Online appendix to "Liquidity traps, debt relief, and macroprudential policy: a mechanism design approach"

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Preliminary; latest version at http://www.columbia.edu/~kd2338/JMP_Appendix.pdf.
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A New Keynesian model

In this section, I present a relatively standard New Keynesian model and show that equilibria in this economy are isomorphic to ZLB-constrained equilibria in the limit as prices become perfectly sticky.

A.1 Households

Households $i = S, B$ have period utility functions $\tilde{U}(C_0^i, h_0^i, \theta_i)$ (at date 0) and $\tilde{u}(C_0^i, h_0^i)$ (in subsequent periods), where C_t^i is consumption and h_t^i is hours worked. I will be interested in a special case where $\tilde{U}(C, h) = U(C - v(h), \theta)$, $\tilde{u}(C, h) = u(C - v(h))$ and we define $c = C - v(h)$ to be net consumption. Each household's real income, excluding transfers, is $\frac{W_t}{P_t} h_t^i + \pi_t - T_t$ where W_t is the nominal wage, P_t is a price index defined shortly, and $\pi_t - T_t$ is total real profits from the monopolistically competitive firms, net of the lump sum transfer used to finance subsidies to the firms. Households trade a nominal bond: d_t^i is the nominal face value of debt which household i promises to repay in period t, and $1 + i_t$ is the nominal interest rate between periods t and $t + 1$. (Since I consider perfect foresight equilibria in the baseline model, allowing households to trade a real bond would make no difference.) Household i solves:

$$\max U_0(C_0^i, h_0^i, \theta_i) + \sum_{t=1}^{\infty} \beta^t \tilde{U}(C_t^i, h_t^i) \tag{1}$$

$$\frac{d_{t+1}^i}{1 + i_t} = d_t^i + P_t C_t^i - T_t - \Pi_t - W_t h_t^i$$

$$d_0^B = d_0^S = 0$$

$$d_{t+1}^i \leq P_{t+1} \phi_t, t = 1, ...$$

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where $\tilde{U}, \tilde{u}$ are strictly concave, strictly increasing in C and decreasing in h, and satisfy $\tilde{U}_{ch} > 0$. Consumption C_t^i is a Dixit-Stiglitz aggregate:

$$C_t^i = \left[\int_0^1 C_t^i(j)^{(\varepsilon-1)/\varepsilon} \, \mathrm{d}j\right]^{\varepsilon/(\varepsilon-1)} \tag{2}$$

with corresponding price index

$$P_t = \left[\int_0^1 p_t(j)^{1-\varepsilon} \, \mathrm{d}j\right]^{1/(1-\varepsilon)} \tag{3}$$

Households' demand for variety j is given by

$$C_t^i(j) = C_t^i \left(\frac{p_t(j)}{P_t}\right)^{-\varepsilon} \tag{4}$$

Real interest rates are defined by the Fisher equation:

$$(1 + i_t) = (1 + r_t)\frac{P_{t+1}}{P_t} \tag{5}$$

Households' labor supply decision satisfies

$$\tilde{u}_c + \frac{W_t}{P_t}\tilde{u}_h = 0,$$

replacing $\tilde{u}$ with $\tilde{U}$ if $t = 0$.

A.2 Firms

There is a continuum of monopolistically competitive firms indexed by $j \in [0, 1]$ who hire labor and produce output using the linear technology $y_t(j) = h_t(j)$. They receive an employment subsidy $\tau = 1/\varepsilon$. In each period t, a fraction $\alpha \in [0, 1]$ of firms are unable to change their price, while $1 - \alpha$ can change their price. The probability of being able to change prices is independent of the firm's current price and the date on which it last adjusted prices. All firms who cannot set prices in period 0 have the same price, P_{-1}. I assume firms discount profits at the riskless nominal interest rate i_t.

Firms who can set their price in period t solve

$$\max_{p_t(j)} \sum_{s=t}^{\infty} \alpha^{s-t} Q_{t,s}(p_t(j) - W_s(1-\tau))Y_s \left(\frac{p_t(j)}{P_s}\right)^{-\varepsilon} \tag{6}$$

where $Q_{t,s} = \left(\prod_{k=t}^{s-1} \frac{1}{1+i_k}\right)$. The firm's first order condition yields

$$
\begin{aligned}
p_t(j) &= \frac{\sum_{s=t}^{\infty} \alpha^{s-t} Q_{t,s} Y_s \left(\frac{P_s}{P_t}\right)^{\varepsilon} W_s}{\sum_{s=t}^{\infty} \alpha^{s-t} Q_{t,s} Y_s \left(\frac{P_s}{P_t}\right)^{\varepsilon}} \\
\frac{p_t(j)}{P_t} &= \frac{\sum_{s=t}^{\infty} \alpha^{s-t} Q_{t,s} Y_s \left(\frac{P_s}{P_t}\right)^{\varepsilon+1} \frac{W_s}{P_s}}{\sum_{s=t}^{\infty} \alpha^{s-t} Q_{t,s} Y_s \left(\frac{P_s}{P_t}\right)^{\varepsilon}} \\
\frac{p_t(j)}{P_t} &= \frac{K_t}{F_t}
\end{aligned}
$$

where we can define K_t and F_t recursively as

$$
K_t = Y_t \frac{W_t}{P_t} + \frac{\alpha}{1+i_t} \Pi_{t+1}^{\varepsilon+1} K_{t+1} \tag{7}
$$

$$
F_t = Y_t + \frac{\alpha}{1+i_t} \Pi_{t+1}^{\varepsilon} F_{t+1} \tag{8}
$$

The aggregate price level evolves according to

$$
\begin{aligned}
P_t^{1-\varepsilon} &= \alpha P_{t-1}^{1-\varepsilon} + (1-\alpha)(p_t^*)^{1-\varepsilon} \\
1 &= \alpha \Pi_t^{\varepsilon-1} + (1-\alpha) \left(\frac{p_t^*}{P_t}\right)^{1-\varepsilon} \\
1 &= \alpha \Pi_t^{\varepsilon-1} + (1-\alpha) \left(\frac{K_t}{F_t}\right)^{1-\varepsilon} \qquad (9)
\end{aligned}
$$

A.3 Monetary policy

The monetary authority sets interest rates according to a Taylor rule, modified to take account of the zero lower bound:

$$
1 + i_t = \max\left\{(1 + r_t^n)\Pi_t^{\phi_\pi}, 1\right\} \tag{10}
$$

where $\phi_\pi > 1$ and r_t^n is the natural rate of interest, defined as the equilibrium real interest rate in the economy with $\alpha = 0$ (perfectly flexible prices).

A.4 Market clearing

Goods and labor markets clear (which ensures that the asset market also clears, by Walras' Law):

$$
\begin{aligned}
C_t^S + C_t^B &= 2Y_t \\
&= 2\left[\int_0^1 y_t(j)^{(\varepsilon-1)/\varepsilon}\,\mathrm{d}j\right]^{\varepsilon/(\varepsilon-1)} \qquad = 2\left[\int_0^1 h_t(j)^{(\varepsilon-1)/\varepsilon}\,\mathrm{d}j\right]^{\varepsilon/(\varepsilon-1)} \\
2\int_0^1 h_t(j)\,\mathrm{d}j &= h_t^S + h_t^B
\end{aligned}
$$

We can combine these conditions as

$$C_t^S + C_t^B = 2Y_t = \frac{h_t^S + h_t^B}{\Delta_t} \tag{11}$$

where we define the measure of price dispersion

$$\Delta_t = \int_0^1 \left(\frac{p_t(j)}{P_t}\right)^{-\varepsilon} \mathrm{d}j \geq 1$$

which evolves according to

$$\Delta_t = (1-\alpha)\left(\frac{K_t}{F_t}\right)^{-\varepsilon} + \alpha \Delta_{t-1} \Pi_t^{\varepsilon} \tag{12}$$

with initial condition $\Delta_{-1} = 1$.

A.5 Equilibrium and isomorphism to ZLB-constrained equilibrium

I define equilibrium in the standard way.

Definition A.1. *An equilibrium is a sequence* $\{C_t^S, C_t^B, h_t^S, h_t^B, d_t^S, d_t^B, \frac{W_t}{P_t}, T_t, Y_t, \Delta_t, \Pi_t, F_t, K_t\}_{t=0}^{\infty}$ *such that:*

1. $\{C_t^i, h_t^i, d_t^i\}$ *solves household i's problem* (1), *for* $i = S, B$
2. $\{\Delta_t, \Pi_t, F_t, K_t\}$ *satisfy* (7), (8), (12), (9)
3. *Interest rates* $\{i_t\}$ *satisfy the modified Taylor rule* (10)
4. *The market clearing conditions* (11) *is satisfied.*

I now present the main result of this Appendix, which states that equilibria of the New Keynesian model are isomorphic to ZLB-constrained equilibrium when prices are fixed and there is no wealth effect on labor supply.

Proposition A.2. *Suppose preferences have the form* $\tilde{U}(C,h) = U(C - v(h), \theta), \tilde{u}(C,h) = u(C - v(h))$, *and suppose* $\alpha = 1$. *Then every equilibrium of the New Keynesian model is isomorphic to a ZLB-constrained equilibrium with* $c_t^i = C_t^i - v(h_t^i)$, $\Pi_t = 1$, $y_t = h_t - v(h_t)$, $y^* = h^* - v(h^*)$, *where* $v'(h^*) = 1$, *and* $i_t = r_t$.

Proof. When $\alpha = 1$, firms never change their prices, so $\Pi_t = \Delta_t = 1, \forall t$. Since there is no inflation, real interest rates equal nominal interest rates, and are given by

$$r_t = \max\{r_t^n, 0\}$$

Using the definition of net consumption and the fact that $\Delta_t = 1$, we can write the resource constraint

$$c_t^S + c_t^B = 2(h_t - v(h_t)) = 2y_t$$

Note that by definition of y^*, we have $y_t \leq y^*$.

Real wages satisfy $\frac{W_t}{P_t} = v'(h_t)$, where $h^t = h_t^S = h_t^B$ is the number of hours supplied by each household. In an economy with $\alpha = 0$, we would have $\frac{W_t}{P_t} = v'(h_t) = 1$, i.e. $h_t = h^*$ and

$y_t = y^*$. We know that whenever $r_t^n \geq 0$, $r_t = r_t^n$ and $y_t = y^*$. It follows that $r_t(y^* - y_t) = 0$, as required. □

Given our assumption that the central bank follows a Taylor rule, equilibrium in the New Keynesian model would almost be isomorphic to the ZLB-constrained equilibrium even if $\alpha < 1$. Any equilibrium of the New Keynesian model with zero inflation ($\Pi_t = 1, \forall t$) is isomorphic to a ZLB-constrained equilibrium. However, whenever the ZLB binds at date 1, and output falls below potential, there is deflation. This causes relative prices to become dispersed. Under Calvo pricing, this dispersion in relative prices is only eliminated in the limit as $t \to \infty$. For this reason, if nothing else, output is below potential even at date 2, and the economy does not immediately reach steady state: $y_2 < y^*$, $y_t \to y^*$ as $t \to \infty$.

Note also that with $\alpha = 1$ and the quasilinear preferences considered here, this Taylor rule is, trivially, optimal monetary policy. Since prices never adjust, the best that monetary policy can do is to set real interest rates equal to the natural rate, ensuring $y_t = y^*$, whenever this does not violate the zero lower bound. There is no advantage to setting $y_t < y^*$ for any $t \geq 2$: this only makes the ZLB tighter.

B Alternative microfoundations

In this section I present two alternative economies providing a microfoundation for this equilibrium concept. The first draws on the extensive literature on rationing or non-Walrasian equilibria. The second is an economy with downward nominal wage rigidity drawing on Schmitt-Grohé and Uríbe [2011].

B.1 Non-Walrasian equilibrium

I briefly show how the ZLB-constrained equilibrium presented in the main text can be interpreted as a rationing or non-Walrasian equilibria (see e.g. Benassy [1993] for a survey of this extensive literature).

As in the main text, households solve

$$\max U(c_0^i, \theta_i) + \sum_{t=1}^{\infty} \beta^t u(c_t^i) \tag{13}$$

$$\text{s.t. } c_t^i = y_t^i - d_t^i + \frac{d_{t+1}^i}{1 + r_t} \tag{14}$$

$$d_0^i = 0, \forall i \tag{15}$$

$$d_{t+1}^i \leq \phi_t, t = 1, ... \tag{16}$$

In a non-Walrasian equilibrium, prices are usually treated as exogenously fixed. Agents send quantity signals indicating how much they would like to supply and demand. They take as given not only prices, but also perceived quantity constraints operating on various markets. In equilibrium, the quantity of each good actually transacted is equal to the minimum of desired supply and desired demand. Agents' perceived quantity constraints are consistent with the amount actually transacted: the total amount of each good sold is rationed between all sellers according to an exogenously given rationing scheme (and the same goes for buyers).

In this economy, households always inelastically supply their total potential output y^*.[1] They receive quantity signals y_t^S, y_t^B indicating how much they are able to sell in each period; given these signals and the path of interest rates, they choose consumption, and send demand signals c_t^S, c_t^B. Output is the minimum of demand and supply:

$$2y_t = \min\{c_t^S + c_t^B, 2y^*\}$$

I assume a proportional rationing scheme: in equilibrium, the perceived quantity constraints are $y_t^B = y_t^B := y_t$. Finally, I assume that interest rates clear markets whenever this does not violate the ZLB. Formally:

Definition B.1. *A* ***non-Walrasian equilibrium*** *is* $\{c_t^i, d_t^i, y_t, r_t\}$ *such that*

1. *agents maximize* (13) *s.t.* (14), (15), (16)
2. *output is the minimum of demand and supply:*

$$2y_t^B = 2y_t^S = 2y_t = \min\{c_t^S + c_t^B, 2y^*\}$$

3. $r_t \geq 0$. *If* $r_t > 0$, $c_t^S + c_t^B = 2y_t = 2y^*$.

When interest rates can adjust to clear markets, they do, and agents sell all of their endowment. When the ZLB prevents interest rates from falling enough to clear markets, agents sell less than their total endowment, and income y_t is the variable that adjusts to clear markets. Clearly, this definition of equilibrium is isomorphic to the ZLB-constrained equilibrium defined in the main text.

B.2 Economy with rigid wages

I now show how the ZLB-constrained equilibrium can be interpreted as an economy with downward nominal wage rigidity.

Households have preferences as in the main text, but now inelastically supply labor $\bar{h}$. Competitive firms hire labor and produce the consumption good using a linear technology, $y_t = h_t$. Nominal wages are downwardly rigid as in Schmitt-Grohé and Uríbe [2011]:

$$W_t \geq \gamma W_{t-1}, h_t \leq \bar{h}, (\bar{h} - h_t)(W_t - \gamma W_{t-1}) \tag{17}$$

where $\gamma > 0$ indexes the degree of nominal rigidity. When the wage rigidity constraint binds, households are equally rationed in the labor market: $h_t^S = h_t^B = h_t$. Since firms are competitive, they set $P_t = W_t$. We define $\Pi_t = \dfrac{P_t}{P_{t-1}}$. Households solve

$$\max U(c_0^i, \theta_i) + \sum_{t=1}^{\infty} \beta^t u(c_t^i) \tag{18}$$

$$\frac{d_{t+1}^i}{1+i_t} = d_t^i + P_t c_t^i - W_t h_t$$

$$d_0^B = d_0^S = 0$$

$$d_{t+1}^i \leq P_{t+1}\phi_t, t = 1, ...$$

[1] I assume that each borrower (or saver) cannot consume her own output, but can consume the endowment of other borrowers and savers.

The monetary authority sets interest rates according to a Taylor rule, modified to take account of the zero lower bound:

$$1 + i_t = \max\left\{(1 + r_t^n)\Pi_t^{\phi_\pi}, 1\right\} \tag{19}$$

where $\phi_\pi > 1$ and r_t^n is the natural rate of interest, defined as the equilibrium real interest rate in the economy with $\gamma = 0$ (no nominal wage rigidity).

The market clearing condition is

$$c_t^S + c_t^B = 2h_t \tag{20}$$

Definition B.2. *An equilibrium is a collection* $\{c_t^S, c_t^B, d_t^S, d_t^B, h_t, \Pi_t, \frac{W_t}{P_t}, i_t\}$ *such that*

1. $\{C_t^i, h_t^i, d_t^i\}$ *solves household i's problem* (18)*, for* $i = S, B$
2. *Firms maximize profits:* $\frac{W_t}{P_t} = 1$
3. *Wages satisfy* (17)
4. *Interest rates* $\{i_t\}$ *satisfy the modified Taylor rule* (19)
5. *The market clearing condition* (20) *is satisfied.*

Proposition B.3. *When* $\gamma = 1$*, every ZLB-constrained equilibrium is an equilibrium of the rigid wage economy, with* $y^* = \bar{h}$.

Proof. Take any ZLB-constrained equilibrium $\{c_t^S, c_t^B, d_t^S, d_t^B, y_t, r_t\}$. Set $i_t = r_t, h_t = y_t, \Pi_t = 1, \frac{W_t}{P_t} = 1$. Since there is no inflation, (17) is satisfied. Since $i_t = r_t$, either interest rates are zero, or they are at a level which ensures $h_t = \bar{h}$, as in the economy with $\gamma = 0$. So the modified taylor rule (19) is satisfied. Finally, by definition of a ZLB constrained equilibrium, households optimize and markets clear. □

Note that unlike in the New Keynesian economy with $\alpha = 1$, here the Taylor rule is not optimal monetary policy. In the economy with rigid wages, there are no costs associated with inflation, and the monetary authority could costlessly circumvent the ZLB by setting $\Pi_2 = \frac{1}{1 + r_1^n}$.

The remainder of this online appendix presents proofs of Propositions and Lemmas in the main text.

C Proof of Proposition 2.3.

Suppose $d_1^B \leq \phi$. I claim that the liquidity constraint never binds. In this case, equilibrium must satisfy each agent's Euler equations and market clearing:

$$u'(c_t^i) = \beta(1 + r_t)u'(c_{t+1}^i),\ i = S, B, t = 1, 2, ...$$
$$c_t^S + c_t^B = 2y^*, t = 1, 2, ...$$

The proposed allocation satisfies all these constraints, and has $d_t^i = d_1^i \leq \phi, \forall t > 1$, so the liquidity constraint is indeed slack.

Even if $d_1^B > \phi$, then since $d_2^B \leq \phi$, an identical argument shows that there exists an equilibrium in which the economy enters steady state at date 2. Clearly, borrowers must be constrained at date 1: if not, we know they would attempt to keep debt constant, $d_2^B = d_1^B$, which violates the liquidity constraint since $d_1^B > \phi$. Substituting the binding liquidity constraint into borrowers' budget constraint, we have

$$c_1^B = y^* - d_1^B + \frac{\phi}{1+r_1}$$

Market clearing means that

$$c_1^S = y^* + d_1^B - \frac{\phi}{1+r_1}, c_2^S = y^* + \frac{r^*}{1+r^*}\phi$$

Since savers are unconstrained, their Euler equation must hold with equality:

$$u'(c_1^S) = \beta(1+r_1)u'(c_2^S)$$
$$u'\left(y^* + d_1^B - \frac{\phi}{1+r_1}\right) = \beta(1+r_1)u'\left(y^* + (1-\beta)\phi\right)$$

This implicitly defines r_1, as claimed. So we are done.

D Proof of Proposition 3.2.

Combining borrowers' and savers' date 0 Euler equations, we have

$$\frac{U_c(c_0^B, \theta_B)}{u'(c_1^B)} = \frac{U_c(c_0^S, \theta_S)}{u'(c_1^S)}$$

As $\theta_B \to \infty$, $c_1^B \to 0$, and Assumption 3.1 guarantees that the ZLB binds.

E Proof of Proposition 3.7.

Part 3. is immediate: if the ZLB does not bind in equilibrium, the transfer required to restore full employment is zero, which is (trivially) incentive compatible.

To prove parts 1 and 2, I show that if the transfer is not incentive compatible given θ_B, it is not incentive compatible given $\theta_B' > \theta_B$. It is sufficient to show that $c_0^B(\theta_B') > c_0^B(\theta_B)$. Suppose not, and $c_0^S(\theta_B') > c_0^S(\theta_B)$; then $r_0(\theta_B') < r_0(\theta_B)$, since we know the ZLB binds in both regimes.

$$1 + r_0 = \frac{U_c(c_0^S, \theta_S)}{\beta u'(\bar{c}_1^S)}$$

Consequently, $d_1(\theta_B') = (1 + r_0(\theta_B'))(y^* - c_0^S(\theta_B')) < (1 + r_0(\theta_B))(y^* - c_0^S(\theta_B)) = d_1(\theta_B)$. In a ZLB-constrained equilibrium, $c_1^B = \bar{c}_1^S + 2\phi - 2d_1$, so $c_1^B(\theta_B') > c_1^B(\theta_B)$. But this contradicts the equilibrium condition

$$\frac{U_c(c_0^B, \theta_B)}{u'(c_1^B)} = \frac{U_c(c_0^S, \theta_S)}{u'(c_1^S)}$$

So c_0^B is increasing in θ_B.

S's gain from mimicking B, given that B receives a transfer which restores full employment and gives him consumption $2y^* - \bar{c}_1^S$, is

$$U(c_0^B, \theta_S) - U(2y^* - c_0^B, \theta_S) + \beta[u(2y^* - \bar{c}_1^S) - u(\bar{c}_1^S)] + \frac{\beta^2}{1-\beta}[u(y^* - (1-\beta)\phi) - u(y^* + (1-\beta)\phi)]$$

which is increasing in c_0^B. It follows that if the transfer is not incentive compatible given θ_B, it is not incentive compatible given $\theta_B' > \theta_B$.

Finally,

$$\lim_{\beta \to 1} \frac{u(y^* - (1-\beta)\phi) - u(y^* + (1-\beta)\phi)}{1-\beta} = -2u'(y^*)\phi$$

so the last two terms in the above expression go to zero as $\beta \to 1, \phi \to 0$. By continuity, part 4 of the Proposition follows.5

F Proof of Proposition 4.1.

The Pareto problem is

$$\max \alpha \left\{ U(c_0^S, \theta_S) + \beta u(c_1^S) + \frac{\beta^2}{1-\beta} u(c_2^S) \right\} + (1-\alpha) \left\{ U(c_0^B, \theta_B) + \beta u(c_1^B) + \frac{\beta^2}{1-\beta} u(c_2^B) \right\} \quad (21)$$

$$\text{s.t. } c_0^S + c_0^B \leq 2y^* \quad \text{(RC0)}$$

$$c_1^S + c_1^B \leq 2y^* \quad \text{(RC1)}$$

$$c_2^S + c_2^B = 2y^* \quad \text{(RC2)}$$

$$c_2^B \geq y^* - (1-\beta)\phi \quad \text{(BC)}$$

$$u'(c_1^S) \geq \beta u'(c_2^S) \quad \text{(ZLB)}$$

$$U(c_0^S, \theta_S) + \beta u(c_1^S) + \frac{\beta^2}{1-\beta} u(c_2^S) \geq U(c_0^B, \theta_S) + \beta u(c_1^B) + \frac{\beta^2}{1-\beta} u(c_2^B) \quad \text{(ICS)}$$

$$U(c_0^B, \theta_B) + \beta u(c_1^B) + \frac{\beta^2}{1-\beta} u(c_2^B) \geq U(c_0^S, \theta_B) + \beta u(c_1^S) + \frac{\beta^2}{1-\beta} u(c_2^S) \quad \text{(ICB)}$$

Lemma F.1. (RC2) *binds.*

Proof. Suppose not: consider the following deviation. Increase both c_2^S and c_2^B, keeping $u(c_2^S) - u(c_2^B)$ fixed; this satisfies all constraints, and increases utility, a contradiction. A corollary is that $c_2^S \geq c_2^B$. □

Lemma F.2. *$u'(c_1^i) > \beta u'(c_2^i)$ for at least one agent.*

Proof. If not, then $c_1^i > c_2^i$ for $i = S, B$; summing, we have $c_1^S + c_1^B > c_2^S + c_2^B = 2y^*$ (by the above result), which is infeasible. □

Lemma F.3. *If* (ZLB) *binds,* (BC) *binds.*

Proof. Suppose by contradiction that (ZLB) binds but (BC) does not. Consider the following deviation: increase c_2^S by $\varepsilon > 0$ and reduce c_2^B by the same amount, and increase c_1^B by δ and reduce c_2^B by the same amount. This deviation is feasible. Choose ε and δ so that

$$u(c_1^S-\delta)+\frac{\beta}{1-\beta}u(c_2^S+\varepsilon)-[u(c_1^B+\delta)+\frac{\beta}{1-\beta}u(c_2^S-\varepsilon)]=u(c_1^S)+\frac{\beta}{1-\beta}u(c_2^S)-[u(c_1^B)+\frac{\beta}{1-\beta}u(c_2^S)]$$

By the Implicit Function Theorem, this defines δ as an increasing function of ε in the neighborhood of $(\delta,\varepsilon)=(0,0)$. To first order, we have

$$\delta\approx\frac{\beta}{1-\beta}\frac{u'(c_2^S)+u'(c_2^B)}{u'(c_1^S)+u'(c_1^B)}$$

To first order, the effect on S's utility is

$$\frac{\beta}{1-\beta}u'(c_1^S)\left[\frac{u'(c_2^S)}{u'(c_1^S)}-\frac{u'(c_2^S)+u'(c_2^B)}{u'(c_1^S)+u'(c_1^B)}\right]$$

By assumption, $\frac{u'(c_2^S)}{u'(c_1^S)}=\frac{1}{\beta}$, which implies that $\frac{u'(c_2^B)}{u'(c_1^B)}<\frac{1}{\beta}$. Thus the change in S's utility is positive. Since by construction the difference between S's utility and B's utility is unchanged, B's utility also increases. Thus the deviation yields a strictly higher value of the objective function, which contradicts the original allocation being optimal. □

Lemma F.4. $c_1^B\leq c_1^S$.

Proof. Suppose by contradiction that $c_1^B>c_1^S$. Then $c_1^S<y^*$; since we know that $c_2^S\geq y^*$, (ZLB) cannot bind, and the following deviation is feasible.. Increase c_1^S by δ, decreasing c_1^B by the same amount, and increase c_1^B by ε, decreasing c_1^S by the same amount. Choose ε and δ as before. Again, this defines δ as an increasing function of ε in the neighborhood of $(\delta,\varepsilon)=(0,0)$. To first order, the effect on S's utility is

$$\frac{\beta}{1-\beta}u'(c_1^S)\left[\frac{u'(c_2^S)+u'(c_2^B)}{u'(c_1^S)+u'(c_1^B)}-\frac{u'(c_2^S)}{u'(c_1^S)}\right]$$

Since $c_1^S<c_2^S$ and $c_1^B>c_2^B$, this expression is positive. So utility increases for both S and B, which contradicts the original allocation being optimal. □

Lemma F.5. *At most one incentive constraint binds.*

Proof. If (by contradiction) (ICS) and (ICB) both hold with equality at an optimum, then subtracting one constraint from the other, we have

$$U(c_0^S,\theta_S)-U(c_0^S,\theta_B)=U(c_0^B,\theta_S)-U(c_0^B,\theta_B)$$

Since $U_{c\theta}>0$, this implies $c_0^S=c_0^B$. Since $c_1^B\leq c_1^S, c_2^B\leq c_2^S$, we must have $c_1^S=c_1^B$ and $c_2^S=c_2^B$ (otherwise, B would clearly prefer S's allocation). Thus (ZLB) is slack. To show that this allocation is not optimal, consider the following deviation: increase c_0^B by $\varepsilon>0$, decreasing c_0^S by the same amount, and increase c_1^S by $\delta>0$, decreasing c_1^B by the same amount. Choose

$$\frac{U_c(c_0^S,\theta_S)}{\beta u'(c_1^S)}<\frac{\delta}{\varepsilon}<\frac{U_c(c_0^B,\theta_B)}{\beta u'(c_1^B)}$$

This deviation increases utility for both agents, and is feasible, because it relaxes both incentive compatibility constraints. This contradicts the assumption that the original allocation was optimal. □

Lemma F.6. (RC0) *binds.*

Proof. Forming the Lagrangian, the first order necessary conditions for a maximum are

$$\begin{aligned}\alpha U_c(c_0^S,\theta_S) - \lambda_0 + \mu_S U_c(c_0^S,\theta_S) - \mu_B U_c(c_0^S,\theta_B) &= 0\\ (1-\alpha)U_c(c_0^B,\theta_B) - \lambda_0 - \mu_S U_c(c_0^B,\theta_S) + \mu_B U_c(c_0^B,\theta_B) &= 0\\ \alpha u'(c_1^S) - \lambda_1 + \zeta u''(c_1^S) + (\mu_S - \mu_B)u'(c_1^S) &= 0\\ (1-\alpha)u'(c_1^B) - \lambda_1 - (\mu_S - \mu_B)u'(c_1^B) &= 0\\ \alpha u'(c_2^S) - \lambda_2 - (1-\beta)\zeta u''(c_2^S) + (\mu_S - \mu_B)u'(c_2^S) &= 0\\ (1-\alpha)u'(c_2^B) - \lambda_2 + \psi - (\mu_S - \mu_B)u'(c_2^B) &= 0\end{aligned}$$

where $\lambda_0, \beta\lambda_1, \frac{\beta^2}{1-\beta}\lambda_2, \psi, \beta\zeta, \mu_S, \mu_B$ are the multipliers on (RC0), (RC1), (RC2), (BC), (ZLB), (ICS), (ICB) respectively.

Since at most one incentive constraint binds, $\mu_S, \mu_B \geq 0$, with at least one equality. It follows that either $\alpha U_c(c_0^S,\theta_S) - \lambda_0 \geq 0$, or $(1-\alpha)U_c(c_0^B,\theta_B) - \lambda_0 \geq 0$, or both. Since $U_c > 0$, this implies $\lambda_0 > 0$. Thus (RC0) binds. □

Lemma F.7. *If* (RC1) *is slack,* (ICS) *and* (ZLB) *both bind.*

Proof. If(RC1) is slack, then $\lambda_1 = 0$, and

$$\begin{aligned}\alpha u'(c_1^S) + \zeta u''(c_1^S) + (\mu_S - \mu_B)u'(c_1^S) &= 0\\ (1-\alpha)u'(c_1^B) - (\mu_S - \mu_B)u'(c_1^S) &= 0\end{aligned}$$

From the second equation, we must have $\mu_S > 0$, so $\mu_B = 0$. From the first equation, since $u''(c_1^S) < 0$, we must have $\zeta > 0$. Thus (ICS) and (ZLB) bind. □

Supose the incentive constraints do not bind. Define the functions $s_t(\alpha), t = 0,1,2$ to solve

$$\begin{aligned}\alpha U_c(s_0(\alpha),\theta_S) &= (1-\alpha)U_c(2y^* - s_0(\alpha),\theta_B)\\ \alpha u'(s_t(\alpha)) &= (1-\alpha)u'(2y^* - s_t(\alpha)), t = 1,2\end{aligned}$$

It is straighforward to show that in a relaxed problem without incentive constraints,

$$\begin{aligned}c_0^S &= 2y^* - c_0^B = g_0(\alpha)\\ c_1^S &= 2y^* - c_1^B = \min\{g_1(\alpha), \bar{c}_1^S\}\\ c_2^S &= 2y^* - c_2^B = \min\{g_2(\alpha), \bar{c}_2^S\}\end{aligned}$$

where $\bar{c}_2^S = y^* + (1-\beta)\phi$, $u'(\bar{c}_1^S) = \beta u'(\bar{c}_2^S)$. This defines c_0^S, c_1^S, c_2^S as increasing functions of α, with c_0^S strictly increasing. S's gain from mimicking B is therefore a decreasing function of α. As $\alpha \to 0$, $U_c(c_0^S,\theta_S) \to \infty$ and $U(c_0^S,\theta_S) \to -\infty$. So there exists $\alpha_S > 0$ such that (ICS) just holds and (ICB) is slack. For all $\alpha < \alpha_S$, this allocation would violate (ICS) but would satisfy (ICB). An identical argument shows that there exists $\alpha_B < 1$ such that (ICB) just holds and (ICS) is slack.

G Proof of Proposition 4.2.

Lemma G.1. *Define the date 1 value function*

$$V(a_1^i) = \max_{\{c_t^i, d_{t+1}^i\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} u(c_t^i) \tag{22}$$

$$s.t. c_1^i = y_1 + a_1^i + \frac{d_2^i}{1+r_1} \tag{23}$$

$$c_t^i = y_t - d_t^i + \frac{d_{t+1}^i}{1+r_t}, t \geq 2 \tag{24}$$

$$d_t^i \leq \phi, t \geq 2 \tag{25}$$

$\{c_t^i, d_{t+1}^i\}_{t=0}^{\infty}$ solves i's problem, given $\{y_t, r_t\}$ and $T(\cdot)$, if and only if:

1. *c_0^i, d_1^i solve*

$$\max_{c_0^i, d_1^i} U(c_0^i, \theta_i) + \beta V(T(d_1^i) - d_1^i)$$

$$s.t.\ c_0^i = y_0 + \frac{d_1^i}{1+r_0}$$

2. *$\{c_t^i, d_{t+1}^i\}_{t=1}^{\infty}$ solve (22), given $a_1^i = T(d_1^i) - d_1^i$.*

Proof. The proof is standard, and is therefore omitted. □

Lemma G.2. *In any equilibrium with transfers, for all $t \geq 2$ and for all i, $r_t = r^* = \beta^{-1} - 1$, $d_t^i = d_2^i$, $c_t^i = c_2^i = y^* - (1-\beta) d_2^i$.*

Proof. First, suppose that households solve a relaxed problem in which $\phi_t = \infty$ for all $t \geq 3$. In this case, household first order conditions yield

$$u'(c_t^i) = \beta(1+r_t) u'(c_{t+1}^i) \phi \text{ for all } t \geq 2$$

I will show that the borrowing constraint does not bind, so households are indeed liquidity unconstrained after date 2.

It is straightforward to see that if $r_t = r^*, \forall t \geq 2$, the proposed allocation uniquely satisfies these first order conditions. Suppose by contradiction that there is also an equilibrium with $r_t > r^*$ for some $t \geq 2$. Then for each household i, $c_t^i < c_{t+1}^i$. Integrating, we have $y_t = \int c_t^i \, \mathrm{d}i < \int c_{t+1}^i \, \mathrm{d}i = y^*$. So $y_t < y^*$, which implies $r_t = 0$ by the definition of ZLB-constrained equilibrium, a contradiction.

Suppose by contradiction that $r_t < r^*$. Then a similar argument implies that $y_{t+1} = \int c_{t+1}^i \, \mathrm{d}i < y^*$ and $r_{t+1} = 0$. Iterating forward, we see that we must have $r_{t+s} = 0, y_{t+s} < y^*$ for all $s \geq 1$. This deflationary equilibrium is clearly Pareto inferior to an equilibrium with $y_t = y^*$, so we can rule this equilibrium out when considering optimal policy.[2]

From the budget constraints, it follows that $c_t^i = c_2^i = y^* - (1-\beta) d_2^i$, $d_{t+1}^i = d_t^i$, for all $t \geq 2$. Since $d_2^i \leq \phi$, households' unconstrained borrowing decisions happen to satisfy the borrowing constraint, as claimed. □

[2]Equivalently, we could append to our definition of equilibrium the condition that $\lim_{t\to\infty} y_t = y^*$.

Lemma G.3. *In the two-type economy, $\{c^i_t\}$ can be implemented as an equilibrium with transfers if and only if there exists r_1 such that*

$$c^S_0 + c^B_0 \leq 2y^* \tag{26}$$

$$r_1 \geq 0, c^S_1 + c^B_1 \leq 2y^*, \text{ with at least one equality} \tag{27}$$

$$c^S_2 + c^B_2 = 2y^* \tag{28}$$

$$u'(c^i_1) \geq \beta(1+r_1)u'(c^i_2),\ c^i_2 \geq y^* - (1-\beta), \text{ with at least one equality, } i = S, B \tag{29}$$

$$U(c^S_0, \theta_S) + \beta u(c^S_1) + \frac{\beta^2}{1-\beta}u(c^S_2) \geq U(c^B_0, \theta_S) + \beta u(c^B_1) + \frac{\beta^2}{1-\beta}u(c^B_2) \tag{30}$$

$$U(c^B_0, \theta_B) + \beta u(c^B_1) + \frac{\beta^2}{1-\beta}u(c^B_2) \geq U(c^S_0, \theta_B) + \beta u(c^S_1) + \frac{\beta^2}{1-\beta}u(c^S_2) \tag{31}$$

Proof. First I show that these conditions are necessary for implementability. Suppose $\{c^i_t, d^i_{t+1}, r_t, y_t\}$ is an equilibrium with transfers, given some policy $T(\cdot)$. (26) and (28) are satisfied by definition. By Lemma G.2, the economy enters a steady state at date 2 with full employment, thus (27) is satisfied. (29) describes necessary conditions for optimality in the household problem. Finally, the incentive compatibility constraints (30), (31) follow from a standard mimicking argument.S's allocation, $\boldsymbol{c^S}$, is feasible for B. If $\boldsymbol{c^B}$ is optimal for B, it must give at least as much utility to B as he would get from $\boldsymbol{c^S}$, which is also feasible. The same argument applies for S.

Next, I show that conditions (26)-(31) are sufficient for implementability. Let $\{c^i_t\}, r_1$ satisfy these conditions. Set $2y_t = c^S_t + c^B_t$ for all t and set $r_t = r^*$ for all $t \geq 2$. Set $d^i_t = \frac{y^* - c^i_2}{1-\beta}, \forall t \geq 2$. If $y_t < y^*$, set $r_0 = 0$, otherwise choose any $r_0 \geq 0$.

It is clear that all equilibrium conditions are satisfied, except, possibly, the condition that for $i = S, B$, c^i_0, d^i_1 solve (22). Let $U^i = U(c^i_0, \theta_i) + \beta u(c^i_1) + \frac{\beta^2}{1-\beta}u(c^i_2)$ be the utility that each agent gets from her allocation. Define $a^i_1 = c^i_1 - y_1 - \frac{d^i_2}{1+r_1}$. For each $i = S, B$, define the set

$$\mathcal{V}^i = \{(c, a) \in \mathbb{R}^2 : U(c, \theta_i) + \beta V(a) \leq U^i\}$$

By construction, $\mathcal{V}^i$ is a closed set and c^i_0, a^i_1 is contained in its boundary. Let

$$\mathcal{V} = \mathcal{V}^S \cap \mathcal{V}^B = \{(c, a) \in \mathbb{R}^2 : U(c, \theta_i) + \beta V(a) \leq U^i, i = S, B\}$$

be the set of allocations which both agents find weakly inferior to their equilibrium allocations. By (30) and (31), the boundary of $\mathcal{V}$ contains c^S_0, a^S_1 and c^B_0, a^B_1. To implement the desired equilibrium, we can offer households any subset of $\mathcal{V}$ which contains both their equilibrium allocations. Let $a(c)$ be any function satisfying

$$(c, a(c)) \in \mathcal{V}, \forall x$$
$$a^i_1 = a(c^i_0), i = S, B$$

It is immediate that

$$c^i_0 \in \arg\max_c U(c, \theta_i) + \beta V(a(c))$$

Define $T(d) = d + a\left(y_0 + \frac{d_1^i}{1+r_0}\right)$. We have immediately that

$$c_0^i, d_1^i \in \arg\max_{c,a} U(c, \theta_i) + \beta V(T(d) - d)$$
$$\text{s.t. } c_0^i = y_0 + \frac{d_1^i}{1+r_0}$$

Since it is clear that these transfer functions satisfy the government budget constraint, we are done. □

Lemma G.4. *In any implementable allocation, $c_0^B \geq c_0^S$, $c_t^B \leq c_t^S$ for $t \geq 1$, and S is unconstrained at date 1.*

Proof. Combining the incentive constraints,

$$U(c_0^B, \theta_B) - U(c_0^B, \theta_S) \geq U(c_0^S, \theta_B) - U(c_0^S, \theta_S)$$

Since $U_{c\theta} > 0$ and $\theta_B > \theta_S$, this implies $c_0^B \geq c_0^S$.

If no agent is constrained at date 1, then $c_1^i = c_t^i$ for $i = S, B, t \geq 1$, and we cannot have $c_1^B > c_1^S$: otherwise S would strictly prefer B's allocation. Suppose B is constrained at date 1. Then by Lemma (G.2),

$$c_2^B = y^* - (1-\beta)\phi \leq y^* + (1-\beta)\phi = c_2^S$$

Since B is constrained and S is not,

$$\frac{u'(c_1^B)}{u'(c_2^B)} > \frac{u'(c_1^S)}{u'(c_2^S)} = \beta(1+r_1)$$

I claim that $\beta(1+r_1) < 1$. If not, then $\frac{u'(c_1^B)}{u'(c_2^B)} > 1, \frac{u'(c_1^S)}{u'(c_2^S)} \geq 1$, which implies $c_2^B > c_1^B$, $c_2^S \geq c_1^S$. Summing, we have $y_1 < y_2 \leq y^*$. But this is a contradiction, since $r_1 > 0$ and we must have $y_1 = y^*$.

Since $\beta(1+r_1) < 1$, it follows that $c_1^S > c_2^S \geq y^*$. Since $c_1^S + c_1^B \leq y^*$, we have immediately that $c_1^B < c_1^S$.

Finally, suppose (by contradiction) that S is constrained at date 1. An identical argument shows that $c_1^S < c_1^B$, $c_t^S \leq c_t^B$ for $t \geq 2$. Since $c_0^B \geq c_0^S$, this is a contradiction, because S would strictly prefer B's allocation. □

Corollary G.5. *In the two-type economy, $\{c_t^i\}$ can be implemented as an equilibrium with transfers if and only if* (26), (28), (30), (ICB) *are satisfied, together with*

$$u'(c_1^S) \geq \beta u'(c_2^S), c_1^S + c_1^B \leq 2y^*, \text{ with at least one equality} \tag{32}$$

$$\frac{u'(c_1^B)}{\beta u'(c_2^B)} \geq \frac{u'(c_1^S)}{\beta u'(c_2^S)}, c_2^B \geq y^* - (1-\beta)\phi, \text{ with at least one equality} \tag{33}$$

Proof. Since S is always unconstrained, $r_1 = \frac{u'(c_1^S)}{\beta u'(c_2^S)}$, and (32) is equivalent to (27). For the same reason, (33) is equivalent to (29) holding for B. Clearly if the allocation is implementable, then S is unconstrained and $c_2^S \geq c_2^B \geq y^* - (1-\beta)\phi$. Take any allocation satisfying the equations: it remains to show that (29) holds for S. By construction, the Euler equation inequality is satisfied, so it is only necessary to show that $c_2^S \geq y^* - (1-\beta)\phi$. Suppose not: then $c_2^B > y^* + (1-\beta)\phi$, and so $\frac{u'(c_1^B)}{\beta u'(c_2^B)} = \frac{u'(c_1^S)}{\beta u'(c_2^S)}$ by (33). Thus $c_2^B > c_2^S$ implies $c_1^B > c_1^S$. But incentive compatibility implies that c_0^B, c_0^S, so the allocation cannot satisfy (30), a contradiction. □

I now show that we can neglect the first and last parts of (33), and the complementary slackness condition in (32), in the Pareto problem.

Lemma G.6. *Suppose $\{c_t^i\}$ solves the relaxed Pareto problem (21). Then it satisfies (33).*

Proof. First, I show that we can never have a solution to (21) with $\frac{u'(c_1^B)}{\beta u'(c_2^B)} < \frac{u'(c_1^B)}{\beta u'(c_2^B)}$. Suppose by contradiction that we have a solution in which this inequality holds. Then by Lemma (G.4), (ZLB) cannot hold with equality, and $\zeta = 0$ in the first order conditions for a maximum. Combining these conditions, we have

$$\frac{u'(c_1^B)}{u'(c_2^B)} = \frac{\lambda_1}{\lambda_2 - \psi} \geq \frac{\lambda_1}{\lambda_2} = \frac{u'(c_1^S)}{u'(c_2^S)},$$

a contradiction.

Next, I show that we can never have $\frac{u'(c_1^B)}{\beta u'(c_2^B)} > \frac{u'(c_1^B)}{\beta u'(c_2^B)}$ and $c_2^B > y^* - (1-\beta)\phi$. If we had such a solution, then by Lemma (F.3), (BC) and (ZLB) will both be slack, and the first order conditions imply $\frac{u'(c_1^S)}{u'(c_2^S)} = \frac{u'(c_1^B)}{u'(c_2^B)}$.

Finally, we know by Lemma (F.7) that if (RC1) is slack, (ZLB) must bind, thus the complementary slackness condition in (32) is satisfied. □

Proposition 4.2 follows.

H Proof of Proposition 4.4.

Lemma H.1. *Suppose $\{c_t^i\}$ solves (21). Define $a_1^i = c_1^i - y_1 - \frac{d_2^i}{1+r_1}$.*

Take any transfer function T and interest rate $r_0 \geq 0$. Define the associated net wealth function

$$a(c) := T((1+r_0)(c-y^*)) - (1+r_0)(c-y^*)$$

Sufficient conditions for T, r_0 to implement $\{c_t^i\}$ are that:

1. *$a(c_0^i) = c_1^i - y_1 + \frac{c_2^i - y^*}{(1+r_1)(1-\beta)}$ for $i = S, B$, and*
2. *for all c,*

$$(c, a(c)) \in \mathcal{V} = \mathcal{V}^S \cap \mathcal{V}^B = \{(c,a) \in \mathbb{R}^2 : U(c,\theta_i) + \beta V(a) \leq U^i, i = S, B\}$$

Proof. Suppose $\{c_t^i\}$ solves (21). Then by Proposition 4.2, for some transfer function $T^*(\cdot)$ and some $\{r_t^*, y_t, d_t^i\}$, $T^*, \{c_t^i, d_{t+1}^i, r_t^*, y_t\}$ is an equilibrium with transfers. Let T, r_0 satisfy the conditions in the Lemma. I will show that if we replace T^* with T and replace r_0^* with r_0, keeping all other variables the same, we have an equilibrium with transfers.

If the conditions in the Lemma are satisfied, then for each i,

$$c_0^i \in \arg\max_c U(c,\theta_i) + \beta V(a(c))$$

In other words,

$$c_0^i, d_1^i \in \arg\max_{c,a} U(c,\theta_i) + \beta V(T(d) - d)$$
$$\text{s.t. } c_0^i = y_0 + \frac{d_1^i}{1+r_0}$$

Defining $d_1^i = (1+r_0)(c_0^i - y^*)$, we have

$$\begin{aligned}\sum_{i=S,B} T(d_1^i) &= \sum_{i=S,B} a_1^i + \sum_{i=S,B} d_1^i \\ &= \sum_{i=S,B}\left(c_1^i - y_1 + \frac{c_2^i - y^*}{(1+r_1)(1-\beta)}\right) + \sum_{i=S,B}(1+r_0)(c_0^i - y^*) \\ &= 0\end{aligned}$$

So the government budget constraint is satisfied. The remaining conditions are satisfied by assumption. □

Lemma H.2. *Let (c,a), (c',a') be two allocations with $c' > c$. If $U(c,\theta_S) + \beta V(a) \geq U(c',\theta_S) + \beta V(a')$, then $U(c,\theta_B) + \beta V(a) > U(c',\theta_B) + \beta V(a')$.*

Proof. This is immediate, since $U_{c\theta} > 0$ and $\theta_B > \theta_S$. □

Lemma H.3. *If* (ICS) *binds, the solution to* (21) *can be implemented with a debt relief transfer function.*

Proof. The $a(c)$ function associated with a debt relief transfer function has the form

$$\begin{aligned}a(c) &= (1+r_0)(c-y^*) - \bar{T} \text{ if } c \leq \underline{c} \\ &= (1+r_0)(\underline{c} - y^*) - \bar{T} \text{ if } c \in [\underline{c}, \bar{c}] \\ &= (1+r_0)(\underline{c} - y^*) - \bar{T} - (1+\tau)(1+r_0)(c-\bar{c}) \text{ if } c > \bar{c}\end{aligned}$$

for some $\bar{T} > 0$, $\bar{c} > \underline{c}$.

Let $\{c_t^i\}$ be a solution to (21) in which (ICS) binds. Set $y_1 = \frac{1}{2}(c_1^S + c_1^B)$, $1 + r_1 = \frac{u'(c_1^S)}{\beta u'(c_2^S)}$ and

$$\begin{aligned}\bar{c} &= c_0^B \\ r_0 &= \frac{U_c(c_0^S,\theta_S)}{\beta u'(c_1^S)} - 1 \\ \tau &= \frac{U_c(c_0^B,\theta_B)}{\beta(1+r_0)u'(c_1^B)} - 1 \\ \bar{T} &= (1+r_0)(y^* - c_0^S) + y_1 - c_1^S + \frac{y^* - c_2^S}{(1+r_1)(1-\beta)}\end{aligned}$$

Clearly neither agent will ever choose $c \in (\underline{c}, \bar{c})$. S prefers c_0^S to any other point $c \leq \underline{c}$, since the budget set is linear in this range and the objective function is concave. By the same arguments, B

prefers c_0^B to any other $c \geq \bar{c}$. Since S is indifferent between c_0^S and c_0^B, S prefers c_0^B (and therefore c_0^S) to any $c > \bar{c}$, by Lemma H.2. Since S is indifferent between these points, B strictly prefers c_0^B to c_0^S, and therefore to any $c < \underline{c}$.

□

The following assumption is sufficient to ensure that the competitive equilibrium is unique.

Assumption H.4. *$-\dfrac{u''(c)}{u'(c)}$ is nonincreasing in c. If $c_0^B > c_0^S$, then $-\dfrac{U_{cc}(c_0^B, \theta_B)}{U_c(c_0^B, \theta_B)} < -\dfrac{U_{cc}(c_0^S, \theta_S)}{U_c(c_0^S, \theta_S)}$.*

Lemma H.5. *$R(\alpha) := \dfrac{U_c(c_0^S(\alpha), \theta_S)}{\beta u'(c_1^S)}$ is decreasing in α on $[\alpha_S, \alpha_B]$. $T(\alpha) := R(\alpha)(y^* - c_0^S(\alpha)) - a_1^S(\alpha)$ is decreasing in α on $[\alpha_S, \alpha_B]$. $T(\alpha_S) > 0 > T(\alpha_B)$. There exists $\bar{\alpha} \in (\alpha_S, \alpha_B)$ such that $T(\bar{\alpha}) = 0$.*

Proof. I will show that $r = \ln R$ is decreasing in $a = \ln \alpha - \ln(1 - \alpha)$. $r(a)$ is defined by

$$r(a) = \ln U_c(c_0^S(a), \theta_S) - \ln u'(c_1^S(a))$$

c_0^S, c_1^S are defined by

$$\begin{aligned} a + \ln U_c(c_0^S(a), \theta_S) &= \ln U_c(2y * - c_0^S(a), \theta_B) \\ a + \ln u'(g_1(a)) &= \ln u'(2y^* - g_1(a)) \\ c_1^S(a) &= \min\{g_1(a), \bar{c}_1^S\} \end{aligned}$$

Define $\Gamma(c, \theta) = -\dfrac{U_{cc}(c, \theta)}{U_c(c, \theta)}$ and $\gamma(c) = -\dfrac{u''(c)}{u'(c)}$.

$$r'(a) = \frac{\gamma(c_1^S)}{\gamma(c_1^S) + \gamma(c_1^B)} \mathbb{1}(g_1(a) < \bar{c}_1^S) - \frac{\Gamma(c_0^S, \theta_S)}{\Gamma(c_0^S, \theta_S) + \Gamma(c_0^B, \theta_B)}$$

Under Assumption H.4, $r'(a) \leq 0$, and R is decreasing in α. Since a_1^S and c_1^S are increasing in α, and $c_0^S < y^*$, $T(\alpha)$ is increasing in α.

When $\alpha = \alpha_S$, $U(c_0^S, \theta_S) + \beta V(a_1^S) = U(c_0^B, \theta_S) + \beta V(a_1^B)$. Since these functions are concave,

$$\begin{aligned} U_c(c_0^S, \theta_S)(c_0^B - c_0^S) + \beta V'(a_1^S)(a_1^B - a_1^S) &> 0 \\ \frac{U_c(c_0^S, \theta_S)}{\beta u'(c_1^S)}(y^* - c_0^S) - a_1^S &> 0 \\ T(\alpha_S) &> 0 \end{aligned}$$

An analogous argument establishes that $T(\alpha_B) < 0$. Finally, since T is clearly continuous, there exists $\bar{\alpha}$ such that $T(\alpha) = 0$. □

Lemma H.6. *If neither incentive constraint binds and $T(\alpha) > 0$, the solution to (21) can be implemented with a debt relief transfer function.*

Proof. The proof proceeds exactly as for Lemma H.3, noting that $\bar{T} = T > 0$. □

This concludes the proof of Proposition 4.4. The proof of Proposition 4.6 is essentially identical, and is therefore omitted.

I Proof of Proposition 4.7.

Suppose the ZLB does not bind in competitive equilibrium. Then we have full employment in all periods and

$$\frac{U_c(c_0^S,\theta_S)}{\beta u'(c_1^S)} = \frac{U_c(c_0^B,\theta_B)}{\beta u'(c_1^B)} = 1 + r_0$$
$$u'(c_1^S) = \beta(1+r_1)u'(c_2^S)$$
$$u'(c_1^B) \geq \beta(1+r_1)u'(c_2^B)$$

Choose α so that $\frac{\alpha}{1-\alpha} = \frac{U_c(c_0^B,\theta_B)}{U_c(c_0^S,\theta_S)}$. It follows that

$$\alpha U_c(c_0^S,\theta_S) = (1-\alpha)U_c(c_0^B,\theta_B)$$
$$\alpha u'(c_1^S) = (1-\alpha)u'(c_1^B)$$
$$\alpha u'(c_1^S) = (1-\alpha)u'(c_1^B) + \psi$$

for some $\psi \geq 0$. So the allocation satisfies the first order sufficient conditions in (21), and is Pareto optimal.

If $\theta_B > \theta^{ZLB}$, we know the ZLB binds and there is underemployment in the non-Walrasian equilibrium. We also know that neither incentive constraint binds in the non-Walrasian equilibrium allocation. Each agent has strictly concave preferences, and strictly prefers her chosen allocation to any other allocation in the budget set. In particular, S strictly prefers $\boldsymbol{c}^S$ to $\boldsymbol{c}^B$. We know from Proposition 4.1 that underemployment can only be optimal if (ICS) binds. So this allocation cannot be Pareto optimal.

To show that debt relief is Pareto improving, consider the following deviation. Increase c_1^B until either the resource constraint binds at date 1, $c_1^B = \bar{c}_1^B$, or (ICS) binds. In the first case, this leads to a Pareto optimal allocation, because any full employment, incentive compatible allocation with $c_1^S = \bar{c}_1^S$ is Pareto optimal. We clearly have $T(\alpha) > 0$, so the allocation can be implemented with debt relief. In the second case, (ICS) binds, so the allocation can be implemented with debt relief.

J Proof of Proposition 4.9.

The borrower-optimal allocation solves

$$\max_{c_0^S,c_1^S,c_2^S,c_0^B,c_1^B,c_2^B} U(c_0^B,\theta_B) + \beta u(c_1^B) + \frac{\beta^2}{1-\beta}u(c_2^B) \tag{34}$$

$$\text{s.t. } U(c_0^S,\theta_S) + \beta u(c_1^S) + \frac{\beta^2}{1-\beta}u(c_2^S) \geq \bar{U}(\theta_S,\theta_B,\phi) \tag{US}$$

(RC0), (RC1), (RC2), (BC), (ZLB), (ICS)

1. Obvious, since the equilibrium is constrained efficient.

2. The allocation is clearly feasible: by construction it satisfies (RC1), by assumption (ICS), and clearly it satisfies the remaining constraints because the savers' consumption, and everyone's date 0 and 2 consumption is the same as in equilibrium, so it is feasible. Any increase in c_1^S is not feasible: it violates (ZLB). Decreasing c_0^B and increasing c_1^B while satisfying resource constraints

and (US) would decrease the borrower's utility: already in equilibrium the agents' marginal rates of substitution between dates 0 and 1 were equal, and now the borrower has more date 1 consumption, so he does not want to increase it further.

3,4,5. Consider the relaxed problem in which we ignore (RC1), (US). (ICS) must bind in the relaxed program (since it corresponds to our regular Pareto problem with $\alpha = 0$). Substiuting constraints into the objective function, we have

$$\max_{c_0^B} U(c_0^B, \theta_B) - U(c_0^B, \theta_S) + U(2y^* - c_0^B, \theta_S) + constants$$

By Assumption 4.8, this function is concave and the first order condition

$$U_c(\hat{c}, \theta_B) = U_c(\hat{c}, \theta_S) + U_c(2y^* - \hat{c}, \theta_S) \tag{35}$$

is necessary and sufficient for a solution. So the solution to this program is $\hat{c}(\theta_B)$. If $\hat{c}(\theta_B) < \underline{c}(\phi)$, this solution violates (RC1), so (RC1) must bind in the true program. If $\hat{c}(\theta_B) > c_0^B$, this solution violates (US), so (US) must bind. In the intermediate range, neither constraint binds.

K Proof of Proposition 5.1.

The Walrasian equilibrium allocation $\{c_0^i(\theta_N), c_1^i(\theta_N)\}_{i\in[0,1]}$, given a function θ_N mapping individuals to types, satisfies:

$$\frac{\theta_N(i)u'(c_0^i(\theta_N))}{u'(c_1^i(\theta_N))} = \frac{u'(c_0^0(\theta_N))}{u'(c_1^0(\theta_N))}, \forall i \in (0,1] \tag{36}$$

$$\int_0^1 c_0^i(\theta_N)\,\mathrm{d}i = y^* \tag{37}$$

$$\int_0^1 c_1^i(\theta_N)\,\mathrm{d}i = y^* \tag{38}$$

We will show that we must have $c_1^0(\theta_N) \to \infty$ as $N \to \infty$. The proof is by contradiction. Suppose $c_1^0(\theta_N)$ does not converge to ∞. Then $\liminf_{N\to\infty} c_1^0(\theta_N) = c^* < \infty$, and the denominator of the right hand side of (36) does not converge to 0.

Suppose the numerator converges to ∞: then $c_0^0(\theta_N) \to 0$. Consider the lifetime utility of household 0:

$$v^0(\theta_N) = u(c_0^0(\theta_N)) + \beta u(c_1^0(\theta_N))$$

Since $\liminf c_0^0(\theta_N) = 0$ and $\liminf c_1^0(\theta_N) = c^* < \infty$, it follows that $\liminf v^0(\theta_N) < u(y^*) + \beta u(y^*)$: for infinitely many N, household 0 gets lower utility than he would under autarky. This cannot be the case in any competitive equilibrium, since autarky is in his budget set. This contradicts the supposition that the numerator $u'(c_0^0(\theta_N))$ converges to ∞. So we have shown that, under the assumption that $c_1^0(\theta_N)$ does not converge to ∞, the right hand side of (36) does not converge to ∞.

Since $\theta_N(i) \to \infty$ for every $i \in (0,1]$, it follows that $\frac{u'(c_0^i(\theta_N))}{u'(c_1^i(\theta_N))} \to 0$. There are two possibilities: either $c_0^i(\theta_N) \to \infty$, or $c_1^i(\theta_N) \to 0$ (or both). If we define

$$S_0 = \{i \in (0,1] : c_0^i(\theta_N) \to \infty\}, S_1 = \{i \in (0,1] : c_1^i(\theta_N) \to 0\},$$

we must have $S_0 \cup S_1 = (0,1]$. S_0 must have measure zero: otherwise the resource constraint (37) cannot be satisfied, given that consumption must be nonnegative. So S_1 must have measure 1. Given the date 1 resource constraint (38), this implies that $c_1^0(\theta_N) \to \infty$.[3] This contradicts our original assumption. So we must have $c_1^0(\theta_N) \to \infty$.

This implies that for large enough N, the date 1 interest rate satisfying the most patient household's Euler equation, $r_1 = \frac{u'(c_1^0(\theta_N))}{\beta u'(y^*)} - 1$, is negative.

L Proof of Lemma 5.2

1, 2 and 3 are standard results. 4 results from differentiating $C_t(v_1, r_1) = X_t(E(v_1, r_1), r_1)$ and using the Envelope Theorem. To prove 5, note that C_1, C_2 are increasing in v_1: thus there exists v_1 low enough that if we considered a relaxed problem without the borrowing constraint, it would be optimal to set $c_2 < \underline{c}_2$, thus the constraint must bind. The second half of 5 then results from duality. To prove 6, note that the first order conditions yield the Euler equation $c_1^{-\sigma} = \beta(1+r_1)c_2^{-\sigma}$ when the borrowing constraint does not bind; substituting this into the budget constraint yields the desired result. To prove 7, note that if X_1 is concave, X_2 is convex. $C_2(v_1, r_1) = X_2(E(v_1, r_1), r_1)$ is the composition of two increasing, convex functions and is convex.

M Proof of Proposition 5.3.

Before proving Proposition 5.3, I verify that it is possible to express incentive compatibility as an integral condition.

Lemma M.1. *u_0, v_1 satisfies*

$$\theta u(c_0(\theta)) + \beta v_1(\theta) \geq \theta u(c_0(\hat{\theta})) + \beta v_1(\hat{\theta}), \forall \theta, \hat{\theta} \tag{39}$$

if and only if

$$v(\theta) = v(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} u_0(z)\, \mathrm{d}z \tag{40}$$

and u_0 is nondecreasing.

Proof. To show that (39) implies (40), we use Theorem 2 in Milgrom and Segal [2002]. Define

$$W(\theta, \hat{\theta}) = \theta u_0(\hat{\theta}) + \beta v_1(\hat{\theta})$$

and suppose $W(\theta, \theta) = v(\theta) = \max_{\hat{\theta}} W(\theta, \hat{\theta})$. For any fixed $\hat{\theta}$, $W(\theta, \hat{\theta})$ is linear in θ, and therefore differentiable and absolutely continuous. We must also show that there exists an integrable function $b : [1, \bar{\theta}] \to \mathbb{R}$ such that $|W_\theta(\theta, \hat{\theta})| \leq b(\theta)$ for all $\theta \in \Theta$ and almost all $\hat{\theta} \in \Theta$. Since $W_\theta(\theta, \hat{\theta}) = u_0(\hat{\theta})$ is increasing in $\hat{\theta}$, we can set $b(\theta) = \max\{|u_0(\underline{\theta})|, |u_0(\bar{\theta})|\}$, and this condition is satisfied. By Theorem 2 in Milgrom and Segal [2002], we have that

$$v(\theta) = v(\underline{\theta}) + \int_{\underline{\theta}}^{\theta} u_0(z)\, \mathrm{d}z$$

[3] Note that since $c_1^i(\theta_N)$ is decreasing in i for any θ_N - lower types consume more at date 1 - if $c_1^i(\theta_N) \to \infty$ for any type i, then $c_1^0(\theta_N) \to \infty$.

as required.

Next, we show that (40) and u_0 nondecreasing imply (39). (The proof follows Mirrlees [1986] Lemma 6.3.) Take any $\hat{\theta}, \theta$: we want to show that

$$v(\theta) := \theta u_0(\theta) + \beta v_1(\theta) \geq \theta u_0(\hat{\theta}) + \beta v_1(\hat{\theta})$$

Since $u_0(\theta)$ is nondecreasing, we have

$$\begin{aligned} v(\theta) - v(\hat{\theta}) &= \int_{\hat{\theta}}^{\theta} u_0(z)\,\mathrm{d}z \\ &\geq \int_{\hat{\theta}}^{\theta} u_0(\hat{\theta})\,\mathrm{d}z = (\theta - \hat{\theta}) u_0(\hat{\theta}) \\ \theta u_0(\theta) + \beta v_1(\theta) &\geq \theta u_0(\hat{\theta}) + \beta v_1(\hat{\theta}) \end{aligned}$$

This completes the proof. □

I now proceed to prove Proposition 5.3. Recall that the social planner's problem is

$$\mathcal{W}^* = \max_{u_0 \in \Omega, \underline{v}, r_1} \underline{v} + \int (1 - A(\theta)) u_0(\theta)\,\mathrm{d}\theta \tag{PP'}$$

$$\text{s.t} \int C_0(u_0(\theta)) f(\theta)\,\mathrm{d}\theta \leq y^* \tag{RC0'}$$

$$\int C_1\left(\beta^{-1}\left[\underline{v} + \int_{\underline{\theta}}^{\theta} u_0(z)\,\mathrm{d}z - \theta u_0(\theta)\right], r_1\right) f(\theta)\,\mathrm{d}\theta \leq y^* \tag{RC1'}$$

$$\int C_2\left(\beta^{-1}\left[\underline{v} + \int_{\underline{\theta}}^{\theta} u_0(z)\,\mathrm{d}z - \theta u_0(\theta)\right], r_1\right) f(\theta)\,\mathrm{d}\theta \leq y^* \tag{RC2'}$$

$$r_1 \geq 0 \tag{ZLB}$$

I need to show that $u_0, \underline{v}, r_1$ solves (PP') if and only if there exist Lagrange multipliers $\lambda_0, \lambda_1, \lambda_2$ such that $u_0, \underline{v}$ solve

$$\begin{aligned} \mathcal{W}^* = \max_{u_0 \in \Omega, \underline{v}} \underline{v} &+ \int (1 - A(\theta)) u_0(\theta)\,\mathrm{d}\theta - \lambda_0 \int C_0(u_0(\theta)) f(\theta)\,\mathrm{d}\theta \\ &- \int M\left(\beta^{-1}\left[\underline{v} + \int_{\underline{\theta}}^{\theta} u_0(z)\,\mathrm{d}z - \theta u_0(\theta)\right] | \lambda_1, \lambda_2, r_1\right) f(\theta)\,\mathrm{d}\theta \end{aligned} \tag{41}$$

where

$$M(v_1 | \lambda_1, \lambda_2, r_1) := \lambda_1 C_1(v_1, r_1) + \lambda_2 C_2(v_1, r_1)$$

and the Lagrange multipliers satisfy the following conditions. If the ZLB is slack, $\dfrac{\lambda_1}{\lambda_2} = (1 + r_1)(1 - \beta)$. If the ZLB binds, $\dfrac{\lambda_1}{\lambda_2} < 1 - \beta$.

Overview. The proof has six steps. First we show that solutions to (PP') also solve a modified problem (PP'') in which we replace the date 1 resource constraint with the aggregate expenditure function. Second, the modified problem can be solved in two stages: first maximize social welfare given r_1, yielding welfare $\mathcal{W}(r)$, and then choose r to maximize $\mathcal{W}(r)$ subject to the ZLB. Third, the first stage of this problem is concave, and Lagrangian theorems apply. Fourth, we can also express the expenditure functions as maximized sub-Lagrangians. Substituting these sub-

Lagrangians into the main Lagrangian, we see that $W(r)$ is also the maximum of an expanded Lagrangian. Fifth, returning to our two stage problem, we can switch the order of maximization, first choosing r to minimize a certain function, subject to the ZLB, and then choosing utilities to maximize social welfare. Sixth, and finally, I show that when the ZLB is slack, one constraint in the planner's problem becomes slack, and the expanded Lagrangian is equivalent to (41), with $\frac{\lambda_1}{\lambda_2} = (1+r_1)(1-\beta)$. When the ZLB binds, we have $\frac{\lambda_1}{\lambda_2} < 1-\beta$.

1. Modified problem. $u_0, \underline{v}, r_1$ solves (PP′) if and only if it solves the modified function (PP″):

$$\mathcal{W}^* = \max_{u_0 \in \Omega, \underline{v}, r_1} \underline{v} + \int (1 - A(\theta)) u_0(\theta) \, \mathrm{d}\theta \tag{PP″}$$

$$\text{s.t.} \int C_0(u_0(\theta)) f(\theta) \, \mathrm{d}\theta \le y^* \tag{42}$$

$$E\left(\beta^{-1}\left[\underline{v} + \int_{\underline{\theta}}^{\theta} u_0(z)\,\mathrm{d}z - \theta u_0(\theta)\right], r_1\right) f(\theta)\,\mathrm{d}\theta \le y^* + \frac{y^*}{(1+r_1)(1-\beta)} \tag{EC}$$

$$\int C_2(v_1(\theta), r_1) f(\theta)\,\mathrm{d}\theta \le y^* \tag{RC2}$$

$$r_1 \ge 0 \tag{43}$$

This is true because the constraint sets are the same in (PP′) and (PP″) are the same. By definition,

$$E(v_1, r_1) = C_1(v_1, r_1) + \frac{C_2(v_1, r_1)}{(1+r_1)(1-\beta)}$$

So (EC) is equivalent to

$$\int \left[C_1(v_1, r_1) + \frac{C_2(v_1, r_1)}{(1+r_1)(1-\beta)}\right] f(\theta)\,\mathrm{d}\theta \le y^* + \frac{y^*}{(1+r_1)(1-\beta)}$$

This is a weighted sum of the constraints (RC1′) and (RC2′). Clearly then, $v_1, r_1 \ge 0$ satisfies (EC) and (RC2) if and only if it satisfies (RC1).

2. Two-stage problem. $\mathcal{W}^* = \max_{r_1 \ge 0} \mathcal{W}(r_1)$, where

$$\mathcal{W}(r) = \max_{u_0 \in \Omega, \underline{v}} \underline{v} + \int (1 - A(\theta)) u_0(\theta)\,\mathrm{d}\theta \tag{44}$$

$$\text{s.t.} \int C_0(u_0(\theta)) f(\theta)\,\mathrm{d}\theta \le y^* \tag{45}$$

$$E\left(\beta^{-1}\left[\underline{v} + \int_{\underline{\theta}}^{\theta} u_0(z)\,\mathrm{d}z - \theta u_0(\theta)\right], r_1\right) f(\theta)\,\mathrm{d}\theta \le y^* + \frac{y^*}{(1+r_1)(1-\beta)}$$

$$\int C_2(v_1(\theta), r_1) f(\theta)\,\mathrm{d}\theta \le y^*$$

3. Lagrangian. There exist Lagrange multipliers $\lambda_0, \lambda_E, \lambda_C$ such that $u_0, \underline{v}$ solve (44) if and only if they solve

$$\mathcal{W}(r) = \max_{u_0 \in \Omega, \underline{v}} \underline{v} + \int (1 - A(\theta)) u_0(\theta) \, \mathrm{d}\theta - \lambda_0 \int C_0(u_0(\theta)) f(\theta) \, \mathrm{d}\theta \tag{46}$$
$$- \lambda_E \int E \left(\beta^{-1} \left[\underline{v} + \int_{\underline{\theta}}^{\theta} u_0(z) \, \mathrm{d}z - \theta u_0(\theta) \right], r_1 \right) f(\theta) \, \mathrm{d}\theta$$
$$- \lambda_C \int C_2 \left(\beta^{-1} \left[\underline{v} + \int_{\underline{\theta}}^{\theta} u_0(z) \, \mathrm{d}z - \theta u_0(\theta) \right], r_1 \right) f(\theta) \, \mathrm{d}\theta$$

Before proving this statement, note for future reference that since by definition $E = C_1 + \frac{C_2}{(1+r_1)(1-\beta)}$, we could equivalently write

$$\mathcal{W}(r) = \max_{u_0 \in \Omega, \underline{v}} \underline{v} + \int (1 - A(\theta)) u_0(\theta) \, \mathrm{d}\theta - \lambda_0 \int C_0(u_0(\theta)) f(\theta) \, \mathrm{d}\theta \tag{47}$$
$$- \lambda_1 \int C_1 \left(\beta^{-1} \left[\underline{v} + \int_{\underline{\theta}}^{\theta} u_0(z) \, \mathrm{d}z - \theta u_0(\theta) \right], r_1 \right) f(\theta) \, \mathrm{d}\theta$$
$$- \lambda_2 \int C_2 \left(\beta^{-1} \left[\underline{v} + \int_{\underline{\theta}}^{\theta} u_0(z) \, \mathrm{d}z - \theta u_0(\theta) \right], r_1 \right) f(\theta) \, \mathrm{d}\theta$$

where we define $\lambda_1 = \lambda_E, \lambda_2 = \frac{\lambda_E}{(1+r_1)(1-\beta)}$. That is, showing necessity and sufficiency for (46) is equivalent to showing necessity and sufficiency for (47).

(44) is a special case of the general problem considered in Luenberger [1969], Sections 8.3-8.4:

$$\inf_{x \in \Omega} f(x)$$
$$\text{s.t. } G(x) \leq 0$$

where X is a linear vector space, Z a normed space, Ω a convex subset of X, P the positive cone in Z, f a real valued convex functional on Ω, G a convex mapping from Ω into Z. We have

$$X = \{\underline{v}, u_0 | \underline{v} \in V(\mathbb{R}_+), u_0 : \Theta \to \mathbb{R}\}$$
$$\Omega = \{\underline{v}, u_0 | \underline{v} \in V, u_0 : \Theta \to U(\mathbb{R}_+), u_0 \text{ non-decreasing}\}$$
$$Z = \mathbb{R}^3, P = \mathbb{R}^3_+$$
$$f(\underline{v}, u_0) = -\underline{v} - \int (1 - A(\theta)) u_0(\theta) \, \mathrm{d}\theta$$
$$G(\underline{v}, u_0) = \begin{bmatrix} \int C_0(u_0(\theta)) f(\theta) \, \mathrm{d}\theta - y^* \\ E(v_1(\theta), r_1) f(\theta) \, \mathrm{d}\theta - y^* - \frac{y^*}{(1+r_1)(1-\beta)} \\ \int C_2(v_1(\theta), r_1) f(\theta) \, \mathrm{d}\theta - y^* \end{bmatrix}$$

Ω is convex, P contains an interior point, and f is convex (since it is linear). C_0 is convex, and E and C_2 are convex in v_1 by Lemma 5.2; since v_1 is a linear function of $\underline{v}, u_0$, and G contains weighted sums of C_0, E, C_2, it follows that G is convex. There exists a point $\underline{v}, u_0 \in \Omega$ such that $G(\underline{v}, u_0) \leq 0$: choose v_1 small enough that $E(v_1, r_1) < y^* - \frac{y^*}{(1+r_1)(1-\beta)}$, $C_2(v_1, r_1) < y^*$, and set $u_0(\theta) = u(y^*/2)$, $\underline{v} = \underline{\theta} u(y^*/2) + \beta v_1$. Then since the hypotheses of Theorem 1 in Luenberger

[1969] p217 are met, it follows that if $u_0, \underline{v}$ solve (44), they solve (46). Conversely, by Theorem 1 in Luenberger [1969] p220, if $u_0, \underline{v}$ solve (46), they solve (44). Finally, as noted above, showing necessity and sufficiency of (46) is equivalent to showing necessity and sufficiency of (47). This completes the proof of the first part of Proposition 5.3. It remains to derive the condition on the Lagrange multipliers.

4. Expenditure minimization sub-Lagrangians. Applying the same theorems to the expenditure minimization problem (EMP), we have that, for each type θ, solutions to the expenditure minimization problem also minimize an appropriately defined Lagrangian:

$$
\begin{aligned}
E(v_1(\theta), r_1) = \min_{c_1(\theta), c_2(\theta)} c_1(\theta) + \frac{c_2(\theta)}{(1+r_1)(1-\beta)} + \psi(\theta)[c_2(\theta) - \underline{c}_2] \\
+\mu(\theta)\left[u(c_1(\theta)) + \frac{\beta}{1-\beta} u(c_2(\theta)) - v_1(\theta)\right]
\end{aligned}
$$

Integrating across all types,

$$
\begin{aligned}
& - \int E\left(\beta^{-1}\left[\underline{v} + \int_{\underline{\theta}}^{\theta} u_0(z)\,\mathrm{d}z - \theta u_0(\theta)\right], r_1\right) f(\theta)\,\mathrm{d}\theta \\
&= - \int \left\{ \min_{c_1(\theta), c_2(\theta)} c_1(\theta) + \frac{c_2(\theta)}{(1+r_1)(1-\beta)} + \psi(\theta)[c_2(\theta) - y^* + (1-\beta)\psi] \right. \\
& \left. \mu(\theta)\left[u(c_1(\theta)) + \frac{\beta}{1-\beta} u(c_2(\theta)) - \beta^{-1}\left(\underline{v} + \int_{\underline{\theta}}^{\theta} u_0(z)\,\mathrm{d}z - \theta u_0(\theta)\right)\right]\right\} f(\theta)\,\mathrm{d}\theta \\
&= \max_{c_1, c_2} \left\{ \int -c_1(\theta) - \frac{c_2(\theta)}{(1+r_1)(1-\beta)} - \psi(\theta)[c_2(\theta) - \underline{c}_2] \right. \\
& \left. - \mu(\theta)\left[u(c_1(\theta)) + \frac{\beta}{1-\beta} u(c_2(\theta)) - \beta^{-1}\left(\underline{v} + \int_{\underline{\theta}}^{\theta} u_0(z)\,\mathrm{d}z - \theta u_0(\theta)\right)\right]\right\} f(\theta)\,\mathrm{d}\theta
\end{aligned}
$$

Substituting into the main Lagrangian, we see that if $u_0, \underline{v}$ solve (44), they also solve

$$
\begin{aligned}
\mathcal{W}(r) = \max_{u_0 \in \Omega, \underline{v}} \underline{v} + \int (1 - A(\theta)) u_0(\theta)\,\mathrm{d}\theta - \lambda_0 \int C_0(u_0(\theta)) f(\theta)\,\mathrm{d}\theta \\
- \lambda_E \max_{c_1, c_2} \left\{ \int -c_1(\theta) - \frac{c_2(\theta)}{(1+r_1)(1-\beta)} - \psi(\theta)[c_2(\theta) - \underline{c}_2] \right. \\
\left. - \mu(\theta)\left[u(c_1(\theta)) + \frac{\beta}{1-\beta} u(c_2(\theta)) - \beta^{-1}\left(\underline{v} + \int_{\underline{\theta}}^{\theta} u_0(z)\,\mathrm{d}z - \theta u_0(\theta)\right)\right]\right\} f(\theta)\,\mathrm{d}\theta \\
- \lambda_C \int C_2\left(\beta^{-1}\left[\underline{v} + \int_{\underline{\theta}}^{\theta} u_0(z)\,\mathrm{d}z - \theta u_0(\theta)\right], r_1\right) f(\theta)\,\mathrm{d}\theta
\end{aligned}
$$

This will be true if and only if $u_0, \underline{v}, c_1, c_2$ solve

$$\begin{aligned}
\mathcal{W}(r) = \max_{u_0 \in \Omega, \underline{v}, c_1, c_2} \underline{v} + \int (1 - A(\theta)) u_0(\theta)\, \mathrm{d}\theta - \lambda_0 \int C_0(u_0(\theta)) f(\theta)\, \mathrm{d}\theta \\
- \lambda_E \left\{ \int -c_1(\theta) - \frac{c_2(\theta)}{(1+r_1)(1-\beta)} - \psi(\theta)[c_2(\theta) - \underline{c}_2] \right. \\
\left. - \mu(\theta) \left[u(c_1(\theta)) + \frac{\beta}{1-\beta} u(c_2(\theta)) - \beta^{-1} \left(\underline{v} + \int_{\underline{\theta}}^{\theta} u_0(z)\, \mathrm{d}z - \theta u_0(\theta) \right) \right] \right\} f(\theta)\, \mathrm{d}\theta \\
- \lambda_C \int C_2 \left(\beta^{-1} \left[\underline{v} + \int_{\underline{\theta}}^{\theta} u_0(z)\, \mathrm{d}z - \theta u_0(\theta) \right], r_1 \right) f(\theta)\, \mathrm{d}\theta
\end{aligned}$$

The economic meaning of this result is that when the ZLB is slack, private and social valuations of date 1 and date 2 consumption coincide, and the planner would not want to distort date 1 or date 2 consumption away from their equilibrium levels, even if it were possible to do so. Since households share the same preferences over date 1 and date 2 consumption, distorting these allocations would not help relax incentive constraints. Instead, it is always optimal to deliver utility in an ex post efficient way at dates 1 and 2.

5. Reversing the order of maximization. Next, note that $\mathcal{W}(r)$ has the form $\mathcal{W}(r) = \max_x f(x) + g(x, r)$, where $x = (u_0, \underline{v}, c_1, c_2)$. Then we have

$$\begin{aligned}
\mathcal{W}^* &= \max_{r \geq 0} \mathcal{W}(r) \\
&= \max_{r \geq 0} \max_x \{ f(x) + g(x, r) \} \\
&= \max_x \max_{r \geq 0} \{ f(x) + g(x, r) \} \\
&= \max_x f(x) + \max_r \{ g(x, r) \}
\end{aligned}$$

Applying this result to the Lagrangian

$$\begin{aligned}
\mathcal{W}^* = \max_{u_0 \in \Omega, \underline{v}, c_1, c_2} \underline{v} + \int (1 - A(\theta)) u_0(\theta)\, \mathrm{d}\theta - \lambda_0 \int C_0(u_0(\theta)) f(\theta)\, \mathrm{d}\theta \\
- \lambda_E \left\{ \int -c_1(\theta) - \frac{c_2(\theta)}{(1+r_1)(1-\beta)} - \psi(\theta)[c_2(\theta) - \underline{c}_2] \right. \\
\left. - \mu(\theta) \left[u(c_1(\theta)) + \frac{\beta}{1-\beta} u(c_2(\theta)) - \beta^{-1} \left(\underline{v} + \int_{\underline{\theta}}^{\theta} u_0(z)\, \mathrm{d}z - \theta u_0(\theta) \right) \right] \right\} f(\theta)\, \mathrm{d}\theta \\
- \min_{r_1 \geq 0} \left\{ \lambda_C \int C_2 \left(\beta^{-1} \left[\underline{v} + \int_{\underline{\theta}}^{\theta} u_0(z)\, \mathrm{d}z - \theta u_0(\theta) \right], r_1 \right) f(\theta)\, \mathrm{d}\theta \right\}
\end{aligned}$$

6. ZLB. When the ZLB is slack, $\min_{r_1 \geq 0} \lambda_C \int C_2\, \mathrm{d}F(\theta) = \min_{r_1 \in \mathbb{R}} \lambda_C \int C_2\, \mathrm{d}F(\theta)$. Since C_2 is increasing in r_1, $r_1 > 0$ can only attain the minimum in this problem when $\lambda_C = 0$. That is, if the ZLB is slack, $\lambda_C = 0$. However, if the ZLB binds at an optimum, $\lambda_C > 0$.

In the Lagrangian (47), we had $\lambda_1 = \lambda_E, \lambda_2 = \dfrac{\lambda_E}{(1+r_1)(1-\beta)} + \lambda_C$. Thus when the ZLB is slack, $\lambda_2 = (1+r_1)(1-\beta)\lambda_1$. When the ZLB binds, $\lambda_2 = (1-\beta)\lambda_1 + \lambda_C > (1-\beta)\lambda_1$. This establishes the desired result.

N Proof of Lemma 5.5.

Definition N.1. *Let f be a real valued functional defined on a vector space X. Define the (one sided) Gateaux differential of f at x with increment h to be*

$$\delta f(x;h) = \lim_{\alpha \downarrow 0} \frac{1}{\alpha}[f(x+\alpha h) - f(x)]$$

If this limit exists for each $h \in X$, f is Gateaux differentiable at x.

Lemma N.2. *Let Ω be a subset of the space of functions mapping $\Theta = [\underline{\theta}, \bar{\theta}] \subset \mathbb{R}_+$ into $\mathbb{R}$. Let $T : \Omega \to \mathbb{R}$ be defined by $T(x) = \int_\Theta \psi(x(\theta)) f(\theta)\, \mathrm{d}\theta$. Suppose that ψ is convex, its left and right hand side derivatives $\psi'_-(x(\theta)), \psi'_+(x(\theta))$ exist and are continuous in $x(\theta)$, and there exists $\varepsilon > 0$ such that $x + \alpha h \in \Omega$. Then*

$$\delta T(x;h) = \int_{\{\theta \in \Theta : h(\theta) > 0\}} \psi'_+(x(\theta)) h(\theta) f(\theta)\, \mathrm{d}\theta + \int_{\{\theta \in \Theta : h(\theta) < 0\}} \psi'_-(x(\theta)) h(\theta) f(\theta)\, \mathrm{d}\theta$$

Note that we can equivalently write this as

$$\delta T(x;h) = \int_\Theta [\psi'_+(x(\theta)) h_+(\theta) + \psi'_-(x(\theta)) h_-(\theta)] f(\theta)\, \mathrm{d}\theta$$

where $h_+(\theta) = \max\{h(\theta), 0\}$ and $h_-(\theta) = \min\{h(\theta), 0\}$.

Proof. The proof is essentially identical to the proof of Lemma A.1 in Amador et al. [2006]. The only reason this result is not a special case of theirs is that ψ may not be differentiable, although its left and right hand derivatives exist.

Define $\Theta_+ = \{\theta \in \Theta : h(\theta) > 0\}$, $\Theta_- = \{\theta \in \Theta : h(\theta) < 0\}$. From the definition of the Gateaux differential,

$$\begin{aligned}
\delta T(x;h) &= \lim_{\alpha \downarrow 0} \frac{1}{\alpha}[T(x+\alpha h) - T(x)] \\
&= \lim_{\alpha \downarrow 0} \int_\Theta \frac{1}{\alpha}[\psi(x(\theta) + \alpha h(x)) - \psi(x(\theta))] f(\theta)\, \mathrm{d}\theta \\
&= \lim_{\alpha \downarrow 0} \int_{\Theta_+} \frac{1}{\alpha}[\psi(x(\theta) + \alpha h(x)) - \psi(x(\theta))] f(\theta)\, \mathrm{d}\theta + \lim_{\alpha \downarrow 0} \int_{\Theta_-} \frac{1}{\alpha}[\psi(x(\theta) + \alpha h(x)) - \psi(x(\theta))] f(\theta)\, \mathrm{d}\theta
\end{aligned}$$

Adding and subtracting $\int_{\Theta_+} \psi'_+(x(\theta)) h_+(\theta) f(\theta)\, \mathrm{d}\theta$ from the first term,

$$\begin{aligned}
&\lim_{\alpha \downarrow 0} \int_{\Theta_+} \frac{1}{\alpha}[\psi(x(\theta) + \alpha h(x)) - \psi(x(\theta))] f(\theta)\, \mathrm{d}\theta \\
&= \int_{\Theta_+} \psi'_+(x(\theta)) h_+(\theta) f(\theta)\, \mathrm{d}\theta + \lim_{\alpha \downarrow 0} \int_{\Theta_+} \left[\frac{1}{\alpha}[\psi(x(\theta) + \alpha h(x)) - \psi(x(\theta))] - \psi'_+(x(\theta)) h_+(\theta) \right] f(\theta)\, \mathrm{d}\theta
\end{aligned}$$

I will show that the last term vanishes. For $\alpha < \varepsilon$, we have

$$\left| \frac{1}{\alpha}[\psi(x(\theta) + \alpha h(x)) - \psi(x(\theta))] - \psi'_+(x(\theta)) h_+(\theta) \right| < \left| \frac{1}{\varepsilon}[\psi(x(\theta) + \varepsilon h(x)) - \psi(x(\theta))] - \psi'_+(x(\theta)) h_+(\theta) \right|$$

for all $\theta \in \Theta_+$, since ψ is convex. As in the proof of Lemma A.1 in Amador et al. [2006], this provides the required integrable bound to apply Lebesgue's Dominated Convergence Theorem, so we have

$$\lim_{\alpha\downarrow 0}\int_{\Theta_+}\left[\frac{1}{\alpha}[\psi(x(\theta)+\alpha h(x))-\psi(x(\theta))]-\psi'_+(x(\theta))h_+(\theta)\right]f(\theta)\,\mathrm{d}\theta$$
$$=\int_{\Theta_+}\lim_{\alpha\downarrow 0}\left[\frac{1}{\alpha}[\psi(x(\theta)+\alpha h(x))-\psi(x(\theta))]-\psi'_+(x(\theta))h_+(\theta)\right]f(\theta)\,\mathrm{d}\theta=0$$

by definition of $\psi'_+(x(\theta))$, noting that for $\theta\in\Theta_+$, and for $\alpha>0$, $x(\theta)+\alpha h(x)>x(\theta)$. It follows that the first term is equal to $\int_{\Theta_+}\psi'_+(x(\theta))h(\theta)f(\theta)\,\mathrm{d}\theta$, as required. An identical argument applies to the second term. So we have

$$\delta T(x;h)=\int_{\Theta_+}\psi'_+(x(\theta))h(\theta)f(\theta)\,\mathrm{d}\theta+\int_{\Theta_-}\psi'_-(x(\theta))h(\theta)f(\theta)\,\mathrm{d}\theta$$

as required. □

Lemma N.3. *Define*

$$M(v_1|\lambda_1,\lambda_2,r_1):=\lambda_1 C_1(v_1,r_1)+\lambda_2 C_2(v_1,r_1)$$

where $\lambda_1,\lambda_2\geq 0$ *and* $\frac{\lambda_1}{\lambda_2}\leq(1+r_1)(1-\beta)$.

1. *M is convex in v_1.*
2. *If $\frac{\lambda_1}{\lambda_2}=(1+r_1)(1-\beta)$, $M(v_1|\lambda_1,\lambda_2,r_1)=\lambda_1 E(v_1,r_1)$, and M is differentiable.*
3. *If $r_1=0$ and $\frac{\lambda_1}{\lambda_2}<(1-\beta)$, M is left and right differentiable, and it is differentiable except at $v_1=\bar{v}_1(0)$.*

Proof. $M=\lambda_1 E_1(v_1,r_1)+\left(\lambda_2-\frac{\lambda_1}{(1+r_1)(1-\beta)}\right)C_2(v_1,r_1)$ is the non-negative-weighted sum of convex functions and is therefore convex. It is immediate that if $\frac{\lambda_1}{\lambda_2}=(1+r_1)(1-\beta)$, and M is differentiable. If $r_1=0$ and $\lambda_2>\frac{\lambda_1}{(1-\beta)}$, M is the weighted sum of a differentiable function E, and a function C_2 which is left- and right-differentiable everywhere, and differentiable except at $\bar{v}_1(0)$. The desired result follows. □

Lemma N.4. *Define the Lagrangian*

$$\mathcal{L}(u_0,\underline{v})=\underline{v}+\int(1-A(\theta))u_0(\theta)\,\mathrm{d}\theta-\lambda_0\int C_0(u_0(\theta))f(\theta)\,\mathrm{d}\theta$$
$$\int M\left(\beta^{-1}\left[\underline{v}+\int_{\underline{\theta}}^{\theta}u_0(z)\,\mathrm{d}z-\theta u_0(\theta)\right]|\lambda_1,\lambda_2,r_1\right)f(\theta)\,\mathrm{d}\theta$$

where

$$M(v_1(\theta)|\lambda_1,\lambda_2,r_1)=\lambda_1 C_1(v_1(\theta),r_1)+\lambda_2 C_2(v_1(\theta),r_1)$$

The Gateaux differential of the Lagrangian is

$$\begin{aligned}\delta\mathcal{L}(u_0,\underline{v};\Delta_0,\underline{\Delta}) =& \underline{\Delta} + \int (1-A(\theta))\Delta_0(\theta)\,\mathrm{d}\theta - \lambda_0 \int C_0'(u_0(\theta))\Delta_0(\theta) f(\theta)\,\mathrm{d}\theta \\ & - \int_{\Theta_+} M_+'(v_1(\theta)|\lambda_1,\lambda_2,r_1)\beta^{-1}\left[\underline{\Delta} + \int_{\underline{\theta}}^{\theta} \Delta_0(z)\,\mathrm{d}z - \theta\Delta_0(\theta)\right] f(\theta)\,\mathrm{d}\theta \\ & - \int_{\Theta_-} M_-'(v_1(\theta)|\lambda_1,\lambda_2,r_1)\beta^{-1}\left[\underline{\Delta} + \int_{\underline{\theta}}^{\theta} \Delta_0(z)\,\mathrm{d}z - \theta\Delta_0(\theta)\right] f(\theta)\,\mathrm{d}\theta\end{aligned}$$

where $v_1(\theta) = \beta^{-1}\left[\underline{v} + \int_{\underline{\theta}}^{\theta} u_0(z)\,\mathrm{d}z - \theta u_0(\theta)\right]$*, and*

$$\begin{aligned}\Theta_+ &= \left\{\theta \in \Theta : \underline{\Delta} + \int_{\underline{\theta}}^{\theta} \Delta_0(z)\,\mathrm{d}z - \theta\Delta_0(\theta) > 0\right\} \\ \Theta_- &= \left\{\theta \in \Theta : \underline{\Delta} + \int_{\underline{\theta}}^{\theta} \Delta_0(z)\,\mathrm{d}z - \theta\Delta_0(\theta) < 0\right\}\end{aligned}$$

Proof. By Lemma N.2, the Gateaux differential of

$$\underline{v} + \int (1-A(\theta))u_0(\theta)\,\mathrm{d}\theta - \lambda_0 \int C_0(u_0(\theta)) f(\theta)\,\mathrm{d}\theta$$

with increment $\Delta_0, \underline{\Delta}$ is

$$\underline{\Delta} + \int (1-A(\theta))\Delta_0(\theta)\,\mathrm{d}\theta - \lambda_0 \int C_0'(u_0(\theta))\Delta_0(\theta) f(\theta)\,\mathrm{d}\theta$$

since C_0 is convex and differentiable. The Gateaux differential of $\int M(v_1(\theta)|\lambda_1,\lambda_2,r_1) f(\theta)\,\mathrm{d}\theta$ with increment $\Delta_1(\theta)$ is

$$\int_{\Theta_+} M_+'(v_1(\theta)|\lambda_1,\lambda_2,r_1)\Delta_1(\theta) f(\theta)\,\mathrm{d}\theta + \int_{\Theta_-} M_-'(v_1(\theta)|\lambda_1,\lambda_2,r_1)\Delta_1(\theta) f(\theta)\,\mathrm{d}\theta$$

where $\Theta_+ = \{\theta \in \Theta : \Delta_1(\theta) > 0\}$, $\Theta_- = \{\theta \in \Theta : \Delta_1(\theta) < 0\}$. This follows because M is convex and both left- and right- differentiable. Defining

$$\Delta_1(\theta) = \beta^{-1}\left[\underline{\Delta} + \int_{\underline{\theta}}^{\theta} \Delta_0(z)\,\mathrm{d}z - \theta\Delta_0(\theta)\right],$$

it follows that the Gateaux differential of

$$\int M\left(\beta^{-1}\left[\underline{v} + \int_{\underline{\theta}}^{\theta} u_0(z)\,\mathrm{d}z - \theta u_0(\theta)\right]|\lambda_1,\lambda_2,r_1\right) f(\theta)\,\mathrm{d}\theta$$

is

$$\begin{aligned}&\int_{\Theta_+} M_+'(v_1(\theta)|\lambda_1,\lambda_2,r_1)\beta^{-1}\left[\underline{\Delta} + \int_{\underline{\theta}}^{\theta} \Delta_0(z)\,\mathrm{d}z - \theta\Delta_0(\theta)\right] f(\theta)\,\mathrm{d}\theta \\ &+ \int_{\Theta_-} M_-'(v_1(\theta)|\lambda_1,\lambda_2,r_1)\beta^{-1}\left[\underline{\Delta} + \int_{\underline{\theta}}^{\theta} \Delta_0(z)\,\mathrm{d}z - \theta\Delta_0(\theta)\right] f(\theta)\,\mathrm{d}\theta\end{aligned}$$

The result then follows. □

O Proof of Lemma 5.5.

Lemma O.1. *Let f be a concave, real valued functional defined on a vector space X. Suppose that f is (one-sided) Gateaux differentiable. Then x_0 maximizes f on the convex set $\Omega \subset X$ if and only if*

$$\delta f(x_0; x - x_0) \leq 0$$

for all $x \in \Omega$.

Proof. The proof of necessity is essentially identical to Luenberger [1969] Theorem 1, p178, except that I use the one-sided Gateaux differential. Suppose x_0 maximizes f. Since Ω is convex, $x_0 + \alpha(x - x_0) \in \Omega$ for $0 \leq \alpha \leq 1$. So for any $x \in \Omega$,

$$\begin{aligned} f(x_0 + \alpha(x - x_0)) - f(x_0) &\leq 0 \\ \frac{1}{\alpha}[f(x_0 + \alpha(x - x_0)) - f(x_0)] &\leq 0 \\ \lim_{\alpha \downarrow \infty} \frac{1}{\alpha}[f(x_0 + \alpha(x - x_0)) - f(x_0)] &\leq 0. \end{aligned}$$

The proof of sufficiency follows Luenberger [1969] Lemma 1, p227. Suppose there exists $x_0 \in \Omega$ such that $\delta f(x_0; x - x_0) \leq 0, \forall x \in \Omega$. Take any $x \in \Omega$. Since f is concave, for any $\alpha \in [0, 1]$,

$$\begin{aligned} f(x_0 + \alpha(x - x_0)) &\geq f(x_0) + \alpha[f(x) - f(x_0)] \\ f(x) - f(x_0) &\leq \frac{1}{\alpha}[f(x_0 + \alpha(x - x_0)) - f(x_0)] \end{aligned}$$

Taking limits, and using the definition of the Gateaux differential,

$$f(x) - f(x_0) \leq \delta f(x_0; x - x_0) \leq 0$$

by assumption. So $f(x_0) \geq f(x)$ for any $x \in \Omega$, and x_0 is a maximum. □

Lemma 5.5 then follows from applying Lemma O.1 to Proposition 5.3, and using the definition of the Gateaux differential in N.2. Note that the Lagrangian is concave.

P Proof of Proposition 5.6.

Lemma P.1. *If $r_1 > 0$ in equilibrium, the equilibrium is constrained efficient.*

Proof. Suppose $c_0, c_1, c_2, r_0, r_1 > 0$ is an equilibrium. Set

$$\begin{aligned} \frac{1}{\lambda_1} &= \int \frac{f(\theta)}{\beta u'(c_1(\theta))} \, \mathrm{d}\theta \\ \frac{\lambda_0}{\lambda_1} &= 1 + r_0 \\ g(\theta) &= \lambda_1, \forall \theta \end{aligned}$$

In a competitive equilibrium, we have $\dfrac{\beta u'(c_1(\theta))}{\theta u'(c_0(\theta))} = \dfrac{1}{1+r_0}$ for all θ. Since the ZLB does not bind, $\lambda(\theta) = \lambda_1$ for all θ. Thus the first order sufficient conditions for optimality are satisfied. □

Lemma P.2. *Suppose $u_0(\theta)$ is continuous and strictly increasing on (θ_1, θ_2), and $\alpha(\theta_1)$ is continuous on (θ_1, θ_2). Suppose also that $\int_\Theta \alpha(\theta)\Delta_0(\theta)\,\mathrm{d}\theta \le 0$ for all functions $\Delta_0 : \Theta \to \mathbb{R}$ such that $u_0 + \Delta_0$ is increasing. Then $\alpha(\theta) = 0$ for all $\theta \in (\theta_1, \theta_2)$.*

Proof. Suppose by contradiction that $\alpha(\theta) > 0$ for some $\theta \in [\theta_1, \theta_2]$. Then by continuity, $\alpha(\theta) > 0$ on some interval $[\theta_1', \theta_2'] \subset [\theta_1, \theta_2]$. Set

$$\Delta_0(\theta) = \frac{(u_0(\theta) - u_0(\theta_1))(u_0(\theta_2) - u_0(\theta))}{(u_0(\theta_2) - u_0(\theta_1))^2} \text{ for } \theta \in [\theta_1', \theta_2'],$$

$\Delta_0(\theta) = 0$ everywhere else. It can be verified that $u_0(\theta) + \Delta_0(\theta)$ is increasing, is an admissible deviation, and is positive for $\theta \in (\theta_1', \theta_2')$. Then we have

$$\int \alpha(\theta)\Delta_0(\theta)\,\mathrm{d}\theta > 0,$$

a contradiction. □

Lemma P.3. *If $u_0(\theta)$ is continuous and strictly increasing and $u_0, \underline{v}, r_1$ solves (PP′), then there exist Lagrange multipliers $\lambda_0, \lambda_1, \lambda_2$ such that*

$$1 = \int M'(v_1(\theta)|\lambda_1, \lambda_2, r_1) f(\theta)\,\mathrm{d}\theta \tag{48}$$

$$\lambda_0 C'(u_0(\theta)) f(\theta) - \beta^{-1}\theta M'(v_1(\theta)|\lambda_1, \lambda_2, r_1) f(\theta) = \int_\theta^{\bar{\theta}} [a(z) - \beta^{-1} M'(v_1(z)|\lambda_1, \lambda_2, r_1) f(z)]\,\mathrm{d}z \tag{49}$$

for all $\theta \in \Theta$, unless $r_1 = 0$ and $\bar{v}_1(0) = v_1(\theta)$. If the ZLB is slack, $\dfrac{\lambda_1}{\lambda_2} = (1 + r_1)(1 - \beta)$. If the ZLB binds, $\dfrac{\lambda_1}{\lambda_2} < 1 - \beta$.

Conversely, if the ZLB is slack and (48), (49) hold for all $\theta \in \Theta$ with u_0 continuous and strictly increasing, and if $u_0, \underline{v}, r_1$ satisfy the constraints in (PP′), then $u_0, \underline{v}, r_1$ solve (PP′).

Proof. Suppose u_0 is continuous and strictly increasing. Then in any incentive compatible allocation, v_1 is continuous and strictly decreasing. It follows that there exists at most one type θ^* such that $v_1(\theta) = \bar{v}_1(r_1)$ (that is, θ is 'just' liquidity constrained). By Lemma N.3, M is differentiable everywhere except at $\bar{v}_1(r_1)$ (and if the ZLB is slack, it is differentiable at this point too). Thus the first-order condition necessary condition for a maximum, stated in Lemma 5.5, can be rewritten as

$$\begin{aligned}\delta\mathcal{L}(u_0, \underline{v}; \Delta_0, \underline{\Delta}) =& \underline{\Delta} + \int (1 - A(\theta))\Delta_0(\theta)\,\mathrm{d}\theta - \lambda_0 \int C_0'(u_0(\theta))\Delta_0(\theta) f(\theta)\,\mathrm{d}\theta \\ & - \int M'(v_1(\theta)|\lambda_1, \lambda_2, r_1)\beta^{-1}\left[\underline{\Delta} + \int_{\underline{\theta}}^{\theta} \Delta_0(z)\,\mathrm{d}z - \theta\Delta_0(\theta)\right] f(\theta)\,\mathrm{d}\theta \le 0\end{aligned}$$

for all $\underline{\Delta}, \Delta_0$ such that $u_0 + \Delta_0$ is increasing. Applying Fubini's Theorem to reverse the order of integration, and rearranging terms, we can rewrite this as

$$\int \Delta_0(\theta)\left\{-\lambda_0 C_0'(u_0(\theta))f(\theta)+\beta^{-1}\theta M'(v_1(\theta))f(\theta)-\int_\theta^{\bar\theta}[a(z)-\beta^{-1}M'(v_1(z))f(z)]\,\mathrm{d}z\right\}\mathrm{d}\theta$$
$$+\underline{\Delta}\left\{1-\int M'(v_1(\theta))f(\theta)\,\mathrm{d}\theta\right\}\leq 0 \text{ for all } \underline{\Delta},\Delta_0 \text{ such that } u_0+\Delta_0\in\Omega$$

where we suppress the dependence of M' on $\lambda_1, \lambda_2, r_1$ to save notation. Clearly, we must have $1 = \int M'(v_1(\theta))f(\theta)\,\mathrm{d}\theta = 0$, so (48) holds. To show that (49) holds, we use Lemma P.2, first setting $[\theta_1, \theta_2] = [\underline{\theta}, \theta^*]$, and then setting $[\theta_1, \theta_2] = [\theta^*, \bar\theta]$. Since

$$\alpha(\theta) := -\lambda_0 C_0'(u_0(\theta))f(\theta)+\beta^{-1}\theta M'(v_1(\theta))f(\theta)+\int_\theta^{\bar\theta}[a(z)-\beta^{-1}M'(v_1(z))f(z)]\,\mathrm{d}z$$

is continuous on $[\underline{\theta}, \theta^*)$ and on $(\theta^*, \bar\theta]$, it follows that $\alpha(\theta) = 0$ for any $\theta \neq \theta^*$. If the ZLB does not bind, M' exists and is continuous everywhere, and we also have $\alpha(\theta^*) = 0$. Then (49) holds.

Finally, if the ZLB is slack, and if $\alpha(\theta) = 0$ everywhere, then clearly $\int \alpha(\theta)\Delta_0(\theta)\,\mathrm{d}\theta = 0$ for all Δ_0, and by Lemma 5.5, $u_0, \underline{v}, r_1$ solve (PP′). □

Lemma P.4. *If $r_1 > 0$, $\int c_1(\theta)f(\theta)\,\mathrm{d}\theta < y^*$ in equilibrium, the equilibrium is constrained inefficient.*

Proof. Suppose by contradiction that such an allocation $u_0, \underline{v}$ solves (PP′) for some non-negative Pareto weights $a(\theta)$ (equivalently, for some differentiable, nondecreasing $A(\theta) = \int_{\underline{\theta}}^{\theta} a(z)\,\mathrm{d}z$). In any competitive equilibrium, allocations and utilities are continuous in θ. Consequently, by Lemma P.3, there must exist Lagrange multipliers $\lambda_0 > 0, \lambda_1 = 0, \lambda_2 > 0$ such that

$$\lambda_0 C'(u_0(\theta))f(\theta)-\beta^{-1}\theta M'(v_1(\theta)|\lambda_1,\lambda_2,r_1)f(\theta)=1-A(\theta)-\int_\theta^{\bar\theta}\beta^{-1}M'(v_1(z)|\lambda_1,\lambda_2,r_1)f(z)]\,\mathrm{d}z$$

Let θ^* be the type who is just liquidity constrained. For $\theta < \theta^*$, this condition states that

$$1-A(\theta)=\lambda_0\frac{f(\theta)}{u'(c_0(\theta))}-\frac{\theta f(\theta)\mu\lambda_2}{\beta u'(c_1(\theta))}+\int_\theta^{\theta^*}\frac{f(z)\mu\lambda_2}{\beta u'(c_1(z))}\,\mathrm{d}z$$

where $\mu := \frac{(1-\beta)\beta^{1/\sigma}}{1-\beta+\beta^{1/\sigma}}$, the date 2 MPC of unconstrained households. As $\theta \to \theta^*$ from below, since all right hand side terms are continuous,

$$1-A(\theta)\to\lambda_0\frac{f(\theta^*)}{u'(c_0(\theta^*))}-\frac{\theta^* f(\theta^*)\mu\lambda_2}{\beta u'(c_1(\theta^*))}$$

For $\theta > \theta^*$,

$$1-A(\theta)=\lambda_0\frac{f(\theta)}{u'(c_0(\theta))}\to\lambda_0\frac{f(\theta^*)}{u'(c_0(\theta^*))}$$

as $\theta \to \theta^*$ from above. These two limits are not the same, which contradicts the hypothesis that $A(\theta)$ was continuous. So the allocation cannot be a solution to (PP′) for any welfare weights, and is not constrained efficient. □

An alternative proof is as follows.

Proof. Consider a deviation, relative to a competitive equilibrium allocation in which the ZLB binds, in which we add the point $(\bar{c}, \bar{a}_1)$ to the budget set. In utility space, this point is $\bar{u}, \bar{v}_1$. If $\bar{u} = u_0(\theta^*)$, there is no deviation. For any $\bar{u} > u_0(\theta^*)$, a set of types $(\theta_1(\bar{u}), \theta_2(\bar{u}))$ (containing θ^*) will be attracted to the deviation, where

$$\theta_1(\bar{u})\bar{u} + \beta\bar{v}_1 = v(\theta_1(\bar{u}))$$
$$\theta_2(\bar{u})\bar{u} + \beta\bar{v}_1 = v(\theta_2(\bar{u}))$$

Their derivatives are

$$\theta_i'(\bar{u}) = -\frac{\theta_i(\bar{u})}{\bar{u} - u_0(\theta_i(\bar{u}))}, i = 1, 2$$

The change in date 0 consumption induced by this deviation is

$$\Delta C_0(\bar{u}, \theta^*) = \int_{\theta_1(\bar{u})}^{\theta^*} [c_0(\bar{u}) - c_0(\theta)] f(\theta)\, \mathrm{d}\theta - \int_{\theta^*}^{\theta_2(\bar{u})} [c_0(\theta) - c_0(\bar{u})] f(\theta)\, \mathrm{d}\theta$$

Taking derivatives,

$$\Delta C_0'(\bar{u}, \theta^*) = c_0'(\bar{u})[F(\theta_2(\bar{u})) - F(\theta_1(\bar{u}))] + f(\theta_1(\bar{u}))\theta_1(\bar{u})\frac{c_0(\bar{u}) - c_0(\theta_i(\bar{u}))}{\bar{u} - u_0(\theta_1(\bar{u}))} - \theta_2(\bar{u})\frac{c_0(\bar{u}) - c_0(\theta_2(\bar{u}))}{\bar{u} - u_0(\theta_2(\bar{u}))}$$

All these terms vanish as $\bar{u} \to u(\theta^*)$, so a small deviation has no first order effect on date 0 consumption.

The change in date 2 consumption induced by this deviation is

$$\Delta C_0(\bar{u}, \theta^*) = \int_{\theta_1(\bar{u})}^{\theta^*} [\underline{c}_2 - c_2(\theta)] f(\theta)\, \mathrm{d}\theta$$

The derivative of the change in date 2 consumption is

$$\begin{aligned}\Delta C_2'(\bar{u}, \theta^*) &= f(\theta_1(\bar{u}))\theta_1(\bar{u})\frac{\underline{c}_2 - c_2(\theta_1(\bar{u}))}{\bar{u} - u_0(\theta_1(\bar{u}))} \\ &= f(\theta_1(\bar{u}))\theta_1(\bar{u})\frac{\underline{c}_2 - c_2(\theta_1(\bar{u}))}{c_0(\theta^*) - c_0(\theta_1(\bar{u}))}\frac{c_0(\theta^*) - c_0(\theta_1(\bar{u}))}{c_0(\bar{u}) - c_0(\theta_1(\bar{u}))}\frac{c_0(\bar{u}) - c_0(\theta_1(\bar{u}))}{\bar{u} - u_0(\theta_1(\bar{u}))}\end{aligned}$$

The first fraction is equal to a negative constant, and the third fraction converges to $C_0'(u(\theta^*)) > 0$. Consider the middle term,

$$\frac{g(\bar{u})}{h(\bar{u})} := \frac{c_0(\theta^*) - c_0(\theta_1(\bar{u}))}{c_0(\bar{u}) - c_0(\theta_1(\bar{u}))}$$

. Both $g(\bar{u})$ and $h(\bar{u})$ converge to zero as $\bar{u} \to u_0(\theta^*)$.

$$g'(\bar{u}) = c_0'(\theta_1(\bar{u}))\frac{\theta_1(\bar{u})}{\bar{u} - u_0(\theta_i(\bar{u}))}$$
$$h'(\bar{u}) = c_0'(\bar{u}) - c_0'(\theta_1(\bar{u}))\frac{\theta_1(\bar{u})}{\bar{u} - u_0(\theta_i(\bar{u}))}$$

So we have

$$\frac{g'(\bar{u})}{h'(\bar{u})} = \frac{-c_0'(\theta_1(\bar{u}))\frac{\theta_1(\bar{u})}{\bar{u}-u_0(\theta_i(\bar{u}))}}{c_0'(\bar{u}) - c_0'(\theta_1(\bar{u}))\frac{\theta_1(\bar{u})}{\bar{u}-u_0(\theta_i(\bar{u}))}} \to 1 \text{ as } \bar{u} \to u(\theta^*).$$

Thus $\Delta C_2'(\bar{u}, \theta^*)$ converges to a positive number as $\bar{u} \to u_0(\theta^*)$.

Putting this all together, we see that a small deviation has no first order effect on C_0, and causes a first order reduction in C_2. It must therefore decrease the value of the Lagrangian. So the original allocation was inefficient.

The only assumption we used here was that $u_0(\theta)$ is continuous and strictly increasing at θ^*. The above argument thus shows that such an allocation can never be optimal when the date 1 resource constraint is slack. □

P.1 Efficiency of piecewise linear equilibria

Definition P.5. *$c_0, c_1, c_2, r_1 = 0$ is a full employment piecewise linear equilibrium (FEPLE) if $\int c_t(\theta) f(\theta) = y^*$, $t = 0, 1, 2$, and there exist $R_B \geq R_S, \theta_B > \theta_S$ such that*

1. *for $\theta > \theta_B$:*

$$R_B = \frac{\theta u'(c_0(\theta))}{\beta u'(c_1(\theta))}$$
$$c_2(\theta) = y^* - (1-\beta)\phi$$
$$u'(c_1(\theta)) \geq \beta u'(c_2(\theta))$$

2. *for $\theta < \theta_S$:*

$$R_S = \frac{\theta u'(c_0(\theta))}{\beta u'(c_1(\theta))}$$
$$c_2(\theta) \geq y^* - (1-\beta)\phi$$
$$u'(c_1(\theta)) \geq \beta u'(c_2(\theta))$$

3. *for $\theta \in [\theta_S, \theta_B]$:*

$$c_0(\theta) = c_0 \text{ constant}$$
$$\frac{\theta u'(c_0(\theta))}{\beta u'(c_1(\theta))} \in (R_S, R_B)$$
$$c_2(\theta) = y^* - (1-\beta)\phi$$
$$u'(c_1(\theta)) = \beta u'(c_2(\theta))$$

Proposition P.6. *Every FEPLE is constrained efficient.*

Proof. Suppose c_0, c_1, c_2, r_0 satisfies the conditions in Definition P.5, and let $u_0 = u(c_0(\theta))$, $\underline{v} = \underline{\theta}u(c_0(\underline{\theta}) + \beta u(c_1(\underline{\theta})) + \frac{\beta^2}{1-\beta}u(c_2(\theta))$ be the associated utilities. The proof proceeds by constructing Lagrange multipliers $\lambda_0, \lambda_1, \lambda_2$ and Pareto weights $a(\theta)$ so that the sufficient conditions in Lemma 5.5 are satisfied.

Note that the proposed allocation has a discontinuity at θ_S. Furthermore, there is a set of types $[\theta_S, \theta_B]$ with positive measure who have $v_1(\theta) = \bar{v}_1(0)$, and for whom $M(v_1(\theta))$ is not differentiable.

$$\lambda_0 = \left[\int \frac{C_0'(u_0(\theta))f(\theta)}{\theta}\,\mathrm{d}\theta\right]^{-1}$$
$$a(\theta) = \frac{\lambda_0 C_0'(u_0(\theta))f(\theta)}{\theta}$$
$$\lambda_1 = \frac{\lambda_0}{R_B}$$
$$\lambda_2 = \beta^{-1/\sigma}\left[\frac{1-\beta+\beta^{1/\sigma}}{R_S(1-\beta)}\lambda_0 - \lambda_1\right]$$

Note that since $R_S < R_B$, $\lambda_1 < (1-\beta)\lambda_2$, as required.

Next, note that by construction, for $\theta < \theta_S$, $\lambda(\theta) = \dfrac{\lambda_0}{R_S}$, and

$$M'(v_1(\theta)) = \frac{\lambda_0}{R_S u'(c_1(\theta))} = \frac{\lambda_0\beta}{\theta u'(c_0(\theta))} = \frac{\beta a(\theta)}{f(\theta)} \tag{50}$$

where the second equality uses the definition of a FEPLE, and the third equality uses the construction of the Pareto weights. Similarly, for $\theta > \theta_B$, $\lambda(\theta) = \dfrac{\lambda_0}{R_B}$ and

$$M'(v_1(\theta)) = \frac{\lambda_0}{R_B u'(c_1(\theta))} = \frac{\lambda_0\beta}{\theta u'(c_0(\theta))} = \frac{\beta a(\theta)}{f(\theta)} \tag{51}$$

Finally, for $\theta \in [\theta_S, \theta_B]$, we have $v_1(\theta) = \bar{v}_1$ and

$$\frac{\beta a(\theta)}{f(\theta)} = \frac{\lambda_0\beta}{\theta u'(c_0(\theta))} \in \left[\frac{\lambda_0}{R_B u'(c_1(\theta))}, \frac{\lambda_0}{R_B u'(c_1(\theta))}\right] = \left[M_-'(\bar{v}_1), M_+'(\bar{v}_1)\right] \tag{52}$$

Define $M'(v_1(\theta)|\Delta_1(\theta)) = M_+'(v_1(\theta))$ if $\Delta_1(\theta) > 0$, $M_-'(v_1(\theta))$ if $\Delta_1(\theta) < 0$. Note that since

$$\underline{\Delta} + \int_{\underline{\theta}}^{\theta} \Delta_0(z)\,\mathrm{d}z = \theta\Delta_0(\theta) + \beta\Delta_1(\theta)$$

we can write the Gateaux differential of the Lagrangian as follows:

$$\begin{aligned}
\delta\mathcal{L}(u_0, \underline{v}; \Delta_0, \underline{\Delta}) &= \int_\Theta a(\theta)[\theta\Delta_0(\theta) + \beta\Delta_1(\theta)] - \lambda_0 \int C_0'(u_0(\theta))\Delta_0(\theta)f(\theta)\,\mathrm{d}\theta \\
&- \int M'(v_1(\theta)|\Delta_1(\theta))f(\theta)\,\mathrm{d}\theta \\
&= \int_\Theta \Delta_0(\theta)\left\{\theta a(\theta) - \lambda_0 C_0'(u_0(\theta))f(\theta)\right\}\mathrm{d}\theta + \int_\Theta \Delta_1(\theta)\left\{\beta a(\theta) - M'(v_1(\theta)|\Delta_1(\theta))f(\theta)\right\}\mathrm{d}\theta
\end{aligned}$$

The first term in curly brackets is zero, by definition of $a(\theta)$. The second term is zero except for

$\theta \in [\theta_S, \theta_B]$, by (50) and (51). So we have

$$\delta\mathcal{L}(u_0, \underline{v}; \Delta_0, \underline{\Delta}) = \int_{\theta_S}^{\theta_B} \Delta_1(\theta) \left\{ \beta a(\theta) - M'(\bar{v}_1 | \Delta_1(\theta)) f(\theta) \right\} \mathrm{d}\theta$$

By (52), $\beta a(\theta) - M'(\bar{v}_1|\Delta_1(\theta))f(\theta)$ is positive when $\Delta_1(\theta) < 0$ and negative when $\Delta_1(\theta) > 0$. It follows that

$$[\beta a(\theta) - M'(\bar{v}_1|\Delta_1(\theta))f(\theta)]\Delta_1(\theta) \leq 0, \forall \theta \in [\theta_S, \theta_B]$$

So we must have $\delta\mathcal{L}(u_0, \underline{v}; \Delta_0, \underline{\Delta}) \leq 0$, as required. □

Alternatively, the Proposition can be proved by showing that a FEPLE solves a relaxed Pareto problem without incentive constraints. Since it also satisfies the incentive constraints, it must solve the restricted Pareto problem (PP').

Proof. Consider the relaxed problem

$$\begin{aligned}
\max_{u_0, v_1, r_1} & \int a(\theta) \left[\theta u_0(\theta) + \beta v_1(\theta) \right] \mathrm{d}\theta \\
\text{s.t.} & \int C_0(u_0(\theta)) f(\theta) \, \mathrm{d}\theta \leq y^* \\
& \int C_1(v_1(\theta), r_1) f(\theta) \, \mathrm{d}\theta \leq y^* \\
& \int C_2(v_1(\theta), r_1) f(\theta) \, \mathrm{d}\theta \leq y^* \\
& r_1 \geq 0
\end{aligned}$$

and suppose that the ZLB binds at an optimum. The same arguments as above demonstrate that solutions to this problem solve a Lagrangian

$$\int a(\theta) \left[\theta u_0(\theta) + \beta v_1(\theta) \right] - \lambda_0 \int C(u_0(\theta)) f(\theta) \, \mathrm{d}\theta - \int M(v_1(\theta))$$

The first order necessary and sufficient conditions for optimality are

$$\begin{aligned}
a(\theta)\theta - \lambda_0 C'(u_0(\theta)) f(\theta) &= 0 \\
a(\theta)\beta - M'(v_1(\theta)) f(\theta) &= 0 \text{ if } v_1(\theta) \neq \bar{v}_1(0) \\
\begin{matrix} a(\theta)\beta - M'_+(v_1(\theta)) f(\theta) \leq 0, \\ a(\theta)\beta - M'_-(v_1(\theta)) f(\theta) \geq 0 \end{matrix} & \text{ if } v_1(\theta) = \bar{v}_1(0)
\end{aligned}$$

As in the previous proof, define

$$\begin{aligned}
\lambda_0 &= \left[\int \frac{C_0'(u_0(\theta)) f(\theta)}{\theta} \, \mathrm{d}\theta \right]^{-1} \\
a(\theta) &= \frac{\lambda_0 C_0'(u_0(\theta)) f(\theta)}{\theta} \\
\lambda_1 &= \frac{\lambda_0}{R_B} \\
\lambda_2 &= \beta^{-1/\sigma} \left[\frac{1 - \beta + \beta^{1/\sigma}}{R_S(1-\beta)} \lambda_0 - \lambda_1 \right]
\end{aligned}$$

By construction, a FEPLE satisfies the first order conditions with these multipliers. So a FEPLE solves a relaxed Pareto problem. As in the previous proof, any FEPLE is incentive compatible. It follows that a FEPLE solves the restricted problem (PP′). □

P.2 Existence of a FEPLE

Let $R_S = (1 + r_0)$. Define $c_0(y_S, y_B, R_B; \theta), a_1(y_S, y_B, R_B; \theta)$ as the solutions to

$$\max_{c_0, a_1} \theta u(c_0) + \beta V(a_1) \tag{53}$$

$$\text{s.t. } a_1 = \bar{a}_1 + R_S(y_S - c_0) \text{ if } c_0 \leq y_S \tag{54}$$

$$a_1 = \bar{a}_1 \text{ if } c_0 \in [y_S, y_B] \tag{55}$$

$$a_1 = \bar{a}_1 - R_B(c_0 - y_B) \text{ if } c_0 \geq y_B \tag{56}$$

where from now on we suppress dependence on r_1, since this will always equal zero (i.e. we write $V(a_1)$ for $V(a_1, 0)$, $\bar{a}_1$ for $\bar{a}_1(0)$, etc.) We also suppress dependence on R_S, since this will always be equal to $1 + r_0$. Define $c_t(y_S, y_B, R_B; \theta) = X_t(a_1(y_S, y_B, R_B; \theta))$ for $t = 1, 2$. Define the aggregate excess demand functions

$$Z_t(y_S, y_B, R_B) = \int c_t(y_S, y_B, R_B; \theta) f(\theta) \, \mathrm{d}\theta - y^*$$

If the ZLB binds in equilibrium, there exist $\bar{y}, R_S = 1 + r_0$ such that $Z_0(\bar{y}, \bar{y}, R_S) = Z_2(\bar{y}, \bar{y}, R_S) = 0$, $Z_1(\bar{y}, \bar{y}, R_S) < 0$.

Lemma P.7. *$Z_t(y_S, y_B, R_B)$ is C^1 in all its arguments for $t = 0, 1, 2$ on the set $\{y_S, y_B, R_B : y_B \geq y_S, R_B \geq R_S\}$. Z_2 is increasing in y_S, decreasing in y_B, and does not depend on R_B.*

Proof. Let s_0, s_1, s_2 be the solution to (53) subject to the constraint that $c_0 \leq y_S$, and define the associated value function $V_S(y_S; \theta)$. Let b_0, b_1, b_2 be the solution subject to the constraint that $c_0 \geq y_B$, and define the associated value function $V_B(y_B, R_B; \theta)$. These programs have continuous, differentiable, concave objective functions and linear constraints; they therefore give rise to continuous policy functions and continuous, differentiable value functions. In addition, the policy functions are differentiable when the inequality constraints $c_0 \leq y_S$, $c_0 \geq y_B$ are slack.

When $c_0 = y_B$, the solution is $y_B, \underline{c}_1, \underline{c}_2$, where $u'(\underline{c}_1) = \beta u'(\underline{c}_2)$. Let $\theta_S(y_S, y_B)$ be the type who is just indifferent between choosing some $c_0 \leq y_S$ and $c_0 = y_B$, and let $\theta_B(y_B, R_B)$ be the highest type who chooses $c_0 = y_B$. θ_S is implicitly defined by

$$V_S(y_S; \theta_S) - \theta_S u(y_B) - \beta V(\bar{a}_1) = 0$$

By the Envelope Theorem, the derivative of this expression with respect to θ_S is $u(s_0(y_S; \theta_S)) - u(y_B) \leq 0$, which is nonzero provided that $s_0(y_S; \theta_S) \neq y_B$, which will be true if $y_B > y_S$. Then by the Implict Function Theorem, this defines θ_S as a C^1 function of y_S, y_B. It can be verified that θ_S is increasing in y_S and decreasing in y_B.

I now show that Z_t is right-differentiable with respect to y_B when $y_B = y_S$. Fix y_S. Define $\hat{\theta}$ by $s_0(\hat{\theta}) = y_S$. Define

$$X(y_B) = \int_{\theta(y_B)}^{\hat{\theta}} [y_B - s_0(\theta)] f(\theta) \, \mathrm{d}\theta$$

where $\theta(y_B)$ is defined by

$$V_S(\theta) = \theta u(y_B) + \beta V(\bar{a}_1)$$

Since $\theta(y_B)$ is continuous, $X(y_B)$ is continuous. For $y_B > y_S$,

$$\theta'(y_B) = \frac{-\theta(y_B)u'(y_B)}{u(y_B) - u(s_0(\theta(y_B)))}$$

by the Implicit Function Theorem. Applying Leibniz's Theorem to X, for $y_B > y_S$ we have

$$\begin{aligned} X'(y_B) &= -\theta'(y_B)[y_B - s_0(\theta(y_B))]f(\theta(y_B)) + \int_{\theta(y_B)}^{\hat{\theta}} f(\theta)\,\mathrm{d}\theta \\ &= \theta(y_B)u'(y_B)f(\theta(y_B))\frac{y_B - s_0(\theta(y_B))}{u(y_B) - u(s_0(\theta(y_B)))} + \int_{\theta(y_B)}^{\hat{\theta}} f(\theta)\,\mathrm{d}\theta \\ \lim_{y_B \downarrow y_S} X'(y_B) &= \hat{\theta} f(\hat{\theta}) \end{aligned}$$

Thus since X is continuous, $X'_+(y_S) = \hat{\theta} f(\hat{\theta})$.

θ_B is explicitly defined by

$$\theta_B = \frac{\beta R_B u'(\bar{c}_1)}{u'(y_B)}$$

Then we have

$$\begin{aligned} Z_t(y_S, y_B, R_B) = &\int_{\underline{\theta}}^{\theta_S(y_S,y_B)} s_t(y_S;\theta)f(\theta)\,\mathrm{d}\theta + [F(\theta_B(y_B,R_B)) - F(\theta_S(y_S,y_B))]c_t^B \\ &+ \int_{\theta_B(y_B,R_B)}^{\bar{\theta}} b_t(y_B,R_B;\theta)f(\theta)\,\mathrm{d}\theta - y^* \end{aligned}$$

Provided that $s_0(y_S;\theta_S) < y_B$, we can apply Leibniz's formula to show that Z_t is C^1. If $s_0(y_S;\theta_S) = y_B$, then we must have $y_B = y_S$.

Next we show that Z_2 is increasing in y_S, decreasing in y_B, and does not depend on R_B. All types with $a_1 \leq \bar{a}_1$ are liquidity constrained, and consume $\underline{c}_2$ at date 2. Thus we have

$$Z_2(y_S, y_B, R_B) = \int_{\underline{\theta}}^{\theta_S(y_S,y_B)} s_2(y_S;\theta)f(\theta)\,\mathrm{d}\theta + [1 - F(\theta_S(y_S,y_B)]\underline{c}_2 - y^* \tag{57}$$

which clearly does not depend on R_B. θ_S is increasing in y_S and decreasing in y_B. Since $s_2(y_S;\theta_S) \geq \underline{c}_2$, it follows that Z_2 is increasing in y_S and decreasing in y_B. □

Since Z_2 does not depend on R_B, we henceforth write $Z_2(y_S, y_B)$.

Lemma P.8. *$Z_2(y_S, y_B) = 0$ defines $y_S = \varphi_S(y_B)$ as a C^1, increasing function of y_B, with $\varphi_S(\bar{y}) = \bar{y}$, $\varphi_S(y_B) \to \infty$ as $y_B \to \infty$.*

Proof. Since Z_2 is C^1, increasing in y_S and decreasing in y_B, the first part follows from the Implicit Function Theorem. Given our assumption that the ZLB binds in equilibrium, there exists $\bar{y}$ such that $Z_2(\bar{y}, \bar{y}) = 0$. To prove the last part, note that since $\bar{c}_2 < y^*$, we must have $\theta_S > \underline{\theta}$ if $Z_2 = 0$. That is, we must have $V_S(y_S;\underline{\theta}) - \theta_S u(y_B) - \beta V(\bar{a}_1) > 0$. As $y_B \to \infty$, this can only be satisfied if $y_S \to \infty$. □

Lemma P.9. *$\Phi_0(y_B, R_B) := Z_0(\varphi(y_B), y_B, R_B)$ is C^1, increasing in y_B and decreasing in R_B.*

Proof. Φ_0 is the composition of C_1 functions and is therefore C_1. It is decreasing in R_B because b_0 is decreasing in R_B.

To see that Φ_0 is increasing in y_B, suppose that $y_B' > y_B$ and $Z_2(y_S, y_B) = Z_2(y_S', y_B') = 0$. Then $y_S' > y_S$. Inspecting (57), we see that we must have $\theta_S' < \theta_S$, since s_2 is increasing in y_S. There are then two effects on Z_0. The increase in y_S and y_B increases date 0 consumption for all types. And the fall in θ_S means that some households switch from a low to a high level of date 0 consumption. The overall effect is to increase Z_0. □

Corollary P.10. *$\Phi_0(y_B, R_B) = 0$ defines $y_B = \varphi_B(R_B)$ as a continuous, increasing function of R_B.*

Proof. This follows immediately by applying the Implicit Function Theorem to the above result. □

Lemma P.11. *$\Phi_1(R_B) = Z_1(\varphi_S(\varphi_B(R_B)), \varphi_B(R_B), R_B)$ is a continuous function of R_B with $\Phi_1(R_S) < 0$, $\lim_{R_B \to \infty} \Phi_1(R_B) > 0$.*

Proof. Φ_1 is the composition of continuous functions, and is therefore continuous. Under the assumption that the ZLB binds in equilibrium, $\varphi_S(\varphi_B(R_S)) = \varphi_B(R_S) = \bar{y}$ and $Z_1(\varphi_S(\varphi_B(R_S)), \varphi_B(R_S), R_S) < 0$. For R_B sufficiently high, all types bunch at y_B and consume $\underline{c}_1 > y^*$ at date 1; it therefore follows that $\Phi_1 > 0$. □

Lemma P.12. *There exist $y_S > \bar{y}, y_B > \bar{y}$, $R_B > R_S$ such that $Z_t(y_S, y_B, R_B) = 0$, $t = 0, 1, 2$.*

Proof. $\Phi_1(R_B)$ is continuous, negative for $R_B = R_S$ and positive for high enough R_B. By the Intermediate Value Theorem, there exists R_B such that $\Phi_1(R_B) = 0$. Define $y_S = \varphi_S(\varphi_B(R_B))$, $y_B = \varphi_B(R_B)$: the result then follows. □

The following proposition is now immediate.

Proposition P.13. *Suppose the ZLB binds in equilibrium. Then there exists a FEPLE.*

Q Proof of Proposition 6.2.

1. Take any solution to the relaxed Pareto problem, and set

$$
\begin{aligned}
\tau(\theta_S) &= 0 \\
1 + \tau(\theta_B) &= \frac{u'(c_1^S)}{u'(c_1^B)} \frac{U_c(c_0^B, \theta_B)}{U_c(c_0^S, \theta_S)} \\
1 + r_0 &= \frac{U_c(c_0^S, \theta_S)}{\beta u'(c_1^S)} \\
1 + r_1 &= \frac{u'(c_1^S)}{\beta u'(c_2^S)} \\
T_1(\theta_S) = -T_1(\theta_B) = (1 + r_0)(c_0^S - y^*) + c_1^S - y^* &+ \frac{c_2^S - y^*}{(1 + r_1)(1 - \beta)}
\end{aligned}
$$

It is straightforward to show that this implements the allocation as an equilibrium with macroprudential taxes. To see that $\tau(\theta_B) = 0$ when the ZLB is slack and > 0 when the ZLB binds note that the first order conditions in the planner's problem have

$$\alpha U_c(c_0^S, \theta_S) - \lambda_0 = 0$$
$$(1-\alpha)U_c(c_0^B, \theta_B) - \lambda_0 = 0$$
$$\alpha u'(c_1^S) - \lambda_1 + \zeta u''(c_1^S) = 0$$
$$(1-\alpha)u'(c_1^B) - \lambda_1 = 0$$

2. Take any solution to the relaxed Pareto problem. Set

$$\phi_0 = \bar{c}_1^S - y^* + \phi$$
$$1 + r_0 = \frac{U_c(c_0^S, \theta_S)}{\beta u'(c_1^S)}$$
$$1 + r_1 = \frac{u'(c_1^S)}{\beta u'(c_2^S)}$$
$$T_0(\theta_S) = -T_0(\theta_B) = c_0^S - y^* + \frac{c_1^S - y^*}{1+r_0} + \frac{c_2^S - y^*}{(1+r_0)(1+r_1)(1-\beta)}$$

If the ZLB is slack, then $c_1^S < \bar{c}_1^S$, and the date 0 debt limit does not bind. If the ZLB binds, then in any solution to the planner's problem, $c_1^S = 2y^* - c_1^B = \bar{c}_1^S$, as required.

R Proof of Proposition 6.3.

The relevant first order conditions in the planner's problem are

$$\alpha U_c(c_0^S, \theta_S) - \lambda_0 + \mu_S U_c(c_0^S, \theta_S) - \mu_B U_c(c_0^S, \theta_B) = 0$$
$$(1-\alpha)U_c(c_0^B, \theta_B) - \lambda_0 - \mu_S U_c(c_0^B, \theta_S) + \mu_B U_c(c_0^B, \theta_B) = 0$$
$$\alpha u'(c_1^S) - \lambda_1 + \zeta u''(c_1^S) + (\mu_S - \mu_B)u'(c_1^S) = 0$$
$$(1-\alpha)u'(c_1^B) - \lambda_1 - (\mu_S - \mu_B)u'(c_1^B) = 0$$

Combining,

$$\frac{\alpha U_c(c_0^S, \theta_S) + \mu_S U_c(c_0^S, \theta_S) - \mu_B U_c(c_0^S, \theta_B)}{(1-\alpha)U_c(c_0^B, \theta_B) - \lambda_0 - \mu_S U_c(c_0^B, \theta_S) + \mu_B U_c(c_0^B, \theta_B)} = \frac{\alpha u'(c_1^S) + \zeta u''(c_1^S) + (\mu_S - \mu_B)u'(c_1^S)}{(1-\alpha)u'(c_1^B) - (\mu_S - \mu_B)u'(c_1^B)}$$
$$\frac{U_c(c_0^S, \theta_S)}{U_c(c_0^B, \theta_B)} \frac{1 + \frac{\mu_S}{\alpha} - \frac{\mu_B U_c(c_0^S, \theta_B)}{\alpha U_c(c_0^S, \theta_S)}}{1 - \frac{\mu_S U_c(c_0^B, \theta_S)}{(1-\alpha)U_c(c_0^B, \theta_B)} + \frac{\mu_B}{1-\alpha}} = \frac{u'(c_1^S)}{u'(c_1^B)} \frac{1 + \frac{\zeta u''(c_1^S)}{\alpha u'(c_1^S)} + \frac{\mu_S - \mu_B}{\alpha}}{1 - \frac{\mu_S - \mu_B}{\alpha}}$$

We have

$$\frac{1 - T'(d_1^B)}{1 - T'(d_1^S)} = \frac{u'(c_1^S)}{u'(c_1^B)} \frac{U_c(c_0^B, \theta_B)}{U_c(c_0^S, \theta_S)} = \frac{1 + \frac{\mu_S}{\alpha} - \frac{\mu_B U_c(c_0^S, \theta_B)}{\alpha U_c(c_0^S, \theta_S)}}{1 + \frac{\zeta u''(c_1^S)}{\alpha u'(c_1^S)} + \frac{\mu_S - \mu_B}{\alpha}} \frac{1 - \frac{\mu_S - \mu_B}{\alpha}}{1 - \frac{\mu_S U_c(c_0^B, \theta_S)}{(1-\alpha)U_c(c_0^B, \theta_B)} + \frac{\mu_B}{1-\alpha}}$$

The Proposition follows immediately.

S Proof of Proposition 6.4.

The proof of Lemma G.3 shows that any incentive compatible allocation can be implemented with a continuous, strictly decreasing function $a(c)$ (and corresponding date 0 interest rate r_0) which gives an agent's date 1 cash on hand as a function of her date 0 consumption. This function is therefore invertible. Define $T_0(d) = a^{-1}(-d) - y^* - \frac{d}{1+r_0}$. The budget set with date 0 transfers is then identical to the budget set with date 1 debt contingent transfers. Since date 1 debt contingent transfers implement efficient allocations, it follows that date 0 transfers also implement efficient allocations.

Consider the equilibrium induced by a debt limit $\phi_0 = \bar{d}_1 = \bar{c}_1^S - y^* - \phi$. If $d_1 < \bar{d}_1$ in equilibrium, the ZLB does not bind, and the equilibrium is constrained efficient. If the debt limit binds, the equilibrium satisfies

$$
\begin{aligned}
U_c(c_0^S, \theta_S) &= \beta(1+r_0)u'(\bar{c}_1^S) \\
U_c(c_0^B, \theta_S) &> \beta(1+r_0)u'(c_1^B) \\
c_t^S + c_t^B &= 2y^*, t = 0, 1 \\
\mathcal{U}(\mathbf{c}^S, \theta_S) &> \mathcal{U}(\mathbf{c}^B, \theta_S) \\
\mathcal{U}(\mathbf{c}^B, \theta_B) &> \mathcal{U}(\mathbf{c}^S, \theta_B)
\end{aligned}
$$

Set $\frac{\alpha}{1-\alpha} = \frac{U_c(c_0^B, \theta_B)}{U_c(c_0^S, \theta_S)}$, $\lambda_0 = \alpha U_c(c_0^S, \theta_S)$, $\lambda_1 = (1-\alpha)u'(c_1^B)$, $\zeta = \frac{\lambda_1 - \alpha u'(c_1^S)}{u''(c^S)} > 0$. Then the allocation satisfies the first order sufficient conditions for a solution to the Pareto problem.

For high enough θ_B, the borrowing constraint binds (in both the equilibrium without policy and under a debt limit). B's gain from the equilibrium with a debt limit is

$$
U(c_0^{B\prime}, \theta_B) - U(c_0^{B\prime}, \theta_B) + \beta[u(2y^* - \bar{c}_1^S) - u(c_1^B)]
$$

where $c_0^{B\prime}$ denotes consumption with a debt limit. In the limit as $\theta_B \to \infty$ and $U_c \to \infty$, borrowers only derive utility from date 0 consumption. Since the debt limit necessarily reduces date 0 consumption, it makes them worse off.

Part 4 of the Proposition follows immediately from our earlier characterization of constrained efficient allocations.

T Proof of Proposition 7.2.

The Pareto problem is

$$\max \alpha \mathcal{U}(\mathbf{c^S}, \theta_S) + (1-\alpha)\mathcal{U}(\mathbf{c^B}, \theta_B) \tag{58}$$

$$\text{s.t. } c_0^S + c_0^B \le 2y^* \tag{RC0}$$

$$c_1^S + c_1^B \le 2y^* \tag{RC1}$$

$$\hat{c}_1^S + \hat{c}_1^B \le 2y^* \tag{RC1'}$$

$$c_2^S + c_2^B = 2y^* \tag{RC2}$$

$$c_2^B \ge y^* - (1-\beta)\phi \tag{BC}$$

$$u'(c_1^S) \ge \beta u'(c_2^S) \tag{ZLB}$$

$$\mathcal{U}(\mathbf{c^S}, \theta_S) \ge \mathcal{U}(\mathbf{c^B}, \theta_S) \tag{ICS}$$

$$\mathcal{U}(\mathbf{c^B}, \theta_B) \ge \mathcal{U}(\mathbf{c^S}, \theta_B) \tag{ICB}$$

where

$$\mathcal{U}(\mathbf{c^i}, \theta) := U(c_0^i, \theta) + \pi\left\{\beta u(c_1^i) + \frac{\beta^2}{1-\beta}u(c_2^i)\right\} + (1-\pi)\frac{\beta}{1-\beta}u(\hat{c}_1^i)$$

Lemma T.1. *(RC2) and (RC1′) bind. $u'(c_1^i) > \beta u'(c_2^S)$ for at least one agent. If (ZLB) binds, (BC) binds. $c_1^B \le c_1^S$.*

Proof. Identical to the proofs of Lemmas 4.1-4.4. □

Lemma T.2. *At most one incentive constraint binds.*

Proof. Suppose by contradiction that $\mathcal{U}(\mathbf{c^S}, \theta_S) = \mathcal{U}(\mathbf{c^B}, \theta_S)$, $\mathcal{U}(\mathbf{c^S}, \theta_S) = \mathcal{U}(\mathbf{c^B}, \theta_S)$. By the same argument as in the proof of Lemma 4.5, $c_0^S = c_0^B$.

Suppose first that $\hat{c}_1^S = \hat{c}_1^B$, $c_1^S = c_1^B$, $c_2^S = c_2^B$. Then the argument in the proof of Lemma 4.5 applies, and we have a contradiction.

If $\mathbf{c^S} \neq \mathbf{c^B}$, then set $c_1^i = \frac{c_1^S + c_1^B}{2}$, $c_2^i = \frac{c_2^S + c_2^B}{2} = y^*$, $\hat{c}_1^i = \frac{\hat{c}_1^S + \hat{c}_1^B}{2} = y^*$. This deviation satisfies all the constraints. Since preferences are strictly concave, this increases utility, contradicting the assumption that the original allocation was optimal. □

Lemma T.3. *(RC0) binds.*

Proof. The proof is essentially identical to the proof of Lemma F.6. Forming the Lagrangian, the first order necessary conditions for a maximum are

$$\alpha U_c(c_0^S, \theta_S) - \lambda_0 + \mu_S U_c(c_0^S, \theta_S) - \mu_B U_c(c_0^S, \theta_B) = 0$$
$$(1-\alpha)U_c(c_0^B, \theta_B) - \lambda_0 - \mu_S U_c(c_0^B, \theta_S) + \mu_B U_c(c_0^B, \theta_B) = 0$$
$$\alpha u'(c_1^S) - \lambda_1 + \zeta u''(c_1^S) + (\mu_S - \mu_B)u'(c_1^S) = 0$$
$$(1-\alpha)u'(c_1^B) - \lambda_1 - (\mu_S - \mu_B)u'(c_1^B) = 0$$
$$\alpha u'(\hat{c}_1^S) - \hat{\lambda}_1 + (\mu_S - \mu_B)u'(\hat{c}_1^S) = 0$$
$$(1-\alpha)u'(\hat{c}_1^B) - \hat{\lambda}_1 - (\mu_S - \mu_B)u'(\hat{c}_1^B) = 0$$
$$\alpha u'(c_2^S) - \lambda_2 - (1-\beta)\zeta u''(c_2^S) + (\mu_S - \mu_B)u'(c_2^S) = 0$$
$$(1-\alpha)u'(c_2^B) - \lambda_2 + \psi - (\mu_S - \mu_B)u'(c_2^B) = 0$$

where $\lambda_0, \beta\pi\lambda_1, \beta(1-\pi)\hat{\lambda}_1, \frac{\beta^2}{1-\beta}\pi\lambda_2, \psi, \beta\zeta, \mu_S, \mu_B$ are the multipliers on (RC0), (RC1), (RC1′), (RC2), (BC), (ZLB), (ICS), (ICB) respectively.

Since at most one incentive constraint binds, $\mu_S, \mu_B \geq 0$, with at least one equality. It follows that either $\alpha U_c(c_0^S, \theta_S) - \lambda_0 \geq 0$, or $(1-\alpha)U_c(c_0^B, \theta_B) - \lambda_0 \geq 0$, or both. Since $U_c > 0$, this implies $\lambda_0 > 0$. Thus (RC0) binds. □

Lemma T.4. *If* (RC1) *is slack,* (ICS) *and* (ZLB) *both bind.*

Proof. Again, this follows directly from the proof of Lemma F.7. □

This concludes the proof of part 1 of Proposition 7.2. Next, I show that ever constrained efficient allocation can be implemented as an equilibrium with transfers.

Lemma T.5. *If an allocation can be implemented as an equilibrium with transfers in the incomplete markets economy, it can be implemented as an equilibrium with transfers in the complete markets economy.*

Proof. Suppose $T(d), \hat{T}(d), r_0$ implement an allocation $\mathbf{c^S}, \mathbf{c^B}$ in the incomplete markets economy. In the incomplete markets economy, consider the transfer functions

$$\begin{aligned} T(d, \hat{d}) &= T(d) \text{ if } d = \hat{d} \\ &= -\infty \text{ if } d \neq \hat{d} \\ \hat{T}(d, \hat{d}) &= \hat{T}(d) \text{ if } d = \hat{d} \\ &= -\infty \text{ if } d \neq \hat{d} \end{aligned}$$

together with the interest rates r_0, $\hat{r}_0 = r_0$. Clearly it is feasible for all households to choose the same allocation as they would in the incomplete markets economy, by setting $\hat{d} = d$. And it can never be optimal for them to do anything else, since this would incur an infinitely large consumption loss. □

With this Lemma in hand, I focus on implementation in the incomplete markets economy, without loss of generality.

Lemma T.6. *Define the date 1 value function $V(a_1^i)$ as in Lemma G.1, and define*

$$\hat{V}(\hat{a}_1^i) = \max_{\{c_t^i, d_{t+1}^i\}_{t=1}^\infty} \sum_{t=1}^{\infty} \beta^{t-1} u(c_t^i) \tag{59}$$

$$s.t. c_1^i = y_1 + \hat{a}_1^i + \frac{d_2^i}{1+r_1} \tag{60}$$

$$c_t^i = y_t - d_t^i + \frac{d_{t+1}^i}{1+r_t}, t \geq 2 \tag{61}$$

$\{c_t^i, d_{t+1}^i\}_{t=0}^\infty$ solves i's problem, given $\{y_t, r_t\}$ and $T(\cdot), \hat{T}(\cdot)$, if and only if:

1. *c_0^i, d_1^i solve*

$$\max_{c_0^i, d_1^i} U(c_0^i, \theta_i) + \beta\pi V(T(d_1^i) - d_1^i) + \beta(1-\pi)\hat{V}(\hat{T}(d_1^i) - d_1^i)$$

$$s.t.\ c_0^i = y_0 + \frac{d_1^i}{1+r_0}$$

2. $\{c_t^i, d_{t+1}^i\}_{t=1}^{\infty}$ *solve* (22), *given* $a_1^i = T(d_1^i) - d_1^i$.

3. $\{\hat{c}_t^i, \hat{d}_{t+1}^i\}_{t=1}^{\infty}$ *solve* (59), *given* $\hat{a}_1^i = \hat{T}(d_1^i) - d_1^i$.

Proof. Again, the proof is standard and is therefore omitted. □

Lemma T.7. *In any equilibrium with transfers:*

1. *for all* $t \geq 2$ *and for all* i, $r_t = r^* = \beta^{-1} - 1$, $d_t^i = d_2^i$, $c_t^i = c_2^i = y^* - (1-\beta)d_2^i$.

2. *for all* $t \geq 1$ *and for all* i, $\hat{r}_t = r^* = \beta^{-1} - 1$, $\hat{d}_t^i = \hat{d}_1^i$, $\hat{c}_t^i = \hat{c}_1^i = y^* - (1-\beta)\hat{d}_1^i$

Proof. The proof is identical to that of Lemma G.2. □

Lemma T.8. *In the two-type economy,* $\{c_t^i\}$ *can be implemented as an equilibrium with transfers if and only if there exists* r_1 *such that*

$$c_0^S + c_0^B \leq 2y^* \tag{62}$$

$$r_1 \geq 0, c_1^S + c_1^B \leq 2y^*, \text{ with at least one equality} \tag{63}$$

$$\hat{c}_1^S + \hat{c}_1^B = 2y^* \tag{64}$$

$$c_2^S + c_2^B = 2y^* \tag{65}$$

$$u'(c_1^i) \geq \beta(1+r_1)u'(c_2^i),\ c_2^i \geq y^* - (1-\beta), \text{ with at least one equality, } i = S, B \tag{66}$$

$$\mathcal{U}(\mathbf{c^S}, \theta_S) \geq \mathcal{U}(\mathbf{c^B}, \theta_S) \tag{67}$$

$$\mathcal{U}(\mathbf{c^B}, \theta_B) \geq \mathcal{U}(\mathbf{c^S}, \theta_B) \tag{68}$$

Proof. As in the proof of Lemma G.3, it is straightforward to show that these conditions are necessary for implementability, and that if an allocation satisfies these conditions, then all equilibrium conditions are satisfied - except, possibly, the condition that for $i = S, B$, c_0^i, d_1^i solve (22). Let $U^i = \mathcal{U}(\mathbf{c^i}, \theta_i)$ be the utility that each agent gets from her allocation. Define $a_1^i = c_1^i - y_1 - \dfrac{d_2^i}{1+r_1}$, $\hat{a}_1^i = \hat{c}_1^i - y^* - \dfrac{\hat{d}_2^i}{1+\hat{r}_1}$. For each $i = S, B$, define the set

$$\mathcal{V}^i = \{(c, a, \hat{a}) \in \mathbb{R}^3 : U(c, \theta_i) + \beta\pi V(a) + \beta(1-\pi)\hat{V}(\hat{a}) \leq U^i\}$$

By construction, $\mathcal{V}^i$ is a closed set and $c_0^i, a_1^i, \hat{a}_1^i$ is contained in its boundary. Let

$$\mathcal{V} = \mathcal{V}^S \cap \mathcal{V}^B = \{(c, a, \hat{a}) \in \mathbb{R}^3 : U(c, \theta_i) + \beta\pi V(a) + \beta(1-\pi)\hat{V}(\hat{a}) \leq U^i, i = S, B\}$$

be the set of allocations which both agents find weakly inferior to their equilibrium allocations. By (67) and (68), the boundary of $\mathcal{V}$ contains $c_0^S, a_1^S, \hat{a}_1^S$ and $c_0^B, a_1^B, \hat{a}_1^B$. To implement the desired equilibrium, we can offer households any subset of $\mathcal{V}$ which contains both their equilibrium allocations. Let $a(c), \hat{a}(c)$ be any functions satisfying

$$(c, a(c), \hat{a}(c)) \in \mathcal{V}, \forall x$$
$$a_1^i = a(c_0^i), \hat{a}_1^i = \hat{a}(c_0^i), i = S, B$$

It is immediate that

$$c_0^i \in \arg\max_c U(c, \theta_i) + \beta\pi V(a(c)) + \beta(1-\pi)\hat{V}(\hat{a}(c))$$

Define $T(d) = d + a\left(y_0 + \frac{d_1^i}{1+r_0}\right)$, $\hat{T}(d) = d + \hat{a}\left(y_0 + \frac{d_1^i}{1+r_0}\right)$. We have immediately that

$$c_0^i, d_1^i \in \arg\max_{c,d} U(c, \theta_i) + \beta\pi V(T(d) - d) + \beta(1-\pi)\hat{V}(\hat{T}(d) - d)$$

$$\text{s.t. } c_0^i = y_0 + \frac{d_1^i}{1+r_0}$$

Since it is clear that these transfer functions satisfy the government budget constraint, we are done. □

Lemma T.9. *In any constrained efficient allocation, S is unconstrained at date 1.*

Proof. Suppose by contradiction that S is constrained at date 1 in the crisis state: then $c_1^S < c_1^B$, $c_t^S \leq c_t^B$ for all $t \geq 2$. Since $c_0^S < c_0^B$, we must have $\hat{c}_1^S > \hat{c}_1^B$, otherwise B consumes more at all dates and states, which cannot be incentive compatible. Suppose then that $\hat{c}_1^S > \hat{c}_1^B$: I will show that this cannot be optimal. Consider the following deviation. Increase c_1^S and decrease c_1^B by $\varepsilon > 0$, and increase $\hat{c}_1^B$ and decrease $\hat{c}_1^S$ by $\hat{\varepsilon} > 0$, where $\varepsilon, \hat{\varepsilon}$ are chosen so that

$$\pi[u(c_1^S + \varepsilon) - u(c_1^B - \varepsilon)] + \frac{1-\pi}{1-\beta}[u(\hat{c}_1^S - \hat{\varepsilon}) + u(\hat{c}_1^B + \hat{\varepsilon})] = \pi[u(c_1^S) - u(c_1^B)] + \frac{1-\pi}{1-\beta}[u(\hat{c}_1^S) + u(\hat{c}_1^B)]$$

By construction, resource and incentive compatibility constraints are satisfied. For small enough ε, S is still borrowing constrained. To first order,

$$\hat{\varepsilon} = \frac{\pi(1-\beta)}{1-\pi} \frac{u'(c_1^S) + u'(c_1^B)}{u'(\hat{c}_1^S) + u'(\hat{c}_1^B)}$$

and the change in each agent's utility is

$$\Delta U^B = -\beta\pi u'(c_1^B)\varepsilon + \beta\pi u'(\hat{c}_1^B) \frac{u'(c_1^S) + u'(c_1^B)}{u'(\hat{c}_1^S) + u'(\hat{c}_1^B)} \varepsilon > 0$$

$$\Delta U^S = \beta\pi u'(c_1^S)\varepsilon - \beta\pi u'(\hat{c}_1^S) \frac{u'(c_1^S) + u'(c_1^B)}{u'(\hat{c}_1^S) + u'(\hat{c}_1^B)} \varepsilon > 0$$

where the inequalities hold because

$$\frac{u'(c_1^S)}{u'(c_1^B)} > \frac{u'(c_1^S) + u'(c_1^B)}{u'(\hat{c}_1^S) + u'(\hat{c}_1^B)} > \frac{u'(\hat{c}_1^S)}{u'(\hat{c}_1^B)}$$

So for some $\varepsilon > 0$, the deviation increases both agents' utilities and satisfies all the constraints. Thus the original allocation cannot have been optimal. □

Corollary T.10. *In the two-type economy, $\{c_t^i\}$ can be implemented as an equilibrium with transfers if and only if* (62), (64), (65), (67), (68) *are satisfied, together with*

$$u'(c_1^S) \geq \beta u'(c_2^S), c_1^S + c_1^B \leq 2y^*, \text{ with at least one equality} \tag{69}$$

$$\frac{u'(c_1^B)}{\beta u'(c_2^B)} \geq \frac{u'(c_1^S)}{\beta u'(c_2^S)}, c_2^B \geq y^* - (1-\beta)\phi, \text{ with at least one equality} \tag{70}$$

Proof. Identical to the proof of Corollary G.5. □

U Proof of Proposition 7.3.

Lemma U.1. *Suppose* $\{c_t^i, \hat{c}_t^i\}$ *solves* (58). *Define* $a_1^i = c_1^i - y_1 - \frac{d_2^i}{1+r_1}$, $\hat{a}_t^i = \hat{c}_1^i - y^* - \frac{\hat{d}_2^i}{1+\hat{r}_1}$.

Take any transfer functions $T, \hat{T}$ *and interest rate* $r_0 \geq 0$. *Define the associated net wealth functions*

$$a(c) := T((1+r_0)(c-y^*)) - (1+r_0)(c-y^*), \hat{a}(c) = \hat{T}((1+r_0)(c-y^*)) - (1+r_0)(c-y^*)$$

Sufficient conditions for $T, \hat{T}, r_0$ *to implement* $\{c_t^i, \hat{c}_t^i\}$ *are that:*

1. $a(c_0^i) = c_1^i - y_1 + \frac{c_2^i - y^*}{(1+r_1)(1-\beta)}$ *for* $i = S, B$, *and*
2. $\hat{a}(c_0^i) = \frac{\hat{c}_1^i - y^*}{1-\beta}$ *for* $i = S, B$,
3. *for all* c,

$$(c, a(c), \hat{a}(c)) \in \mathcal{V} = \mathcal{V}^S \cap \mathcal{V}^B = \{(c, a, \hat{a}) \in \mathbb{R}^3 : U(c, \theta_i) + \beta\pi V(a) + \beta(1-\pi)\hat{V}(\hat{a}) \leq U^i, i = S, B\}$$

Proof. Suppose $\{c_t^i, \hat{c}_t^i\}$ solves (58), and is therefore implementable. Let $T, \hat{T}, r_0$ satisfy the conditions in the Lemma; I show that $T, \hat{T}, r_0$ implement this allocation.

If the conditions in the Lemma are satisfied, then for each i,

$$c_0^i \in \arg\max_c U(c, \theta_i) + \beta\pi V(a(c)) + \beta(1-\pi)\hat{V}(\hat{a}(c))$$

That is,

$$c_0^i, d_1^i \in \arg\max_{c,d} U(c, \theta_i) + \beta\pi V(T(d) - d) + \beta(1-\pi)\hat{V}(\hat{T}(d) - d)$$
$$\text{s.t. } c_0^i = y_0 + \frac{d_1^i}{1+r_0}$$

Defining $d_1^i = (1+r_0)(c_0^i - y^*)$, we have

$$\begin{aligned}
\sum_{i=S,B} T(d_1^i) &= \sum_{i=S,B} a_1^i + \sum_{i=S,B} d_1^i \\
&= \sum_{i=S,B} \left(c_1^i - y_1 + \frac{c_2^i - y^*}{(1+r_1)(1-\beta)}\right) + \sum_{i=S,B} (1+r_0)(c_0^i - y^*) \\
&= 0 \\
\sum_{i=S,B} \hat{T}(d_1^i) &= \sum_{i=S,B} \hat{a}_1^i + \sum_{i=S,B} d_1^i \\
&= \sum_{i=S,B} \frac{\hat{c}_1^i - y^*}{1-\beta} + \sum_{i=S,B} (1+r_0)(c_0^i - y^*) \\
&= 0
\end{aligned}$$

So the government budget constraints are satisfied. The remaining conditions are satisfied by assumption. □

Assumption U.2. *$u'(y^* + (1-\beta)\phi)\phi$ is increasing in ϕ.*

Lemma U.3. *In any constrained efficient allocation, $a_1^B \geq \hat{a}_1^B$, $a_1^S \leq \hat{a}_1^S$.*

Recall the necessary conditions for a maximum:

Proof.

$$[\alpha + \mu_S - \mu_B]u'(c_1^S) + \zeta u''(c_1^S) = \lambda_1 = [1 - \alpha - \mu_S + \mu_B]u'(c_1^B)$$
$$[\alpha + \mu_S - \mu_B]u'(\hat{c}_1^S) = \hat{\lambda}_1 = [1 - \alpha - \mu_S + \mu_B]u'(\hat{c}_1^B)$$
$$[\alpha + \mu_S - \mu_B]u'(c_2^S) - (1-\beta)\zeta u''(c_2^S) = \lambda_2 = [1 - \alpha - \mu_S + \mu_B]u'(c_1^B) + \psi$$

It is clear that if the ZLB is slack and $\zeta = 0$, $c_1^S = \hat{c}_1^S$, $c_1^B = \hat{c}_1^B$, while if the ZLB binds, $c_1^B \geq \hat{c}_1^B$, $c_1^S \leq \hat{c}_1^S$. Finally, if the borrowing constraint

If the borrowing constraint is slack, clearly optimal allocations are the same in the two states, and $a_1^i = \hat{a}_1^i, i = S, B$. If the borrowing constraint binds but the ZLB is slack, then if (by contradiction) $a_1^i = \hat{a}_1^i$ for $i = S, B$ in the optimal allocation, we would have $c_1^B < \hat{c}_1^B$. To see this, note that borrowers' consumption when the constraint binds is

$$c^B(\phi) = y^* + a + \beta \frac{u'(y^* + (1-\beta)\phi)\phi}{u'(c_1^B)}$$

By Assumption U.2, $c_1^B(\phi)$ is increasing in ϕ. In the non-crisis state, borrowers roll over their debt, and $\hat{c}_1^B = c(-\hat{a}_1^B)$. In the crisis state, $\phi < -\hat{a}_1^B$, and $c_1^B = c(\phi) < \hat{c}_1^B$. We know it is optimal to smooth consumption across states. To implement this, we must have $a_1^B > \hat{a}_1^B$, and thus by market clearing $a_1^S < \hat{a}_1^S$.

Finally, when the ZLB binds, it is optimal to give the borrowers even higher consumption than in the non-crisis state, but their pre-transfer income is (weakly) lower, because we may have $y_1 < y^*$. Thus again, we must have $a_1^B > \hat{a}_1^B$ and $a_1^S < \hat{a}_1^S$. □

Lemma U.4. *$R(\alpha) := \dfrac{U_c(c_0^S(\alpha), \theta_S)}{\beta[\pi u'(c_1^S(\alpha)) + (1-\pi)u'(\hat{c}_1^S(\alpha))]}$ is decreasing in α on $[\alpha_S, \alpha_B]$. $T(\alpha) := R(\alpha)(y^* - c_0^S(\alpha)) - a_1^S(\alpha)$ is decreasing in α on $[\alpha_S, \alpha_B]$. There exits $\bar{\alpha} \in (\alpha_S, \alpha_B)$ such that $T(\bar{\alpha}) = 0$.*

Proof. $R(\alpha)$ is defined by

$$R(\alpha) = \frac{U_c(c_0^S(\alpha), \theta_S)}{\beta[\pi u'(c_1^S(\alpha)) + (1-\pi)u'(\hat{c}_1^S(\alpha))]}$$
$$\alpha U_c(c_0^S(\alpha), \theta_S) = (1-\alpha)U_c(2y^* - c_0^S(\alpha), \theta_S)$$
$$\alpha u'(g_1(\alpha)) = (1-\alpha)u'(2y^* - g_1(\alpha))$$
$$\alpha u'(\hat{c}_1^S(\alpha)) = (1-\alpha)u'(2y^* - \hat{c}_1^S(\alpha))$$
$$c_1^S(\alpha) = \min\{g_1(\alpha), \bar{c}_1^S\}$$

Under Assumption H.4, R is decreasing in α. Since a_1^S and c_1^S are increasing in α, and $c_0^S < y^*$, $T(\alpha)$ is increasing in α.

When $\alpha = \alpha_S$, $U(c_0^S, \theta_S) + \beta\pi V(a_1^S) + \beta(1-\pi)\hat{V}(\hat{a}_1^S) = U(c_0^B, \theta_S) + \beta\pi V(a_1^B) + \beta(1-\pi)\hat{V}(\hat{a}_1^B)$. Since these functions are concave,

$$
\begin{aligned}
U_c(c_0^S, \theta_S)(c_0^B - c_0^S) + \beta\pi V'(a_1^S)(a_1^B - a_1^S) + \beta(1-\pi)\hat{V}'(\hat{a}_1^S)(\hat{a}_1^B - \hat{a}_1^S) > 0 \\
R(\alpha)(c_0^B - c_0^S) + \frac{\pi V'(a_1^S)}{\pi V'(a_1^S) + (1-\pi)\hat{V}'(\hat{a}_1^S)}(a_1^B - a_1^S) + \frac{(1-\pi)\hat{V}'(\hat{a}_1^S)}{\pi V'(a_1^S) + (1-\pi)\hat{V}'(\hat{a}_1^S)}(\hat{a}_1^B - \hat{a}_1^S) \\
R(\alpha)(c_0^B - c_0^S) + a_1^S - a_1^B > 0 \\
T(\alpha) > 0
\end{aligned}
$$

where the third line uses Lemma U.3. An analogous argument establishes that $T(\alpha_B) < 0$. Finally, since T is clearly continuous, there exists $\bar{\alpha}$ such that $T(\alpha) = 0$. □

V Proof of Proposition 7.4.

If the borrowing constraint does not bind in equilibrium, $y_t = y^*$ in all periods and the economy enters steady state at date 1, $c_1^i = \hat{c}_1^i, i = S, B$, and

$$
\begin{aligned}
\frac{U_c(c_0^i, \theta_i)}{\beta u'(c_1^i)} = \frac{U_c(c_0^i, \theta_i)}{\beta u'(\hat{c}_1^i)} = 1 + r_0 = 1 + \hat{r}_0, i = S, B \\
u'(c_1^i) = \beta(1 + r_1)u(c_2^i), u'(\hat{c}_1^i) = \beta(1 + \hat{r}_1)u(\hat{c}_2^i), i = S, B
\end{aligned}
$$

Choose α so that $\frac{\alpha}{1-\alpha} = \frac{U_c(c_1^S, \theta_S)}{U_c(c_0^B, \theta_B)}$. It follows that the allocation satisfies the first order sufficient conditions for an optimum.

Suppose the borrowing constraint binds, but the ZLB is slack. In the complete markets economy, we have

$$
\begin{aligned}
\frac{U_c(c_0^i, \theta_i)}{\beta u'(c_1^i)} = 1 + r_0, i = S, B \\
\frac{U_c(c_0^i, \theta_i)}{\beta u'(\hat{c}_1^i)} = 1 + \hat{r}_0, i = S, B \\
u'(c_1^i) \geq \beta(1 + r_1)u(c_2^i), i = S, B
\end{aligned}
$$

hoose α so that $\frac{\alpha}{1-\alpha} = \frac{U_c(c_1^S, \theta_S)}{U_c(c_0^B, \theta_B)}$. It follows that the allocation satisfies the first order sufficient conditions for an optimum.

Choose α so that $\frac{\alpha}{1-\alpha} = \frac{U_c(c_0^B, \theta_B)}{U_c(c_0^S, \theta_S)}$. It follows that

$$
\begin{aligned}
\alpha U_c(c_0^S, \theta_S) = (1-\alpha)U_c(c_0^B, \theta_B) \\
\alpha u'(c_1^S) = (1-\alpha)u'(c_1^B) + \psi\alpha u'(\hat{c}_1^S) = (1-\alpha)u'(\hat{c}_1^B)
\end{aligned}
$$

for some $\psi \geq 0$. So the allocation satisfies the first order sufficient conditions in (58), and is Pareto optimal.

In the incomplete markets economy, $c_1^B < \hat{c}_1^B$. But in any solution to the planner's problem,

$c_1^B = \hat{c}_1^B$. So the incomplete markets equilibrium cannot be efficient. To see that debt relief is Pareto improving, take an equilibrium without policy and increase c_1^B while decreasing c_1^S. This clearly leads to a Pareto improvement, and can be implemented with debt relief.

Finally, if the ZLB binds in equilibrium (in either economy), $y_1 < y^*$, which cannot be optimal since neither incentive constraint binds.

W Proof of Proposition 7.8.

The Pareto problem is

$$\max_{\boldsymbol{x^S},\boldsymbol{x^B}} \alpha\mathcal{U}(\boldsymbol{x^S},\theta_S) + (1-\alpha)\mathcal{U}(\boldsymbol{x^B},\theta_B) \tag{71}$$

$$\text{s.t } x_0^S + x_0^B \leq 0 \tag{72}$$

$$x_1^S + x_1^B \leq 0 \tag{73}$$

$$x_2^S + x_2^B = 0 \tag{74}$$

$$u_1'(x_1^S,\theta_S) \geq \beta u_2'(x_2^S,\theta_S) \tag{75}$$

$$x_2^B \geq -(1-\beta)\phi \tag{76}$$

$$\frac{u_1'(x_1^B,\theta_B)}{\beta u_2'(x_1^B)} \geq \frac{u_1'(x_1^S,\theta_S)}{\beta u_2'(x_1^S)} \tag{77}$$

$$(x_2^B + (1-\beta)\phi)\left(\frac{u_1'(x_1^B,\theta_B)}{\beta u_2'(x_1^B)} - \frac{u_1'(x_1^S,\theta_S)}{\beta u_2'(x_1^S)}\right) = 0 \tag{78}$$

$$\mathcal{U}(\boldsymbol{x^B},\theta_B) \geq \mathcal{U}(\boldsymbol{x^S},\theta_B) \tag{79}$$

$$\mathcal{U}(\boldsymbol{x^S},\theta_S) \geq \mathcal{U}(\boldsymbol{x^B},\theta_S) \tag{80}$$

where

$$\mathcal{U}(\boldsymbol{x},\theta) := u_0(x_0,\theta) + \beta u_1(x_1,\theta) + \frac{\beta^2}{1-\beta}u_2(x_2,\theta)$$

There are two new constraints, (77) and (78). These constraints impose that if the borrowing constraint does not bind, agents must have the same marginal rate of substitution between date 1 and date 2 consumption. If neither incentive constraint binds, these new constraints do not bind for the planner, since it is already optimal to give agents the same MRS. However, if one incentive constraint binds, the planner might want to distort allocations away from the first best, giving agents different MRSs, in order to make incentive compatibility hold.[4] Then these new constraints, which restrict MRSs to be the same, will bind.

Lemma W.1. *$u_1'(x_1^i,\theta_i) > \beta u_2'(x_2^i,\theta_i)$ for at least one agent.*

Proof. If not, then $x_1^i > x_2^i$ for $i = S, B$; summing, we have $x_1^S + x_1^B > x_2^S + x_2^B = 0$, which is infeasible. □

Lemma W.2. *If (75) binds, (76) binds.*

[4] In the baseline model considered thoughout the paper, there was no difference between agents' preferences between dates 1 and 2. Thus the planner had no motive to distort the MRS between these two dates.

Proof. Suppose by contradiction that (75) binds but (76) does not. By (77), we must have $\frac{u_1'(x_1^B,\theta_B)}{\beta u_2'(x_1^B)} = \frac{u_1'(x_1^S,\theta_S)}{\beta u_2'(x_1^S)} = 1$ (since (75) binds), contradicting Lemma W.1. □

Lemma W.3. *At most one incentive constraint binds.*

Proof. First I show that if both incentive compatibility constraints hold, the allocation is weakly inferior to the autarchic allocation $x_t^i = 0, \forall i, t$. Then I show that this allocation itself cannot be optimal.

If both (80) and (79) bind, then for $i = S, B$,

$$\mathcal{U}(\mathbf{x}^S, \theta_i) = \mathcal{U}(\mathbf{x}^B, \theta_i)$$
$$\mathcal{U}(\mathbf{0}, \theta_i) \geq \mathcal{U}(\frac{1}{2}(\mathbf{x}^S + \mathbf{x}^B), \theta_i) \geq \mathcal{U}(\mathbf{x}^S, \theta_i)$$

To show that autarky is not optimal, consider the following deviation: set $x_0^B = -x_0^S = \varepsilon_0 > 0$, $x_1^S = -x_0^B = \varepsilon_1 > 0$, $x_2^S = -x_2^B = \varepsilon_2$, choosing $\varepsilon_0, \varepsilon_1$ so that

$$\frac{u_0'(0,\theta_S)}{\beta u_1'(0,\theta_S)} < \frac{\delta}{\varepsilon} < \frac{u_0'(0,\theta_B)}{\beta u_1'(0,\theta_B)}$$

and choosing ε_2 to satisfy the agents' Euler equations. This deviation increases utility for both agents, and is feasible, because it relaxes both incentive compatibility constraints. □

Lemma W.4. (RC0) *binds.*

Proof. Identical to the proof of Lemma F.6. □

Lemma W.5. *If* (RC1) *is slack,* (ICS) *and* (ZLB) *both bind.*

Proof. Identical to the proof of Lemma F.7. □

The proof that every solution to this Pareto problem can be implemented as an equilibrium with transfers has the same structure as the corresponding proof in the baseline model. I show that household optimality conditions can be expressed in recursive form, show that the economy enters steady state at date 2, show that we can represent optimality conditions using incentive compatibility constraints, and then show that certain constraints do not bind.

Definition W.6. *An* ***equilibrium with transfers*** *is a collection* $\{x_t^i, d_{t+1}^i, z_t^i, r_t\}$ *such that, given a policy* $T(\cdot)$*:*

1. *for each* $i = S, B$*,* $\{x_t^i, d_{t+1}^i\}$ *solves agent i's problem, given* $\{z_t^i, r_t\}$ *and given policy* $T(\cdot)$*:*

$$\max u_0(x_0^i, \theta_i) + \sum_{t=1}^{\infty} \beta^t u_t(x_t^i, \theta_i) \tag{81}$$
$$\text{s.t. } \frac{d_2^i}{1+r_1} = d_1^i + x_1^i + z_1^i - T(d_1^i) \tag{82}$$
$$\frac{d_{t+1}^i}{1+r_t} = d_t^i + x_t^i + z_t^i, \forall t \neq 1 \tag{83}$$
$$d_t^i \leq \phi, t \geq 2 \tag{84}$$
$$d_0^i = 0 \tag{85}$$

2. *for all t,*

$$x_t^S + x_t^B + 2z_t = 0 \tag{86}$$

$$r_t \geq 0, z_t \geq 0, r_t z_t = 0 \tag{87}$$

3. *the government budget constraint is satisfied:*

$$T(d_1^S) + T(d_1^B) = 0 \tag{88}$$

Lemma W.7. *Define the date 1 value function*

$$V_i(a_1^i, \theta_i) = \max_{\{x_t^i, d_{t+1}^i\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} u_t(x_t^i, \theta_i) \tag{89}$$

$$s.t. x_1^i = a_1^i - z_1 + \frac{d_2^i}{1+r_1} \tag{90}$$

$$x_t^i = -d_t^i - z_t + \frac{d_{t+1}^i}{1+r_t}, t \geq 2 \tag{91}$$

$$d_t^i \leq \phi, t \geq 2 \tag{92}$$

$\{x_t^i, d_{t+1}^i\}_{t=0}^{\infty}$ *solves i's problem, given* $\{z_t, r_t\}$ *and* $T(\cdot)$*, if and only if:*

1. x_0^i, d_1^i *solve*

$$\max_{c_0^i, d_1^i} u(x_0^i, \theta_i) + \beta V_i(T(d_1^i) - d_1^i)$$

$$s.t.\ x_0^i = \frac{d_1^i}{1+r_0} - z_0$$

2. $\{x_t^i, d_{t+1}^i\}_{t=1}^{\infty}$ *solve* (89)*, given* $a_1^i = T(d_1^i) - d_1^i$.

Proof. Again, the proof is standard and is therefore omitted. □

Recall the following assumptions:

Assumption W.8. *For all* t, θ*, there exists* $x_t(\theta)$ *such that* $u_t(\cdot, \theta)$ *is* C^2 *on* $(x_t(\theta), \infty)$*, with* $u_t' > 0$*,* $u_t'' < 0$*,* $\lim_{x \to x_t(\theta)} u_t'(x, \theta) = +\infty$*,* $\lim_{x \to x_t(\theta)} u_t(x, \theta) = -\infty$*.*

Assumption W.9. $\dfrac{u_0'(x_0, \theta)}{\beta u_1'(x_1, \theta)}$ *is increasing in* θ*.*

Assumption W.10. $\dfrac{u_{t+1}'(x, \theta)}{u_t'(x, \theta)} = 1$*, for all* θ, x*,* $t \geq 1$*.*

Lemma W.11. *In any equilibrium with transfers, for all* $t \geq 2$ *and for all i,* $r_t = r^* = \beta^{-1} - 1$*,* $d_t^i = d_2^i$*,* $x_t^i = x_2^i = -(1-\beta) d_2^i$*.*

Proof. First, suppose that households solve a relaxed problem in which $\phi_t = \infty$ for all $t \geq 3$. In this case, household first order conditions yield

$$u_t'(x_t^i, \theta_i) = \beta(1+r_t) u_{t+1}'(x_{t+1}^i) \phi \text{ for all } t \geq 2$$

I will show that the borrowing constraint does not bind, so households are indeed liquidity unconstrained after date 2.

If $r_t = r^*, \forall t \geq 2$, the proposed allocation uniquely satisfies these first order conditions, by assumption W.10. Suppose by contradiction that there is also an equilibrium with $r_t > r^*$ for some $t \geq 2$. Then for each household i, $x_t^i < x_{t+1}^i$. Integrating, we have $z_t = -\int x_t^i \, \mathrm{d}i > -\int x_{t+1}^i \, \mathrm{d}i = 0$. So $z_t > 0$, which implies $r_t = 0$ by the definition of equilibrium, a contradiction.

Suppose by contradiction that $r_t < r^*$. Then a similar argument implies that $z_{t+1} = -\int x_{t+1}^i \, \mathrm{d}i > 0$ and $r_{t+1} = 0$. Iterating forward, we see that we must have $r_{t+s} = 0, z_{t+s} > 0$ for all $s \geq 1$. This deflationary equilibrium is clearly Pareto inferior to an equilibrium with $z_t = 0$, so we can rule this equilibrium out when considering optimal policy.

From the budget constraints, it follows that $x_t^i = x_2^i = -(1-\beta)d_2^i$, $d_{t+1}^i = d_t^i$, for all $t \geq 2$. Since $d_2^i \leq \phi$, households' unconstrained borrowing decisions happen to satisfy the borrowing constraint, as claimed. □

Lemma W.12. *If i is constrained at date 1, $x_t^i \leq x_t^j, \forall t \geq 1$. If $\phi > 0$, the inequality is strict.*

Proof. It follows immediately from Lemma W.11 that if i is constrained, he consumes less than j in steady state:

$$x_2^i = -(1-\beta)\phi \leq (1-\beta)\phi = x_2^j$$

with strict inequality if $\phi > 0$. Since i is constrained and j is not, we have

$$\frac{u_1'(x_1^i, \theta_i)}{u_2'(x_2^i, \theta_i)} > \frac{u_1'(x_1^j, \theta_i)}{u_2'(x_2^j, \theta_i)} = \beta(1+r_1)$$

I claim that $\beta(1+r_1) < 1$. If not, then $\frac{u_1'(x_1^j, \theta_i)}{u_2'(x_2^j, \theta_i)}, \frac{u_1'(x_1^i, \theta_i)}{u_2'(x_2^i, \theta_i)} \geq 0$, which implies $x_2^i > x_1^i$, $x_2^j > x_1^j$ by Assumption W.10. Summing, we have $x_1^i + x_1^j < x_2^i + x_2^j \leq 0$. But this is a contradiction, since $r_1 > 0$ and we must have full employment.

Since $\beta(1+r_1) < 1$, it follows that $x_1^j > x_2^j \geq 0$. Since $x_1^i + x_1^j \leq 0$, we have immediately that $x_1^i < x_1^j$. □

Lemma W.13. *In any equilibrium with transfers:*

1. *If $x_0^i > x_0^j$, then $x_t^i \leq x_t^j, \forall t \geq 0$, with at least one strict inequality*
2. *If $x_0^i = x_0^j$, then $x_t^i = x_t^j, \forall t$*

Proof. Suppose $x_0^i > x_0^j$. If no agent is constrained at date 1, then the economy enters steady state and $x_1^i = x_t^i$, for all $t \geq 1$. We must have $x_1^i < x_1^j$, otherwise j would strictly prefer i's allocation. If j is constrained, we know from Lemma W.12 that i must consume more than j at every date $t \geq 1$. So j consumes less in every period, which is impossible, since then j would prefer i's allocation. If i is constrained, then j consumes more than i at every date $t \geq 1$, which is what we wanted to show.

Suppose $x_0^i = x_0^j$. If either agent is liquidity constrained at date 1, that agent consumes less in every subsequent period, and would rather choose the other agent's allocation. If neither agent is constrained, the economy enters steady state and $x_t^i = x_1^i = x_j^i = x_t^j$ in every period. Since there is full employment in steady state, $x_t^i = x_t^j = 0$. □

Lemma W.14. *In the two-type economy,* $\{c_t^i\}$ *can be implemented as an equilibrium with transfers if and only if there exists* r_1 *such that*

$$x_0^S + x_0^B \leq 0 \tag{93}$$

$$r_1 \geq 0, x_1^S + x_1^B \leq 0, \text{ with at least one equality} \tag{94}$$

$$x_2^S + x_2^B = 0 \tag{95}$$

$$u_1'(x_1^i, \theta_1) \geq \beta(1+r_1)u_2'(x_2^i, \theta_2), x_2^i \geq -(1-\beta)\phi, \text{ with at least one equality, } i = S, B \tag{96}$$

$$\mathcal{U}(\boldsymbol{x^B}, \theta_B) \geq \mathcal{U}(\boldsymbol{x^S}, \theta_B) \tag{97}$$

$$\mathcal{U}(\boldsymbol{x^S}, \theta_S) \geq \mathcal{U}(\boldsymbol{x^B}, \theta_S) \tag{98}$$

Proof. First I show that these conditions are necessary for implementability. Suppose $\{x_t^i, d_{t+1}^i, r_t, z_t\}$ is an equilibrium with transfers, given some policy $T(\cdot)$. (93) and (95) are satisfied by definition. By Lemma W.11, the economy enters a steady state at date 2 with full employment, thus (94) is satisfied. (29) describes necessary conditions for optimality in the household problem. Finally, the incentive compatibility constraints (98), (97) follow from a standard mimicking argument.

Next, I show that conditions (93)-(97) are sufficient for implementability. Let $\{x_t^i\}, r_1$ satisfy these conditions. Set $2z_t = -x_t^S - x_t^B$ for all t and set $r_t = r^*$ for all $t \geq 2$. Set $d_t^i = \dfrac{-x_2^i}{1-\beta}, \forall t \geq 2$. If $z_t > 0$, set $r_0 = 0$, otherwise choose any $r_0 \geq 0$.

It is clear that all equilibrium conditions are satisfied, except, possibly, the condition that for $i = S, B$, x_0^i, d_1^i solve (89). Let $U^i = \mathcal{U}(\boldsymbol{x^i}, \theta_i)$ be the utility that each agent gets from her allocation. Define $a_1^i = x_1^i + z_1 - \dfrac{d_2^i}{1+r_1}$. For each $i = S, B$, define the set

$$\mathcal{V}^i = \{(x, a) \in \mathbb{R}^2 : u_0(x, \theta_i) + \beta V_i(a) \leq U^i\}$$

By construction, $\mathcal{V}^i$ is a closed set and x_0^i, a_1^i is contained in its boundary. Let

$$\mathcal{V} = \mathcal{V}^S \cap \mathcal{V}^B = \{(x, a) \in \mathbb{R}^2 : u_0(x, \theta_i) + \beta V_i(a) \leq U^i, i = S, B\}$$

be the set of allocations which both agents find weakly inferior to their equilibrium allocations. By (98) and (97), the boundary of $\mathcal{V}$ contains x_0^S, a_1^S and x_0^B, a_1^B. To implement the desired equilibrium, we can offer households any subset of $\mathcal{V}$ which contains both their equilibrium allocations. Let $a(x)$ be any function satisfying

$$(x, a(x)) \in \mathcal{V}, \forall x$$
$$a_1^i = a(x_0^i), i = S, B$$

It is immediate that

$$x_0^i \in \arg\max_x u_0(x, \theta_i) + \beta V_i(a(x))$$

Define $T(d) = d + a\left(\frac{d_1^i}{1+r_0} - z_0\right)$. We have immediately that

$$x_0^i, d_1^i \in \arg\max_{x,a} u_0(x, \theta_i) + \beta V_i(T(d) - d)$$
$$\text{s.t. } x_0^i = \frac{d_1^i}{1+r_0} - z_0$$

Since it is clear that these transfer functions satisfy the government budget constraint, we are done. □

Lemma W.15. *Suppose $\theta_B > \theta_S$. Let $\boldsymbol{b}, \boldsymbol{s} = \{b_t\}_{t=0}^{\infty}, \{s_t\}_{t=0}^{\infty}$ be two allocations with $b_0 > s_0$, $b_t \leq s_t, \forall t \geq 1$, with strict inequality for $t = 1$. If $U(\boldsymbol{b}, \theta_S) \geq U(\boldsymbol{s}, \theta_S)$, then $U(\boldsymbol{b}, \theta_B) > U(\boldsymbol{s}, \theta_B)$.*

Proof. Suppose not, and $U(\boldsymbol{b}, \theta_B) \leq U(\boldsymbol{s}, \theta_B)$. By continuity, there exists $\bar{\theta} \in [\theta_S, \theta_B)$ such that $U(\boldsymbol{b}, \bar{\theta}) = U(\boldsymbol{s}, \bar{\theta})$. There exists some isoutility curve $\{(x_0(\tau), x_1(\tau), ...) | \tau \in [0,1]\}$ linking $\boldsymbol{s}$ and $\boldsymbol{b}$, with $\boldsymbol{x}(0) = \boldsymbol{s}$, $\boldsymbol{x}(1) = \boldsymbol{b}$, $x_0'(\tau) > 0$, $x_t'(\tau) \leq 0, \forall t \geq 1$, with at least one strict inequality, such that $U(\boldsymbol{x}(\tau), \bar{\theta})$ is constant for all $\tau \in [0,1]$. That is, for all $\tau \in [0,1]$, we have

$$\frac{d}{d\tau} U(\boldsymbol{x}(\tau), \bar{\theta}) = 0$$
$$u_0'(x_0(\tau), \bar{\theta}) x_0'(\tau) + \sum_{t=1}^{\infty} \beta^t u_t'(x_t(\tau), \bar{\theta}) x_t'(\tau) = 0$$
$$x_0'(\tau) + \sum_{t=1}^{\infty} \beta^t \frac{u_t'(x_t(\tau), \bar{\theta})}{u_0'(x_0(\tau), \bar{\theta})} x_t'(\tau) = 0$$
$$x_0'(\tau) + \sum_{t=1}^{\infty} \beta^t \frac{u_t'(x_t(\tau), \theta_B)}{u_0'(x_0(\tau), \theta_B)} x_t'(\tau) > 0$$

where the last line uses Assumption W.9. Multiplying by $u_0'(x_0(\tau), \theta_B)$ and integrating, we have

$$u_0'(x_0(\tau), \theta_B) x_0'(\tau) + \sum_{t=1}^{\infty} \beta^t u_t'(x_t(\tau), \theta_B) x_t'(\tau) > 0$$
$$\int_0^1 u_0'(x_0(\tau), \theta_B) x_0'(\tau) \, \mathrm{d}\tau + \sum_{t=1}^{\infty} \beta^t \int_0^1 u_t'(x_t(\tau), \theta_B) x_t'(\tau) \, \mathrm{d}\tau > 0$$
$$U(\boldsymbol{b}, \theta_B) - U(\boldsymbol{s}, \theta_B) > 0$$

a contradiction. So we must have $U(\boldsymbol{b}, \theta_B) > U(\boldsymbol{s}, \theta_B)$. □

Lemma W.16. *In any implementable allocation, $x_0^B \geq x_0^S$. If $x_0^B > x_0^S$, then $x_t^B \leq x_t^S$, for all $t \geq 1$, with at least one strict inequality. If $x_0^B = x_0^S$, then $x_t^B = x_t^S$ for all t. S is unconstrained at date 1.*

Proof. The first part follows from Lemma W.13 and Lemma W.15. Suppose by contradiction that there exists an implementable allocation in which S is constrained. Then by Lemma W.12, $x_t^S \leq x_t^B$, for all $t \geq 1$. We know that $x_0^B \geq x_0^S$. If any one of these inequalities is strict, S prefers B's allocation, which contradicts the assumption that the allocation is implementable. If $x_t^S = x_t^B$ for all t, S is unconstrained. □

Corollary W.17. *$\{x_t^i\}$ can be implemented as an equilibrium with transfers if and only if (93), (95), (98), (79) are satisfied, together with*

$$u_1'(x_1^S,\theta_S) \geq \beta u_2'(x_2^S,\theta_S), x_1^S + x_1^B \leq 0, \text{ with at least one equality} \tag{99}$$

$$\frac{u_1'(x_1^B,\theta_B)}{\beta u_2'(x_2^B,\theta_B)} \geq \frac{u_1'(x_1^S,\theta_S)}{\beta u_2'(x_2^S,\theta_S)}, x_2^B \geq -(1-\beta)\phi, \text{ with at least one equality} \tag{100}$$

Proof. The proof follows exactly the proof of Corollary G.5. □

Lemma W.18. *Suppose $\{x_t^i\}$ solves the Pareto problem (71). Then (99) and (100) are satisfied.*

Proof. (100) is satisfied by construction. (99) is satisfied by Lemma W.5. □

Next, I show that debt relief implements some constrained efficient allocations.

Assumption W.19. *The economy has a unique equilibrium.*

Fix θ_S, θ_B. Let $\tilde{x}_t^i(\alpha)$ denote allocations which solve a relaxed Pareto problem without incentive constraints and without the ZLB. Define the net transfer from borrowers to savers in this relaxed Pareto problem as

$$\tilde{T}(\alpha) = \frac{u_0'(\tilde{x}_0^S(\alpha),\theta_S)}{\beta u_1'(\tilde{x}_1^S(\alpha),\theta_S)}\tilde{x}_0^B(\alpha) + \tilde{a}_1^B(\alpha)$$

Since the ZLB binds in the original Pareto problem if x_1^S is large enough, and x_1^S is increasing in α, there exists α_{ZLB} (which may equal 1) such that the ZLB binds if $\alpha > \alpha_{ZLB}$. Assumption W.19 implies that $\tilde{T}(\alpha) = 0$ has at most one solution in $[0, \alpha_{ZLB}]$ (otherwise both solutions would be competitive equilibria.

When the ZLB binds, the solution to the relaxed Pareto problem is $\tilde{x}_0^i(\alpha), \bar{x}_1^i, \bar{x}_2^i, i = S, B$, where $\bar{x}_2^S = (1-\beta)\phi$, $u_1'(\bar{x}_1^S,\theta_S) = \beta u_2'(\bar{x}_2^S,\theta_S)$, $\bar{x}_t^B = -\bar{x}_t^S, t = 1,2$. Define

$$T_{ZLB}(\alpha) = \frac{u_0'(\tilde{x}_0^S(\alpha),\theta_S)}{\beta u_1'(\bar{x}_1^S,\theta_S)}\tilde{x}_0^B(\alpha) + \bar{a}_1^B;$$

it follows that T_{ZLB} is strictly decreasing in α. Define $T(\alpha) = T_{ZLB}(\alpha)$ if $\alpha \leq \alpha_{ZLB}$, $T(\alpha) = \tilde{T}(\alpha)$ if $\alpha > \alpha_{ZLB}$. An identical argument to that in the proof of Lemma H.5 shos that $T(\alpha_S) > 0$, $T(\alpha_B) < 0$. It follows that there exists $\bar{\alpha}$ such that $T(\bar{\alpha}) = 0$, $T(\alpha) > 0$ for $\alpha < \bar{\alpha}$, $T(\alpha) < 0$ for $\alpha > \bar{\alpha}$.

Lemma W.20. *Constrained efficient allocations with $T(\alpha) > 0$ can be implemented with debt relief. Constrained efficent allocations with $T(\alpha) < 0$ can be implemented with a savings subsidy.*

Proof. As in Lemma H.3. □

Part 3 of the Proposition follows.

Finally, the proof of part 4 is essentially identical to the proof of Proposition 4.7 presented above, and is omitted.

X Proof of Proposition 7.9.

Let

$$\mathcal{U}(c^i, h^i, \theta_i) := \theta_i U(c_0^i, h_0^i) + \beta U(c_1^i, h_1^i) + \frac{\beta^2}{1-\beta} U(c_2^i, h_2^i)$$

We have a Pareto problem

$$\max \alpha \mathcal{U}(\mathbf{c^S}, \mathbf{h^S}, \theta_S) + (1-\alpha)\mathcal{U}(\mathbf{c^B}, \mathbf{h^B}, \theta_B) \tag{101}$$

$$c_t^S + c_t^B = h_t^S + h_t^B, t = 0,1,2 \tag{102}$$

$$U_c(c_t^S, h_t^S) + U_h(c_t^S, h_t^S) = 0, t = 0,2 \tag{103}$$

$$U_c(c_t^B, h_t^B) + U_h(c_t^B, h_t^B) = 0, t = 0,2 \tag{104}$$

$$U_c(c_1^S, h_1^S) \geq \beta U_c(c_2^S, h_2^S) \tag{105}$$

$$\frac{U_c(c_1^B, h_1^B)}{\beta U_c(c_2^B, h_2^B)} \geq \frac{U_c(c_1^S, h_1^S)}{\beta U_c(c_2^S, h_2^S)} \tag{106}$$

$$\left(\frac{U_c(c_1^B, h_1^B)}{\beta U_c(c_2^B, h_2^B)} - \frac{U_c(c_1^S, h_1^S)}{\beta U_c(c_2^S, h_2^S)}\right)(c_2^B - h_2^B - \frac{r^*}{1+r^*}\phi) = 0 \tag{107}$$

$$c_2^B \geq h_2^B - (1-\beta)\phi \tag{108}$$

$$\mathcal{U}(c^S, h^S, \theta_S) \geq \mathcal{U}(c^B, h^B, \theta_S) \tag{109}$$

$$\mathcal{U}(c^B, h^B, \theta_B) \geq \mathcal{U}(c^S, h^S, \theta_B) \tag{110}$$

I will confine attention to the case in which the borrowing constraint binds. Note that I impose that labor supply must be efficient at date 0, as well as in the steady state.

Definition X.1. *An equilibrium with transfers is a collection* $\{c_t^i, d_t^i, h_t^i, r_t, w_t, \pi_t\}_{t=0}^{\infty}$ *such that, given a policy* $T(d), \tau(d)$,

1. *for each* i*, given* $\{r_t, w_t\}$ *and given policy,* $\{c_t^i, d_t^i, h_t^i\}$ *solves*

$$\max \theta_i U(c_0^i, h_0^i) + \sum_{t=1}^{\infty} \beta^t U(c_t^i, h_t^i)$$

$$s.t.\ c_t^i + d_t^i = w_t h_t^i + \pi_t + \frac{d_{t+1}^i}{1+r_t}, t \neq 1$$

$$c_1^i + d_1^i = T(d_1^i) + (1-\tau(d_1^i))w_1 h_1^i + \pi_1 + \frac{d_2^i}{1+r_1}$$

$$d_t^i \leq \phi, t \geq 2$$

$$d_0^i = 0$$

2. *firms' profits are* $2\pi_t = (1-w_t)(h_t^S + h_t^B)$, $\forall t$

3. *markets clear:*

$$c_t^S + c_t^B = h_t^S + h_t^B, \forall t$$

4. $w_t \leq 1$, $r_t \geq 0$, $r_t(1-w_t) = 0$.

Lemma X.2. *Define the date 1 value function*

$$V(a_1^i) = \max_{\{c_t^i, h_t^i, d_{t+1}^i\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} U(c_t^i, h_t^i) \tag{111}$$

$$s.t. c_1^i = a_1^i + (1 - \tau(a_t^i)) w_1 h_1 + \pi_1 + \frac{d_2^i}{1 + r_1} \tag{112}$$

$$c_t^i = w_t h_t + \pi_t - d_t^i + \frac{d_{t+1}^i}{1 + r_t}, t \geq 2 \tag{113}$$

$$d_t^i \leq \phi, t \geq 2 \tag{114}$$

$\{c_t^i, h_t^i, d_{t+1}^i\}_{t=0}^{\infty}$ *solves i's problem, given* $\{w_t, r_t\}$ *and* $T(\cdot), \tau(\cdot)$, *if and only if:*

1. c_0^i, h_0^i, d_1^i *solve*

$$\max_{c_0^i, h_0^i, d_1^i} \theta_i U(c_0^i, h_0^i) + \beta V(T(d_1^i) - d_1^i)$$

$$s.t.\ c_0^i = w_0 h_0 + \frac{d_1^i}{1 + r_0}$$

2. $\{c_t^i, h_t^i, d_{t+1}^i\}_{t=1}^{\infty}$ *solve* (111), *given* $a_1^i = T(d_1^i) - d_1^i$.

Proof. Again, the proof is standard and is therefore omitted. □

Lemma X.3. *In any equilibrium with transfers, for all* $t \geq 2$ *and for all* i, $r_t = r^* = \beta^{-1} - 1$, $w_t = 1$, $d_t^i = d_2^i$, $c_t^i = c_2^i = h_2 - (1 - \beta) d_2^i$.

Proof. First, suppose that households solve a relaxed problem in which $\phi_t = \infty$ for all $t \geq 3$. In this case, household first order conditions yield

$$U_c(c_t^i, h_t^i) = \beta(1 + r_t) U_c(c_{t+1}^i, h_{t+1}^i) \text{ for all } t \geq 2$$

I will show that the borrowing constraint does not bind, so households are indeed liquidity unconstrained after date 2.

If $r_t = r^*, \forall t \geq 2$, then $w_t = 1, \forall t \geq 2$, and $-U_h = -U_c$, which defines h as a function $h(c)$ of c. Since $\beta(1 + r_t) = 1$, marginal utility must be constant, so consumption and labor supply must also be constant. Then budget constraints impose that $c_t^i = c_2^i = h_2 - (1 - \beta) d_2^i$, as claimed.

Suppose by contradiction that there is also an equilibrium with $r_t > r^*$ for some $t \geq 2$. Then for each household i, $U_c(c_t^i, h_t^i) > c_{t+1}^i$. Integrating, we have $y_t = \int c_t^i \, \mathrm{d}i < \int c_{t+1}^i \, \mathrm{d}i = y^*$. So $y_t < y^*$, which implies $r_t = 0$ by the definition of ZLB-constrained equilibrium, a contradiction.

Suppose by contradiction that $r_t < r^*$. Then a similar argument implies that $y_{t+1} = \int c_{t+1}^i \, \mathrm{d}i < y^*$ and $r_{t+1} = 0$. Iterating forward, we see that we must have $r_{t+s} = 0, y_{t+s} < y^*$ for all $s \geq 1$. This deflationary equilibrium is clearly Pareto inferior to an equilibrium with $y_t = y^*$, so we can rule this equilibrium out when considering optimal policy.[5]

From the budget constraints, it follows that $c_t^i = c_2^i = y^* - (1 - \beta) d_2^i$, $d_{t+1}^i = d_t^i$, for all $t \geq 2$. Since $d_2^i \leq \phi$, households' unconstrained borrowing decisions happen to satisfy the borrowing constraint, as claimed. □

[5]Equivalently, we could append to our definition of equilibrium the condition that $\lim_{t \to \infty} y_t = y^*$.

Lemma X.4. *$\{c_t, h_t\}$ can be implemented as an equilibrium with transfers if and only if there exists $r_1 \geq 0$ such that*

$$c_t^i = c_2^i, h_t^i = h_2^i, t > 2 \quad (115)$$

$$c_t^S + c_t^B = h_t^S + h_t^B, \forall t \quad (116)$$

$$U_c(c_2^i, h_2^i) + U_h(c_2^i, h_2^i) = 0, i = S, B \quad (117)$$

$$-\frac{U_h(c_0^S, h_0^S)}{U_c(c_0^S, h_0^S)} = -\frac{U_h(c_0^B, h_0^B)}{U_c(c_0^B, h_0^B)} \quad (118)$$

$$U_c(c_1^i, h_1^i) \geq \beta(1 + r_1)U_c(c_2^i, h_2^i), c_2^i \geq h_2^i - (1 - \beta)\phi, \text{ with at least one equality, } i = S, B \quad (119)$$

$$U(c^S, h^S, \theta_S) \geq U(c^B, h^B, \theta_S) \quad (120)$$

$$U(c^B, h^B, \theta_B) \geq U(c^S, h^S, \theta_B) \quad (121)$$

Proof. Take any equilibrium with transfers. From Lemma X.3, we know the first three conditions hold. (118) follows from households' date 0 first order conditions, given that they face the same wage. (119) follows from households' date 1 problem. Finally, the incentive compatibility conditions (120), (121) hold by a standard mimicking argument.

Next, we show that these conditions are sufficient for the allocation to be implementable. Suppose we have an allocation $\{c_t, h_t\}$ and associated $r_1 \geq 0$ such that these conditions hold. Set $r_t = r^*, w_t = 1$ for all $t \geq 2$. Set $w_0 = -\frac{U_h(c_0^S, h_0^S)}{U_c(c_0^S, h_0^S)}$. If $w_0 < 1$, set $r_0 = 0$; otherwise, choose any $r_0 \geq 0$. Set $w_1 = 1$ and let $\tau(a)$ be any continuous function such that $\tau(a_1^i) = 1 + \frac{U_h(c_1^i, h_1^i)}{U_c(c_1^i, h_1^i)}$, $i = S, B$, where for each i, we define

$$a_1^i = c_1^i - h_1^i + \frac{c_2^i - h_2^i}{(1 + r_1)(1 - \beta)}$$

It remains to show that each household solves (111). Given prices, and given the transfer function τ, the argument in Lemma G.3 applies directly, and the incentive compatibility conditions (120), (121) imply this. So we are done. □

As in the previous sections, it is straightforward to show that any solution to the Pareto problem satisfies the conditions in Lemma X.4.

Y Proof of Proposition 7.10.

If the borrowing constraint binds, $c_2^S, c_2^B, h_2^S, h_2^B$ are pinned down by (104), (102) and (108). Let $W(x)$ solve

$$W(x) = U(c, h)$$
$$U_c(c, h) + U_h(c, h) = 0 c - h = x$$

Then we can write the Pareto problem as

$$\max_{x_0^S, c_1^S, h_1^S, x_0^B, c_1^B, h_1^B} \alpha\theta_S W(x_0^S) + \alpha\beta U(c_1^S, h_1^S) + (1-\alpha)\theta_B W(x_0^B) + (1-\alpha)\beta U(c_1^B, h_1^B)$$

$$x_0^S + x_0^B = 0$$
$$c_1^S + c_1^B = h_1^S + h_1^B$$
$$U_c(c_1^S, h_1^S) \geq \beta U_c(c_2^S, h_2^S)$$
$$\theta_S W(x_0^S) + \beta U(c_1^S, h_1^S) + \Delta \geq \theta_S W(x_0^B) + \beta U(c_1^B, h_1^B)$$
$$\theta_B W(x_0^B) + \beta U(c_1^B, h_1^B) \geq \theta_B W(x_0^S) + \beta U(c_1^S, h_1^S) + \Delta$$

where $\Delta := \dfrac{\beta^2}{1-\beta}[U(c_2^S, h_2^S) - U(c_2^B, h_2^B)]$ is fixed. First order necessary and sufficient conditions for an optimum are

$$(\alpha\theta_S + \mu_S - \mu_B)W'(x_0^S) = \lambda_0$$
$$((1-\alpha)\theta_B - \mu_S + \mu_B)W'(x_0^B) = \lambda_0$$
$$(\alpha + \mu_S - \mu_B)U_c(c_1^S, h_1^S) + \zeta U_{cc}(c_1^S, h_1^S) = \lambda_1$$
$$-(\alpha + \mu_S - \mu_B)U_h(c_1^S, h_1^S) + \zeta U_{ch}(c_1^S, h_1^S) = \lambda_1$$
$$(1 - \alpha - \mu_S + \mu_B)U_c(c_1^B, h_1^B) = \lambda_1$$
$$-(1 - \alpha - \mu_S + \mu_B)U_h(c_1^B, h_1^B) = \lambda_1$$

First, suppose no incentive constraints bind, $\mu_S = \mu_B = 0$. The first two equations then define x_0^S, x_0^B as (respectively) strictly increasing and strictly decreasing functions of α. S's date 1 utility is also weakly increasing in α. S's net gain from choosing his own allocation is

$$\theta_S[W(x_0^S(\alpha)) - W(x_0^B(\alpha))] + \beta[U_1^S(\alpha) - U_1^B(\alpha)] + \Delta$$

which is increasing in α, and is positive for sufficiently small α, so there exists $\alpha_S > 0$ such that ICS binds if $\alpha < \alpha_S$. An analogous argument shows that there exists α_B such that ICB binds if $\alpha > \alpha_B$.

If $\lambda_1 > 0$, we have

$$(1 + \tau(d_1^S)) = -\frac{U_h(c_1^S, h_1^S)}{U_c(c_1^S, h_1^S)} = \frac{\alpha + \mu_S - \mu_B + \zeta \frac{U_{cc}(c_1^S, h_1^S)}{U_c(c_1^S, h_1^S)}}{\alpha + \mu_S - \mu_B + \zeta \frac{U_{ch}(c_1^S, h_1^S)}{U_h(c_1^S, h_1^S)}}$$

Under the regularity condition $\dfrac{U_{ch}}{U_h} > \dfrac{U_{cc}}{U_c}$, the labor wedge $\tau(d_1^S) > 0$. With quasilinear preferences, $\dfrac{U_{ch}}{U_h} = \dfrac{U_{cc}}{U_c}$, and the labor wedge is zero.

If $\lambda_1 = 0$, $\alpha + \mu_S - \mu_B = 1$ and $U_c(c_1^S, h_1^S) + \zeta U_{cc}(c_1^S, h_1^S) = -U_h(c_1^S, h_1^S) - \zeta U_{ch}(c_1^S, h_1^S) = 0$, and so

$$(1 + \tau(d_1^S)) = \frac{-U_h}{U_c} = 1 + \zeta \frac{U_{cc} + U_{ch}}{U_c}$$

Under the regularity condition that $U_{cc} + U_{ch} < 0$, again the labor wedge is positive.

To prove part 3, note that if B faces a positive labor wedge, it must be that $\lambda_1 = 0$, which in turn can only be the case if $\mu_S > 0$ and ICS binds. (Specifically, it can only be the case if $\alpha + \mu_S - \mu_B = 1$.) To prove part 4, note that when $U_{ch} = 0$, $\lambda_1 = -(\alpha + \mu_S - \mu_B) U_h(c_1^S, h_1^S)$, which is positive when $\alpha + \mu_S - \mu_B = 1$. So we cannot have $\lambda_1 = 0$.

Z Proof of Proposition 7.11.

Again, it simplifies matters to directly assume uniqueness of equilibrium in the absence of policy.

Assumption Z.1. *The economy has a unique equilibrium.*

Fix θ_S, θ_B. First consider a relaxed Pareto problem without incentive constraints in which the ZLB never binds. In this case, labor supply is always efficient ($U_c + U_h = 0, \forall i, t$) and each household obtains utility $W(x) = \max_h U(x + h, h)$ in each period. Thus when the ZLB does not bind, the economy with endogenous labor supply is isomorphic to a special case of the economy with persistent types, and the argument presented there establishes that $\tilde{T}(\alpha) = 0$ has at most one solution in $[0, \alpha_{ZLB}]$, where

$$\tilde{T}(\alpha) = \frac{\theta_S U_c(\tilde{c}_0^S(\alpha), \tilde{h}_0^S(\alpha))}{\beta U_c(\tilde{c}_1^S(\alpha), \tilde{h}_1^S(\alpha))} \tilde{x}_0^B(\alpha) + \tilde{a}_1^B(\alpha)$$

and where a tilde denotes the solution to the relaxed Pareto problem. When the ZLB binds, $U_c(c_1^S, h_1^S)$ is fixed by the ZLB constraint, and the same argument above shows that $T_{ZLB}(\alpha)$ (defined in the obvious way) is decreasing. Define $T(\alpha) = T_{ZLB}(\alpha)$ if $\alpha \leq \alpha_{ZLB}$, $T(\alpha) = \tilde{T}(\alpha)$ if $\alpha > \alpha_{ZLB}$. Again, an identical argument to that in the proof of Lemma H.5 shows that $T(\alpha_S) > 0$, $T(\alpha_B) < 0$. It follows that there exists $\bar{\alpha}$ such that $T(\bar{\alpha}) = 0$, $T(\alpha) > 0$ for $\alpha < \bar{\alpha}$, $T(\alpha) < 0$ for $\alpha > \bar{\alpha}$.

Lemma Z.2. *Constrained efficient allocations with $T(\alpha) > 0$ can be implemented with debt relief. Constrained efficent allocations with $T(\alpha) < 0$ can be implemented with a savings subsidy.*

Proof. As in Lemma H.3. □

Again, the proof that debt relief is Pareto improving at the ZLB is essentially identical to the proof of Proposition 4.7 presented above, and is omitted.

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