$$

$$\boxed{f(1,2) = \frac{(1+s(2))^2}{(1+s(1))} - 1}$$

$s(1) = R = 5,00\%$.

for $s(2)$: $B(0,2) = \frac{100}{(1+s(2))^2} \Rightarrow s(2) = 6.00\%$.

$$\boxed{f(1,2) = 7.01\%}$$

(c) caplet: $T = 2$ Yr

payoff $= 100 \max(\text{rate} - 6\%, 0)$.

$$\begin{tikzcd}[column sep=small]
 & 8.76 \\
5.00 \arrow[ur, "\frac{1}{2}"] \arrow[dr, "\frac{1}{2}"'] & \\
 & 5.31
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}[column sep=small]
 & 2.76 \\
C_0 \arrow[ur, "\frac{1}{2}"] \arrow[dr, "\frac{1}{2}"'] & \\
 & 0
\end{tikzcd}$$

$\underline{t=0}$ $\underline{t=1}$ caplet:

principal amount = 100 \$

tenor = 1 Yr

$$\boxed{C_0 = \frac{1}{1+R}\left[\frac{1}{2}\left(\frac{2.76}{1+R_u} + \frac{0}{1+R_d}\right)\right]}$$

$$\boxed{C_0 = 1.21\,\$}$$