Robust Sensitivity Analysis for Stochastic Systems

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We study a worst-case approach to measure the sensitivity to model misspecification in the performance analysis of stochastic systems. The situation of interest is when only minimal parametric information is available on the form of the true model. Under this setting, we post optimization programs that compute the worst-case performance measures, subject to constraints on the amount of model misspecification measured by Kullback-Leibler (KL) divergence. Our main contribution is the development of infinitesimal approximations for these programs, resulting in asymptotic expansions of their optimal values in terms of the divergence. The coefficients of these expansions can be computed via simulation, and are mathematically derived from the representation of the worst-case models as changes of measure that satisfy a well-defined class of functional fixed point equations.

Key words: sensitivity analysis; model uncertainty; nonparametric method; robust optimization

MSC2000 Subject Classification: Primary: 65C05, 62G35; Secondary: 90C31, 46G05

OR/MS subject classification: Primary: decision analysis–sensitivity, simulation–statistical analysis, statistics–nonparametric; Secondary: simulation–efficiency

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1. Introduction Any performance analysis of stochastic systems requires model assumptions that, to various extent, deviate from the truth. Understanding how these model errors affect the analysis is of central importance in stochastic modeling.

This paper concerns the robust approach to measure the impacts of model errors: Given a baseline model that is believed to reasonably approximate the truth, without any specific information on how it is misrepresented, an optimization program is imposed to evaluate the worst-case performance measure among all models that are close to the baseline in the sense of some nonparametric statistical distance, such as Kullback-Leibler (KL).

The main contribution of this paper is to bring in a new line of infinitesimal analysis for the worst-case optimization described above. Namely, taking the viewpoint that the true model is within a small neighborhood of the baseline model, we conduct an asymptotic expansion on the worst-case objective value as the statistical distance that defines the neighborhood shrinks to zero. The primary motivation for this asymptotic analysis is to handle the difficulty in direct solution of these worst-case optimizations in the context of stochastic systems driven by standard i.i.d. input processes (being non-convex and infinite-dimensional). In particular, the coefficients of our expansions are computable via simulation, hence effectively converting the otherwise intractable optimizations into simulation problems. This approach thus constitutes a tractable framework for nonparametric sensitivity analysis as the expansion coefficients capture the worst-case effect on a performance measure when the model deviates from the truth in the nonparametric space.

2. Formulation and Highlights Define a *performance measure* $E[h(\mathbf{X}_T)]$, where $h(\cdot)$ is a real-valued *cost function*, $\mathbf{X}_T = (X_1, X_2, \ldots, X_T)$ is a sequence of i.i.d. random objects each lying on the domain $\mathcal{X}$, and T is the time horizon. We assume that the cost function h can be evaluated given its argument, but does not necessarily have closed form. For example, $h(\mathbf{X}_T)$ can be the waiting time of the 100-th customer in a queueing system, where $\mathbf{X}_T$ is the sequence of interarrival and service time pairs.

Our premise is that there is a *baseline* model that is believed to approximately describe each i.i.d. X_t. The probability distribution that governs the baseline model is denoted P_0. Correspondingly, the baseline performance measure is $E_0[h(\mathbf{X}_T)]$, where $E_0[\cdot]$ is the expectation under the product measure $P_0^T = P_0 \times P_0 \times \cdots \times P_0$. On the other hand, we denote P_f as the distribution that governs the true model (which is unknown), and analogously, $E_f[\cdot]$ as the expectation under the product measure P_f^T.

We are interested in the worst (or best)-case optimizations

$$\begin{array}{ll} \max & E_f[h(\mathbf{X}_T)] \\ \text{subject to} & D(P_f \| P_0) \leq \eta \\ & X_t \overset{i.i.d.}{\sim} P_f \quad \text{for } t = 1, \ldots, T \\ & P_f \in \mathcal{P}_0 \end{array} \tag{1}$$

and

$$\begin{array}{ll} \min & E_f[h(\mathbf{X}_T)] \\ \text{subject to} & D(P_f\|P_0) \leq \eta \\ & X_t \overset{i.i.d.}{\sim} P_f \quad \text{for } t = 1, \ldots, T \\ & P_f \in \mathcal{P}_0. \end{array} \tag{2}$$

Here P_f is the decision variable. The space $\mathcal{P}_0$ denotes the set of all distributions absolutely continuous with respect to the baseline P_0. The constraint $D(P_f\|P_0) \leq \eta$ represents the η-neighborhood surrounding P_0, using KL divergence as the notion of distance, i.e.

$$D(P_1\|P_2) := \int \log \frac{dP_1}{dP_2} dP_1 = E_2 \left[\frac{dP_1}{dP_2} \log \frac{dP_1}{dP_2} \right]$$

where (dP_1/dP_2) is the likelihood ratio, equal to the Radon-Nikodym derivative of P_1 with respect to P_2, and $E_2[\cdot]$ is the expectation under P_2. In brief, the pair of optimizations (1) and (2) describes the most extreme performance measures among any P_f within η units of KL divergence from the baseline P_0.

Note that we use the notation $X_t \overset{i.i.d.}{\sim} P_f$ in (1) and (2) to highlight the assumption that X_t's are i.i.d. each with distribution P_f. This i.i.d. property deems (1) and (2) non-convex and difficult to solve in general.

The major result of this paper stipulates that when letting η go to 0, under mild assumptions on h, the optimal values of (1) and (2) can each be expressed as

$$\max / \min E_f[h(\mathbf{X}_T)] = E_0[h(\mathbf{X}_T)] + \zeta_1(P_0, h)\sqrt{\eta} + \zeta_2(P_0, h)\eta + \cdots \tag{3}$$

where $\zeta_1(P_0, h), \zeta_2(P_0, h), \ldots$ is a sequence of coefficients that can be written explicitly in terms of h and P_0.

To avoid redundancy, in the following discussion we will focus on the maximization formulation, and will then point out the adaptation to minimization formulation briefly.

3. Main Results

3.1 Single-Variable Case Consider the special case $T = 1$, namely the cost function h depends only on a single variable $X \in \mathcal{X}$ in formulation (1):

$$\begin{array}{ll} \max & E_f[h(X)] \\ \text{subject to} & D(P_f\|P_0) \leq \eta \\ & X \sim P_f \\ & P_f \in \mathcal{P}_0. \end{array} \tag{4}$$

The result for this particular case will constitute an important building block in our later development.

We make two assumptions. First, for $X \sim P_0$, we impose a finite exponential moment condition on $h(X)$:

Assumption 3.1 *The variable $h(X)$ has finite exponential moment in a neighborhood of 0 under P_0, i.e. $E_0[e^{\theta h(X)}] < \infty$ for $\theta \in (-r, r)$ for some $r > 0$.*

Second, we impose the following non-degeneracy condition:

Assumption 3.2 *The variable $h(X)$ is non-constant under P_0.*

The first assumption on the light-tailedness of $h(X)$ is in particular satisfied by any bounded $h(\mathbf{X}_T)$, which handles all probability estimation problems for instance. The second assumption ensures that the baseline distribution P_0 is not a "locally optimal" model, in the sense that there always exists an opportunity to upgrade the value of the performance measure by rebalancing the probability measure.

Under the above assumptions, we can get a very precise understanding of the objective value when η is small:

Theorem 3.1 *Let $T=1$ in formulation (1), with $h(\cdot):\mathcal{X}\to\mathbb{R}$. Suppose Assumptions 3.1 and 3.2 hold. Denote $\psi(\beta)=\log E_0[e^{\beta h(X)}]$ as the logarithmic moment generating function of $h(X)$. When $\eta>0$ is within a sufficiently small neighborhood of 0, the optimal value of (1) is given by*

$$\max E_f[h(X)]=\psi'(\beta^*) \tag{5}$$

where β^* is the unique positive solution to the equation $\beta\psi'(\beta)-\psi(\beta)=\eta$. This implies

$$\max E_f[h(X)]=E_0[h(X)]+\sqrt{2Var_0(h(X))}\eta^{1/2}+\frac{1}{3}\frac{\kappa_3(h(X))}{Var_0(h(X))}\eta+O(\eta^{3/2}) \tag{6}$$

where $Var_0(h(X))$ and $\kappa_3(h(X))$ are the variance and the third order cumulant of $h(X)$ under P_0 respectively, i.e.

$$Var_0(h(X))=E_0[(h(X)-E_0[h(X)])^2]$$
$$\kappa_3(h(X))=E_0[(h(X)-E_0[h(X)])^3].$$

We shall explain how to obtain Theorem 3.1. The first step is to transform the decision variables from the space of measures to the space of functions. Recall that P_f is assumed to be absolutely continuous with respect to P_0, and hence the likelihood ratio $L:=dP_f/dP_0$ exists. Via a change of measure, the optimization problem (4) can be rewritten as a maximization over the likelihood ratios, i.e. (4) is equivalent to

$$\begin{array}{ll} \max & E_0[h(X)L(X)] \\ \text{subject to} & E_0[L(X)\log L(X)]\leq\eta \\ & L\in\mathcal{L} \end{array} \tag{7}$$

where $\mathcal{L}:=\{L\in\mathcal{L}_1(P_0):E_0[L]=1,\ L\geq 0 \text{ a.s. }\}$, and we denote $\mathcal{L}_1(P_0)$ as the $\mathcal{L}_1$-space with respect to the measure P_0 (we sometimes suppress the dependence of X in $L=L(X)$ for convenience when no confusion arises). The key now is to find an optimal solution L^*, and investigate its asymptotic relation with η. To this end, consider the Lagrangian relaxation

$$\max_{L\in\mathcal{L}} E_0[h(X)L]-\alpha(E_0[L\log L]-\eta) \tag{8}$$

where α is the Lagrange multiplier. The solution of (8) is characterized by the following proposition:

Proposition 3.1 *Under Assumption 3.1, when $\alpha>0$ is sufficiently large, there exists a unique optimizer of (8) given by*

$$L^*(x)=\frac{e^{h(x)/\alpha}}{E_0[e^{h(X)/\alpha}]}. \tag{9}$$

This result is known (e.g. [25], [42]); for completeness we provide a proof in the appendix. With this proposition, we can prove Theorem 3.1:

Proof of Theorem 3.1. By the sufficiency result in Chapter 8, Theorem 1 in [37] (shown in Theorem A.1 in the Appendix), suppose that we can find $\alpha^*\geq 0$ and $L^*\in\mathcal{L}$ such that L^* maximizes (8) for $\alpha=\alpha^*$ and $E_0[L^*\log L^*]=\eta$, then L^* is the optimal solution for (7). We will show later that when η is close to 0, we can indeed obtain such α^* and L^*. For now, assuming that such α^* and L^* exist and that α^* is sufficiently large, the proof of (5) is divided into the following two steps:

Relation between η and α^*. By Proposition 3.1, L^* satisfies (9) with $\alpha=\alpha^*$. We have

$$\begin{aligned}\eta=E_0[L^*\log L^*]&=\frac{E_0[h(X)L^*]}{\alpha^*}-\log E_0[e^{h(X)/\alpha^*}]\\ &=\frac{\beta^*E_0[h(X)e^{\beta^*h(X)}]}{E_0[e^{\beta^*h(X)}]}-\log E_0[e^{\beta^*h(X)}]=\beta^*\psi'(\beta^*)-\psi(\beta^*)\end{aligned} \tag{10}$$

where we define $\beta^*=1/\alpha^*$, and $\psi(\beta)=\log E_0[e^{\beta h(X)}]$ is the logarithmic moment generating function of $h(X)$.

Relation between the optimal objective value and α^*. The optimal objective value is

$$E_0[h(X)L^*] = \frac{E_0[h(X)e^{h(X)/\alpha^*}]}{E_0[e^{h(X)/\alpha^*}]} = \frac{E_0[h(X)e^{\beta^* h(X)}]}{E_0[e^{\beta^* h(X)}]} = \psi'(\beta^*) \tag{11}$$

This gives the form in (5). We are yet to show the existence of a sufficiently large $\alpha^* > 0$ such that the corresponding L^* in (9) satisfies $E_0[L^* \log L^*] = \eta$. To this end, we use Taylor's expansion to write

$$\begin{aligned}
\beta\psi'(\beta) - \psi(\beta) &= \sum_{n=0}^{\infty} \frac{1}{n!}\kappa_{n+1}\beta^{n+1} - \sum_{n=0}^{\infty}\frac{1}{n!}\kappa_n\beta^n \\
&= \sum_{n=1}^{\infty}\left[\frac{1}{(n-1)!} - \frac{1}{n!}\right]\kappa_n\beta^n = \sum_{n=2}^{\infty}\frac{1}{n(n-2)!}\kappa_n\beta^n \\
&= \frac{1}{2}\kappa_2\beta^2 + \frac{1}{3}\kappa_3\beta^3 + \frac{1}{8}\kappa_4\beta^4 + O(\beta^5)
\end{aligned} \tag{12}$$

where $\kappa_n = \psi^{(n)}(0)$ is the n-th cumulant of $h(X)$ under P_0, and the remainder $O(\beta^5)$ is continuous in β. By Assumption 3.2, we have $\kappa_2 > 0$. Thus for small enough η, (12) reveals that there is a small $\beta^* > 0$ that is a root to the equation $\eta = \beta\psi'(\beta) - \psi(\beta)$. Moreover, this root is unique. This is because by Assumption 3.2, $\psi(\cdot)$ is strictly convex, and hence $(d/d\beta)(\beta\psi'(\beta) - \psi(\beta)) = \beta\psi''(\beta) > 0$ for $\beta > 0$, so that $\beta\psi'(\beta) - \psi(\beta)$ is strictly increasing.

Since $\alpha^* = 1/\beta^*$, this shows that for any sufficiently small η, we can find a large $\alpha^* > 0$ such that the corresponding L^* in (9) satisfies (10), or in other words $E_0[L^* \log L^*] = \eta$.

Next, using (12), we can invert the relation

$$\eta = \frac{1}{2}\kappa_2\beta^{*2} + \frac{1}{3}\kappa_3\beta^{*3} + \frac{1}{8}\kappa_4\beta^{*4} + O(\beta^{*5})$$

to get

$$\begin{aligned}
\beta^* &= \sqrt{\frac{2\eta}{\kappa_2}}\left(1 + \frac{2}{3}\frac{\kappa_3}{\kappa_2}\beta^* + \frac{1}{4}\frac{\kappa_4}{\kappa_2}\beta^{*2} + O(\beta^{*3})\right)^{-1/2} \\
&= \sqrt{\frac{2\eta}{\kappa_2}}\left(1 - \frac{1}{3}\frac{\kappa_3}{\kappa_2}\beta^* + O(\beta^{*2})\right) \\
&= \sqrt{\frac{2}{\kappa_2}}\eta^{1/2} - \frac{2}{3}\frac{\kappa_3}{\kappa_2^2}\eta + O(\eta^{3/2}).
\end{aligned}$$

As a result, (11) can be expanded as

$$\begin{aligned}
E_0[h(X)L^*] = \psi'(\beta^*) &= \kappa_1 + \kappa_2\beta^* + \kappa_3\frac{\beta^{*2}}{2} + O(\beta^{*3}) \\
&= \kappa_1 + \kappa_2\left(\sqrt{\frac{2}{\kappa_2}}\eta^{1/2} - \frac{2}{3}\frac{\kappa_3}{\kappa_2^2}\eta + O(\eta^{3/2})\right) + \frac{\kappa_3}{2}\left(\frac{2}{\kappa_2}\eta + O(\eta^{3/2})\right) + O(\eta^{3/2}) \\
&= \kappa_1 + \sqrt{2\kappa_2}\eta^{1/2} + \frac{1}{3}\frac{\kappa_3}{\kappa_2}\eta + O(\eta^{3/2})
\end{aligned}$$

which gives (6).

□

3.2 Finite Horizon Problems We now state our main result on formulation (1) for $T > 1$. This requires correspondences of Assumptions 3.1 and 3.2. The finite exponential moment condition is now stated as follows:

ASSUMPTION 3.3 *The cost function h satisfies $|h(\mathbf{X}_T)| \leq \sum_{t=1}^T \Lambda_t(X_t)$ for some deterministic functions $\Lambda_t(\cdot) : \mathcal{X} \to \mathbb{R}$, where each of the $\Lambda_t(X_t)$'s possesses finite exponential moment under P_0, i.e. $E_0[e^{\theta\Lambda_t(X)}] < \infty$ for θ in a neighborhood of zero.*

To state our second assumption, we introduce a function $g(\cdot) := \mathcal{G}(h)(\cdot)$ where $\mathcal{G}$ is a functional acted on h and $g := \mathcal{G}(h)$ maps from $\mathcal{X}$ to $\mathbb{R}$. This function $g(x)$ is defined as the sum of individual conditional expectations of $h(\mathbf{X}_T)$ over all time steps, i.e.

$$g(x) = \sum_{t=1}^{T} g_t(x) \tag{13}$$

where $g_t(x)$ is the individual conditional expectation at time t, given by

$$g_t(x) = E_0[h(\mathbf{X}_T)|X_t = x]. \tag{14}$$

Our second assumption is a non-degeneracy condition imposed on the random variable $g(X)$ for $X \sim P_0$:

Assumption 3.4 *The random variable $g(X)$ is non-constant under P_0.*

The following is our main result:

Theorem 3.2 *Under Assumptions 3.3 and 3.4, the optimal value of* (1) *satisfies*

$$\max E_f[h(\mathbf{X}_T)] = E_0[h(\mathbf{X}_T)] + \sqrt{2Var_0(g(X))}\eta^{1/2} + \frac{1}{Var_0(g(X))}\left(\frac{1}{3}\kappa_3(g(X)) + \nu\right)\eta + O(\eta^{3/2}) \tag{15}$$

where $Var_0(g(X))$ and $\kappa_3(g(X))$ are the variance and the third order cumulant of $g(X)$ respectively, and

$$\nu = E_0[(G(X,Y) - E_0[G(X,Y)])(g(X) - E_0[g(X)])(g(Y) - E_0[g(Y)])]. \tag{16}$$

Here $g(\cdot)$ is defined in (13) *and* (14), *and $G(\cdot,\cdot)$ is a function derived from h that is defined as*

$$G(x,y) = \sum_{t=1}^{T} \sum_{s=1,\dots,T, s\neq t} G_{ts}(x,y) \tag{17}$$

where

$$G_{ts}(x,y) = E_0[h(\mathbf{X}_T)|X_t = x, X_s = y]. \tag{18}$$

Also, X and Y are independent random variables each having distribution P_0.

The proof of Theorem 3.2 is laid out in Section 5.

3.3 Extension to Random Time Horizon Problems Theorem 3.2 can be generalized to some extent to problems involving a random time horizon τ. Consider the cost function $h(\mathbf{X}_\tau)$ that depends on the sequence $\mathbf{X}_\tau = (X_1, X_2, \dots, X_\tau)$. Formulation (1) is replaced by

$$\begin{array}{ll} \max & E_f[h(\mathbf{X}_\tau)] \\ \text{subject to} & D(P_f \| P_0) \leq \eta \\ & X_t \overset{i.i.d.}{\sim} P_f \quad \text{for } t = 1, 2, \dots \\ & P_f \in \mathcal{P}_0 \end{array} \tag{19}$$

where $E_f[\cdot]$ is the corresponding expectation with respect to $\mathbf{X}_\tau$.

To state the result in this direction, we impose either a boundedness or an independence condition on τ:

Assumption 3.5 *The random time τ is a stopping time with respect to $\{\mathcal{F}_t\}_{t\geq 1}$, a filtration that supersets the filtration generated by the sequence $\{X_t\}_{t\geq 1}$, namely $\{\mathcal{F}(X_1, \dots, X_t)\}_{t\geq 1}$. Moreover, τ is bounded a.s. by a deterministic time T. The cost function h satisfies $|h(\mathbf{X}_\tau)| \leq \sum_{t=1}^{T} \Lambda_t(X_t)$ a.s. for some deterministic functions $\Lambda_t(\cdot)$, where $\Lambda_t(X_t)$ each possesses finite exponential moment, i.e. $E_0[e^{\theta\Lambda_t(X)}] < \infty$, for θ in a neighborhood of zero.*

Assumption 3.6 *The random time τ is independent of the sequence $\{X_t\}_{t\geq 1}$, and has finite second moment under P_0, i.e. $E_0\tau^2 < \infty$. Moreover, the cost function $h(\mathbf{X}_\tau)$ is bounded a.s..*

Next, we also place a non-degeneracy condition analogous to Assumption 3.4. We define $\tilde{g}:\mathcal{X}\to\mathbb{R}$ as

$$\tilde{g}(x)=\sum_{t=1}^{\infty}\tilde{g}_t(x) \tag{20}$$

where $\tilde{g}_t(x)$ is given by

$$\tilde{g}_t(x)=E_0[h(\mathbf{X}_\tau);\tau\geq t|X_t=x]. \tag{21}$$

Our non-degeneracy condition is now imposed on the function $\tilde{g}$ acted on $X\sim P_0$:

Assumption 3.7 *The random variable $\tilde{g}(X)$ is non-constant under P_0.*

We have the following theorem:

Theorem 3.3 *With either Assumption 3.5 or 3.6 in hold, together with Assumption 3.7, the optimal value of* (19) *satisfies*

$$\max E_f[h(\mathbf{X}_\tau)]=E_0[h(\mathbf{X}_\tau)]+\sqrt{2Var_0(\tilde{g}(X))}\eta^{1/2}+\frac{1}{Var_0(\tilde{g}(X))}\left(\frac{1}{3}\kappa_3(\tilde{g}(X))+\tilde{\nu}\right)\eta+O(\eta^{3/2}) \tag{22}$$

where

$$\tilde{\nu}=E_0[(\tilde{G}(X,Y)-E_0[\tilde{G}(X,Y)])(\tilde{g}(X)-E_0[\tilde{g}(X)])(\tilde{g}(Y)-E_0[\tilde{g}(Y)])]. \tag{23}$$

Here $\tilde{g}(x)$ is defined in (20) *and* (21)*, and $\tilde{G}(x,y)$ is defined as $\tilde{G}(x,y)=\sum_{t=1}^{\infty}\sum_{\substack{s\geq 1\\ s\neq t}}\tilde{G}_{ts}(x,y)$, where $\tilde{G}_{ts}(x,y)$ is given by*

$$\tilde{G}_{ts}(x,y)=E_0[h(\mathbf{X}_\tau);\tau\geq t\wedge s|X_t=x,X_s=y].$$

When τ is a finite deterministic time, Theorem 3.3 reduces to Theorem 3.2. Further relaxation of Assumptions 3.5 or 3.6 to more general stopping times is out of the scope of the present work and will be left elsewhere.

3.4 Discussions We close this section with some discussions:

1. Similar results to Theorems 3.1, 3.2 and 3.3 hold if maximization formulation is replaced by minimization. Under the same assumptions, the first order term in all the expansions above will have a sign change for minimization formulation, while the second order term will remain the same. For example, the expansion for Theorem 3.2 becomes

$$\min E_f[h(\mathbf{X}_T)]=E_0[h(\mathbf{X}_T)]-\sqrt{2Var_0(g(X))}\eta^{1/2}+\frac{1}{Var_0(g(X))}\left(\frac{1}{3}\kappa_3(g(X))+\nu\right)\eta+O(\eta^{3/2}). \tag{24}$$

For Theorem 3.1, the change in (5) for the minimization formulation is that β^* becomes the unique negative solution of the same equation.

These changes can be seen easily by merely replacing h by $-h$ in the analysis.

2. The function $g(\cdot)$ defined in (13) is the Gateaux derivative of $E_0[h(\mathbf{X}_T)]$ with respect to the distribution P_0, viewing $E_0[h(\mathbf{X}_T)]$ as a functional of P_0. To illustrate what we mean, consider a perturbation of the probability distribution from P_0 to a mixture distribution $(1-\epsilon)P_0+\epsilon Q$ where Q is a probability measure on $\mathcal{X}$ and $0<\epsilon<1$. Under suitable integrability conditions, one can check that

$$\frac{d}{d\epsilon}\int h(x_1,\dots,x_T)\prod_{t=1}^{T}d((1-\epsilon)P_0(x_t)+\epsilon Q(x_t))\bigg|_{\epsilon=0}=\int g(x)d(Q(x)-P_0(x)). \tag{25}$$

In the statistics literature, the function $g(x)-E_0[g(X)]$ has been known as the influence function [20] in which $X_1,\dots,X_T$ would play the role of i.i.d. data. Influence functions have been used in measuring the effect on given statistics due to outliers or other forms of data contamination [20, 21].

3. Our asymptotic expansions suggest that the square root of KL divergence is the correct scaling of the first order model misspecification effect. We will also show, in Section 7, that our first order expansion coefficients dominate any first order parametric derivatives under a suitable rescaling from Euclidean distance to KL divergence.

4. Our results can be generalized to situations with multiple random sources and when one is interested in evaluating the model misspecification effect from one particular source. To illustrate, consider $E[h(\mathbf{X}_T, \mathbf{Y})]$ where $\mathbf{Y}$ is some random object potentially dependent of the i.i.d. sequence $\mathbf{X}_T = (X_t)_{t=1,\dots,T}$. Suppose the model for Y is known and the focus is on assessing the effect of model misspecification for X_t. Theorem 3.2 still holds with $h(\mathbf{X}_T)$ replaced by $E[h(\mathbf{X}_T, \mathbf{Y})|\mathbf{X}_T]$, where $E[\cdot]$ is with respect to the known distribution of $\mathbf{Y}$. This modification can be seen easily by considering $E_f[E[h(\mathbf{X}_T, \mathbf{Y})|\mathbf{X}_T]]$ as the performance measure and $E[h(\mathbf{X}_T, \mathbf{Y})|\mathbf{X}_T]$ as the cost function. Analogous observations apply to Theorems 3.1 and 3.3.

5. KL divergence is a natural choice of statistical distance, as it has been used in model selection in statistics (e.g. in defining Akaike Information Criterion [1]), possesses information theoretic properties [32, 31, 30, 13], and is transformation invariant [12]. Nevertheless, there are other possible choices of statistical distances, such as those in the ϕ-divergence class [41].

4. Connections to Past Literatures Here we briefly review two lines of past literatures that are related to our work. First, the worst-case optimization and the use of statistical distance that we consider is related to robust control [25] and distributionally robust optimization [7, 14]. These literatures consider decision making when full probabilistic description of the underlying model is not available. The problems are commonly set in terms of a minimax objective, where the maximum is taken over a class of models that is believed to contain the truth, often called the uncertainty set [19, 34, 4]. The use of statistical distance such as KL divergence in defining uncertainty set is particularly popular for dynamic control problems [38, 28, 42], economics [22, 23, 24], finance [9, 10, 17], queueing [29], and dynamic pricing [35]. In particular, [18] proposes the use of simulation, which they called *robust Monte Carlo*, in order to approximate the solutions for a class of worst-case optimizations that arise in finance. Nevertheless, in all the above literatures, the typical focus is on the tractability of optimization formulations, which often include convexity. Instead, this paper provides a different line of analysis using asymptotic approximations for formulations that are intractable via developed methods yet arise naturally in stochastic modeling.

The second line of related literatures is sensitivity analysis. The surveys [33], [15], §VII in [3] and §7 in [16] provide general overview on different methods for derivative estimation in classical sensitivity analysis, which focus on parametric uncertainty. Another notable area is perturbation analysis of Markov chains. These results are often cast as Taylor series expansions in terms of the perturbation of the transition matrix (e.g. [45, 11, 26]), where the distances defining the perturbations are typically matrix norms on the transition kernels rather than statistical distances defined between distributions. We also note the area of variance-based global sensitivity analysis [44]. This often involves the estimation of the variance of conditional expectations on some underlying parameters, which resembles to some extent the form of the first order coefficient in our main theorems. The randomness in this framework can be interpreted from a Bayesian [43, 39] or data-driven [47, 2, 5] perspective, or in the context of Bayesian model averaging, the posterior variability among several models [48]. All these are nonetheless parametric-based.

5. Mathematical Developments for Finite Horizon Problems In this section we lay out the analysis of the worst-case optimization for finite time horizon problems when $T > 1$, leading to Theorem 3.2. Leveraging the idea in Section 3.1, we first write the maximization problem (1) in terms of likelihood ratio L:

$$\begin{array}{ll} \max & E_0[h(\mathbf{X}_T)\underline{L}_T] \\ \text{subject to} & E_0[L(X)\log L(X)] \leq \eta \\ & L \in \mathcal{L} \end{array} \tag{26}$$

where for convenience we denote $\underline{L}_T = \prod_{t=1}^T L(X_t)$, and X as a generic variable that is independent of $\{X_t\}_{t=1,\dots,T}$ and having identical distribution as each X_t. We will follow the recipe from Section 3.1 to prove Theorem 3.2:

(i) Consider the Lagrangian relaxation of (26), and characterize its optimal solution.

(ii) Find the optimality conditions for (26) in terms of the Lagrange multiplier and the Lagrangian relaxation.

(iii) Using these conditions, expand the optimal value of (26) in terms of the Lagrange multiplier and subsequently η.

The main technical challenge on implementing the above scheme is the product form $\underline{L}_T$ that appears in the objective function in (26). In this regard, our key development is a characterization of the optimal solution of the Lagrangian relaxation via a fixed point equation on a suitable functional space.

For technical reason, we will look at an equivalent problem with a modified space of L and will introduce a suitable norm and metric. Let $\Lambda(x)=\sum_{t=1}^T\Lambda_t(x)$ where $\Lambda_t(x)$ is defined in Assumption 3.3. Define

$$\mathcal{L}(M)=\{L\in\mathcal{L}:E_0[\Lambda(X)L(X)]\leq M\}$$

for $M>0$, and the associated norm $\|L\|_\Lambda:=E_0[(1+\Lambda(X))L(X)]$ and metric

$$\|L-L'\|_\Lambda=E_0[(1+\Lambda(X))|L(X)-L'(X)|]. \tag{27}$$

It is routine to check that $\mathcal{L}(M)$ is complete. We have the following observation:

Lemma 5.1 *For any $\eta\leq N$ for some small N, formulation* (26) *is equivalent to*

$$\begin{array}{ll}\max & E_0[h(\mathbf{X}_T)\underline{L}_T]\\ \textit{subject to} & E_0[L(X)\log L(X)]\leq\eta\\ & L\in\mathcal{L}(M)\end{array} \tag{28}$$

for some large enough $M>0$, independent of η.

The proof of Lemma 5.1 is left to Section 9. From now on we will focus on (28). Its Lagrangian relaxation is given by

$$\max_{L\in\mathcal{L}(M)}E_0[h(\mathbf{X}_T)\underline{L}_T]-\alpha(E_0[L\log L]-\eta). \tag{29}$$

Our optimality characterization for (29) is:

Proposition 5.1 *Under Assumption 3.3, when $\alpha>0$ is large enough, the unique optimal solution of* (29) *satisfies*

$$L(x)=\frac{e^{g^L(x)/\alpha}}{E_0[e^{g^L(X)/\alpha}]} \tag{30}$$

where $g^L(x)=\sum_{t=1}^Tg_t^L(x)$ and $g_t^L(x)=E_0\left[h(\mathbf{X}_T)\prod_{\substack{1\leq r\leq T\\ r\neq t}}L(X_r)\middle|X_t=x\right]$.

The form in (30) can be guessed from a heuristic differentiation with respect to L. To see this, consider further relaxation of the constraint $E_0[L]=1$ in (29):

$$E_0[h(\mathbf{X}_T)L(X_1)L(X_2)\cdots L(X_T)]-\alpha E_0[L\log L]+\alpha\eta+\lambda E_0[L]-\lambda. \tag{31}$$

There are T factors of L in the first term. A heuristic "product rule" of differentation is to sum up the derivative with respect to each L factor, keeping all other L's unchanged. To do so, we condition on X_t to write

$$E_0[h(\mathbf{X}_T)L(X_1)L(X_2)\cdots L(X_T)]=E_0\left[E_0\left[h(\mathbf{X}_T)\prod_{\substack{1\leq r\leq T\\ r\neq t}}L(X_r)\middle|X_t\right]L(X_t)\right]$$

and

$$\frac{d}{dL(x)}E_0\left[h(\mathbf{X}_T)\prod_{t=1}^TL(X_t)\right]\text{ “ = ” }\sum_{t=1}^TE_0\left[h(\mathbf{X}_T)\prod_{\substack{1\leq r\leq T\\ r\neq t}}L(X_r)\middle|X_t=x\right]. \tag{32}$$

So the Euler-Lagrange equation is

$$\sum_{t=1}^{T} E_0\left[h(\mathbf{X}_T)\prod_{\substack{1\leq r\leq T\\ r\neq t}} L(X_r)\middle| X_t = x\right] - \alpha\log L(x) - \alpha + \lambda = 0 \tag{33}$$

which gives

$$L(x) \propto \exp\left\{\sum_{t=1}^{T} E_0\left[h(\mathbf{X}_T)\prod_{\substack{1\leq r\leq T\\ r\neq t}} L(X_r)\middle| X_t = x\right]\Big/\alpha\right\}.$$

The constraint $E_0[L] = 1$ then gives the expression (30). The "product rule" (32) can be readily checked for finitely supported X. The following shows an instance when $T = 2$:

Example 5.1 *Consider two i.i.d. random variables X_1 and X_2, and a cost function $h(X_1, X_2)$. The variables X_1 and X_2 have finite support on $1, 2, \ldots, n$ under P_0. Denote $p(x) = P_0(X_1 = x)$ for $x = 1, 2, \ldots, n$. The objective value in (26) in this case is*

$$E_0[h(X_1, X_2)L(X_1)L(X_2)] = \sum_{x_1=1}^{n}\sum_{x_2=1}^{n} h(x_1, x_2)p(x_1)p(x_2)L(x_1)L(x_2).$$

Now differentiate with respect to each $L(1), L(2), \ldots, L(n)$ respectively. For $i = 1, \ldots, n$, we have

$$\begin{aligned}\frac{d}{dL(i)}E_0[h(X_1, X_2)] &= \sum_{x_1\neq i} h(x_1, i)p(x_1)p(i)L(x_1) + \sum_{x_2\neq i} h(i, x_2)p(i)p(x_2)L(x_2) + 2h(i, i)p(i)^2L(i)\\ &= E_0[h(X_1, i)L(X_1)] + E_0[h(i, X_2)L(X_2)].\end{aligned}$$

This coincides with the product rule (32) discussed above.

5.1 Outline of Argument of Proposition 5.1 The proof of Proposition 5.1 centers around an operator $\mathcal{K} : \mathcal{L}(M)^{T-1} \to \mathcal{L}(M)^{T-1}$ as follows. First, we define a function derived from h as

$$S_h(\mathbf{x}_T) = \sum_{\mathbf{y}\in\mathcal{S}_T} h(\mathbf{y}) \tag{34}$$

where $\mathcal{S}_T$ is the symmetric group of all permutations of $\mathbf{x}_T = (x_1, \ldots, x_T)$. The summation in (34) has $T!$ number of terms. Obviously, by construction the value of S_h is invariant to any permutation of its arguments.

Denote $\mathbf{L} = (L_1, \ldots, L_{T-1}) \in \mathcal{L}(M)^{T-1}$. We now define a mapping $K : \mathcal{L}(M)^{T-1} \to \mathcal{L}(M)$ given by

$$K(L_1, \ldots, L_{T-1})(x) := \frac{e^{E_0[S_h(X, X_1, \ldots, X_{T-1})\prod_{t=1}^{T-1} L_t(X_t)|X=x]/(\alpha(T-1)!)}}{E_0[e^{E_0[S_h(X, X_1, \ldots, X_{T-1})\prod_{t=1}^{T-1} L_t(X_t)|X]/(\alpha(T-1)!)}]} \tag{35}$$

where $X, X_1, X_2, \ldots, X_{T-1}$ are i.i.d. random variables with distribution P_0. Then for a given $\mathbf{L}$, define

$$\begin{aligned}\tilde{L}_1 &= K(L_1, \ldots, L_{T-1})\\ \tilde{L}_2 &= K(\tilde{L}_1, L_2, \ldots, L_{T-1})\\ \tilde{L}_3 &= K(\tilde{L}_1, \tilde{L}_2, L_3, \ldots, L_{T-1})\\ &\vdots\\ \tilde{L}_{T-1} &= K(\tilde{L}_1, \ldots, \tilde{L}_{T-2}, L_{T-1}).\end{aligned} \tag{36}$$

Finally, the operator $\mathcal{K}$ on $\mathcal{L}(M)^{T-1}$ is defined as

$$\mathcal{K}(\mathbf{L}) = (\tilde{L}_1, \ldots, \tilde{L}_{T-1}). \tag{37}$$

The following shows the main steps for the proof of Proposition 5.1.

Step 1: Contraction Mapping. We have:

Lemma 5.2 *Under Assumption 3.3, when α is sufficiently large, the operator $\mathcal{K}:\mathcal{L}(M)^{T-1}\to\mathcal{L}(M)^{T-1}$ defined in* (37) *is well-defined, closed and a contraction, using the metric $d(\cdot,\cdot):\mathcal{L}(M)^{T-1}\times\mathcal{L}(M)^{T-1}\to\mathbb{R}_+$ defined as*

$$d(\mathbf{L},\mathbf{L}')=\max_{t=1,\dots,T}E_0[(1+\Lambda(X))|L_t(X)-L_t'(X)|]$$

where $\mathbf{L}=(L_1,\dots,L_{T-1}),\mathbf{L}'=(L_1',\dots,L_{T-1}')\in\mathcal{L}(M)^{T-1}$. Hence there exists a unique fixed point $\mathbf{L}^*$ that satisfies $\mathcal{K}(\mathbf{L}^*)=\mathbf{L}^*$. Moreover, all components of $\mathbf{L}^*$ are identical.

This leads to a convergence result on the iteration driven by the mapping K:

Corollary 5.1 *With Assumption 3.3 and sufficiently large α, starting from any $L^{(1)},\dots,L^{(T-1)}\in\mathcal{L}(M)$, the iteration $L^{(k)}=K(L^{(k-T+1)},\dots,L^{(k-1)})$ for $k\geq T$, where $K:\mathcal{L}(M)^{T-1}\to\mathcal{L}(M)$ is defined in* (35)*, converges to L^* in $\|\cdot\|_\Lambda$-norm, where L^* is the identical component of the fixed point $\mathbf{L}^*$ of $\mathcal{K}$. Moreover, $L^*=K(L^*,\dots,L^*)$.*

Step 2: Monotonicity of the Objective Value under Iteration of K. We shall consider the objective in (29) multiplied by $T!$, i.e.

$$T!(E_0[h(\mathbf{X}_T)\underline{L}_T]-\alpha E_0[L\log L])=E_0[S_h(\mathbf{X}_T)\underline{L}_T]-\alpha T!E_0[L\log L]. \tag{38}$$

Iterations driven by the mapping K possess a monotonicity property on this scaled objective:

Lemma 5.3 *With Assumption 3.3 and sufficiently large α, starting from any $L^{(1)},\dots,L^{(T)}\in\mathcal{L}(M)$, construct the sequence $L^{(k+1)}=K(L^{(k-T+2)},\dots,L^{(k)})$ for $k\geq T$, where K is defined in* (35)*. Then*

$$E_0\left[S_h(\mathbf{X}_T)\prod_{t=1}^T L^{(k+t-1)}(X_t)\right]-\alpha(T-1)!\sum_{t=1}^T E_0[L^{(k+t-1)}\log L^{(k+t-1)}] \tag{39}$$

is non-decreasing in k, for $k\geq 1$.

Step 3: Convergence of the Objective Value to the Optimum. Finally, we have the convergence of (39) to the scaled objective (38) evaluated at any identical component of the fixed point of $\mathcal{K}$:

Lemma 5.4 *With Assumption 3.3 and sufficiently large α, starting from any $L^{(1)},\dots,L^{(T)}\in\mathcal{L}(M)$, we have*

$$\begin{aligned}&E_0\left[S_h(\mathbf{X}_T)\prod_{t=1}^T L^{(k+t-1)}(X_t)\right]-\alpha(T-1)!\sum_{t=1}^T E_0[L^{(k+t-1)}\log L^{(k+t-1)}]\\ \to\quad &E_0\left[S_h(\mathbf{X}_T)\prod_{t=1}^T L^*(X_t)\right]-\alpha T!E_0[L^*\log L^*]\end{aligned}$$

where $L^{(k)}$ is defined by the same recursion in Lemma 5.3, and L^* is any identical component of $\mathbf{L}^*\in\mathcal{L}(M)^{T-1}$, the fixed point of $\mathcal{K}$ defined in (37)*.*

These lemmas will be proved in Section 9.1. Once they are established, Proposition 5.1 follows immediately:

Proof of Proposition 5.1. Given any $L\in\mathcal{L}(M)$, Lemmas 5.3 and 5.4 together conclude that

$$E_0\left[S_h(\mathbf{X}_T)\prod_{t=1}^T L(X_t)\right]-\alpha(T-1)!\sum_{t=1}^T E_0[L\log L]\leq E_0\left[S_h(\mathbf{X}_T)\prod_{t=1}^T L^*(X_t)\right]-\alpha T!E_0[L^*\log L^*]$$

by defining $L^{(1)}=\cdots=L^{(T)}=L$ and using the recursion defined in the lemmas. Here L^* is the identical component of the fixed point of $\mathcal{K}$. By Corollary 5.1, $L^*=K(L^*,\dots,L^*)$ so L^* satisfies (30). This concludes Proposition 5.1. □

5.2 Asymptotic Expansions The characterization of L^* in Proposition 5.1 can be used to obtain asymptoptic expansion of L^* in terms of α^*. The proof of Theorem 3.2, as outlined in the recipe at the beginning of this section, then follows from an elaboration of the machinery developed in Section 3.1. Details are provided in Section 9.2.

6. Extension to Random Time Horizon Problems We shall discuss here the extension to random time horizon problems under Assumption 3.5, using the result in Theorem 3.2:

PROOF OF THEOREM 3.3 UNDER ASSUMPTION 3.5. First, formulation (19) can be written in terms of likelihood ratio:

$$\begin{array}{ll} \max & E_0[h(\mathbf{X}_\tau)\underline{L}_\tau] \\ \text{subject to} & E_0[L(X)\log L(X)] \leq \eta \\ & L \in \mathcal{L} \end{array} \tag{40}$$

where $\underline{L}_\tau = \prod_{t=1}^{\tau} L(X_t)$. Under Assumption 3.5, $\tau \leq T$ a.s., for some $T > 0$. Hence the objective in (40) can also be written as $E_0[h(\mathbf{X}_\tau)\underline{L}_\tau] = E_0[h(\mathbf{X}_\tau)\underline{L}_T]$ by the martingale property of $\underline{L}_t$.

This immediately falls back into the framework of Theorem 3.2, with the cost function now being $h(\mathbf{X}_\tau)$. For this particular cost function, we argue that the $g(x)$ and $G(x,y)$ in Theorem 3.2 are indeed in the form stated in Theorem 3.3. To this end, we write

$$g(x) = \sum_{t=1}^{T} E_0[h(\mathbf{X}_\tau)|X_t = x] = \sum_{t=1}^{T} E_0[h(\mathbf{X}_\tau);\tau \geq t|X_t = x] + \sum_{t=1}^{T} E_0[h(\mathbf{X}_\tau);\tau < t|X_t = x]. \tag{41}$$

Consider the second summation in (41). Since $h(\mathbf{X}_\tau)I(\tau < t)$ is $\mathcal{F}_{t-1}$-measurable, it is independent of X_t. As a result, the second summation in (41) is constant. Similarly, we can write

$$\begin{aligned} G(x,y) &= \sum_{t=1}^{T} \sum_{\substack{s=1,\dots,T \\ s\neq t}} E_0[h(\mathbf{X}_\tau)|X_t = x, X_s = y] \\ &= \sum_{t=1}^{T} \sum_{\substack{s=1,\dots,T \\ s\neq t}} E_0[h(\mathbf{X}_\tau);\tau \geq t\wedge s|X_t = x, X_s = y] + \sum_{t=1}^{T} \sum_{\substack{s=1,\dots,T \\ s\neq t}} E_0[h(\mathbf{X}_\tau);\tau < t\wedge s|X_t = x, X_s = y] \end{aligned} \tag{42}$$

and the second summation in (42) is again a constant. It is easy to check that the first and second order coefficients in Theorem 3.2 are translation invariant to $g(x)$ and $G(x,y)$ respectively, i.e. adding a constant in $g(x)$ or $G(x,y)$ does not affect the coefficients. Therefore Theorem 3.3 follows immediately. □

The proof of Theorem 3.3 under Assumption 3.6 builds on the above argument by considering a sequence of truncated random time $\tau \wedge T,\ T = 1,2,\ldots$. We defer its details to Section 9.3.

7. Bounds on Parametric Derivatives The coefficients in our expansions in Section 3 dominate any parametric derivatives in the following sense:

PROPOSITION 7.1 *Suppose P_0 lies in a parametric family P^θ with $\theta \in \Theta \subset \mathbb{R}$, say $P_0 = P^{\theta_0}$ where $\theta_0 \in \Theta^\circ$. Denote $E^\theta[\cdot]$ as the expectation under P^θ. Assume that*

(i) *P^θ is absolutely continuous with respect to P^{θ_0} for θ in a neighborhood of θ_0.*

(ii) *$D(\theta,\theta_0) := D(P^\theta\|P^{\theta_0}) \to 0$ as $\theta \to \theta_0$.*

(iii) *For any η in a neighborhood of 0 (but not equal to 0), $D(\theta,\theta_0) = \eta$ has two solutions $\theta^+(\eta) > \theta_0$ and $\theta^-(\eta) < \theta_0$; moreover,*

$$\frac{d}{d\theta}\sqrt{D(\theta,\theta_0)}\Big|_{\theta=\theta_0^+} > 0$$

and

$$\frac{d}{d\theta}\sqrt{D(\theta,\theta_0)}\Big|_{\theta=\theta_0^-} < 0.$$

(iv) *$\frac{d}{d\theta}E^\theta[h(\mathbf{X})]\Big|_{\theta=\theta_0}$ exists.*

Then

$$\left|\frac{d}{d\theta}E^\theta[h(\mathbf{X})]\Big/\frac{d}{d\theta}\sqrt{D(\theta,\theta_0)}\Big|_{\theta=\theta_0^\pm}\right| \leq \sqrt{2Var_0(\zeta(X))}$$

where ζ is the function h, g or $\tilde{g}$ in Theorems 3.1, 3.2 and 3.3 respectively, depending on the structure of $\mathbf{X}$ that is stated in each theorem under the corresponding assumptions.

This proposition states the natural property that the first order expansion coefficients of worst-case optimizations dominate the parametric derivative taken in the more restrictive parametric model space. The proof is merely a simple application of the first principle of differentiation:

PROOF. We consider only the setting in Theorem 3.2, as the others are similar. Let $P_0 = P^{\theta_0}$. Denote $E_{f^+(\eta)}[h(\mathbf{X}_T)]$ as the optimal value of (1) and $E_{f^-(\eta)}[h(\mathbf{X}_T)]$ as the optimal value of (2), when η is in a neighborhood of 0. Under our assumptions, $P^{\theta^\pm(\eta)}$ with $D(\theta^\pm(\eta), \theta_0) = \eta$ are feasible solutions to both programs (1) and (2), and hence the quantity $E^{\theta^\pm(\eta)}[h(\mathbf{X})]$ satisfies $E^{\theta^\pm(\eta)}[h(\mathbf{X}_T)] \leq E_{f^+(\eta)}[h(\mathbf{X}_T)]$ and $E^{\theta^\pm(\eta)}[h(\mathbf{X}_T)] \geq E_{f^-(\eta)}[h(\mathbf{X}_T)]$. This implies

$$\frac{E_{f^-(\eta)}[h(\mathbf{X}_T)] - E_0[h(\mathbf{X}_T)]}{\sqrt{\eta}} \leq \frac{E^{\theta^\pm(\eta)}[h(\mathbf{X}_T)] - E_0[h(\mathbf{X}_T)]}{\sqrt{\eta}} \leq \frac{E_{f^+(\eta)}[h(\mathbf{X}_T)] - E_0[h(\mathbf{X}_T)]}{\sqrt{\eta}}.$$

Taking the limit as $\sqrt{\eta} \to 0$, the upper and lower bounds converge to $\sqrt{2Var_0(g(X))}$ by Theorem 3.2 (and discussion point 1 in Section 3.4). Moreover, the quantities

$$\lim_{\sqrt{\eta}\to 0} \frac{E^{\theta^\pm(\eta)}[h(\mathbf{X}_T)] - E_0[h(\mathbf{X}_T)]}{\sqrt{\eta}} = \frac{d}{d\sqrt{\eta}} E^{\theta^\pm(\eta)}[h(\mathbf{X}_T)]\Big|_{\sqrt{\eta}=0}$$

become $\frac{d}{d\theta} E^\theta[h(\mathbf{X}_T)] \Big/ \frac{d}{d\theta}\sqrt{D(\theta,\theta_0)}\Big|_{\theta=\theta_0^+}$ and $\frac{d}{d\theta} E^\theta[h(\mathbf{X}_T)] \Big/ \frac{d}{d\theta}\sqrt{D(\theta,\theta_0)}\Big|_{\theta=\theta_0^-}$ respectively, by using chain rule and implicit function theorem. This concludes the proposition. □

8. Numerical Examples We demonstrate some numerics of our results, in particular Theorem 3.2, using an example of multi-server queue. Consider a first-come-first-serve Markovian queue with s number of servers. Customers arrive according to a Poisson process with rate $0.7s$ and enact i.i.d. exponential service times with rate 1. Whenever the service capacity is full, newly arriving customers have to wait. We assume the system is initially empty. Our focus is to assess the effect if service times deviate from the exponential assumption. More concretely, let us consider our performance measure as the tail probability of the waiting time for the 100-th customer larger than a threshold 1.

To quantify the sensitivity of the exponential assumption for the service times, we compute the first order coefficient $\sqrt{2Var_0(g(X))}$ in Theorem 3.2, where P_0 is $Exp(1)$ and $g(\cdot)$ is computed by sequentially conditioning on the service time of customers 1 through 100, as defined in (13). We tabulate, for $s = 1, \ldots, 5$, the point and interval estimates of the baseline performance measures and the first order coefficients $\sqrt{2Var_0(g(X))}$ in Table 1. Moreover, for each s, we calculate the ratio between the first order coefficient and the baseline performance measure as an indicator of the relative impact of model misspecification:

$$\text{Relative model misspecification impact} := \frac{\text{Magnitude of first order coefficient}}{\text{Performance measure}}$$

	Baseline performance measure		First order coefficient		Relative impact
Number of servers	Mean	95% C.I.	Mean	95% C.I.	
1	0.519	$(0.518, 0.520)$	1.685	$(1.566, 1.805)$	3.248
2	0.316	$(0.315, 0.316)$	1.689	$(1.556, 1.822)$	5.353
3	0.200	$(0.199, 0.201)$	1.446	$(1.318, 1.573)$	7.239
4	0.129	$(0.128, 0.129)$	1.217	$(1.079, 1.355)$	9.460
5	0.084	$(0.083, 0.084)$	0.957	$(0.856, 1.058)$	11.462

Table 1: Simulation results for the performance measures and the first order coefficients in Theorem 3.2 for the tail probability of waiting time of the 100-th customer in $M/M/s$ systems with different server capacities

Table 1 shows that the tail probability of the waiting time for the 100-th customer decreases from 0.52 to 0.08 as the number of servers increases from 1 to 5. The first order coefficient in Theorem 3.2 also decreases in general from 1.69 when $s = 1$ to 0.96 when $s = 5$. The relative effect of model misspecification, on the other hand, increases from 3.25 to 11.46 as s increases from 1 to 5.

Figure 1 further depicts the first order approximations for the worst-case deviations $E_0[h(\mathbf{X}_T)] \pm \sqrt{2Var_0(g(X))\eta}$ for different levels of η that represents the KL divergence. The solid line in the figure plots the baseline tail probability computed from using 1,000,000 samples for each $s = 1, \ldots, 5$. The dashed lines then show the approximate worst-case upper and lower bounds as η increases. To get a sense on the magnitude of η, $\eta = 0.005$ is equivalent to around 10% discrepancy in service rate if the model is *known* to lie in the family of exponential distribution; this can be seen by expressing the KL divergence in terms of service rate to see that roughly $KL\ divergence \approx (\%\ discrepancy\ in\ service\ rate)^2/2$ for small discrepancy. In fact, a service rate of 1.1 corresponds to $\eta = 0.0044$.

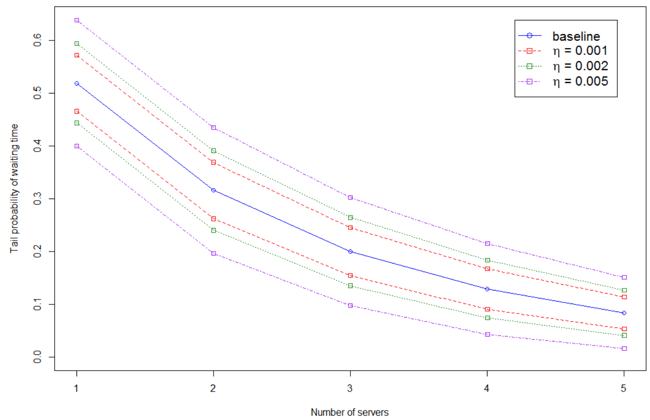

Figure 1: Upper and lower first order approximations of the worst-case performance measures under different levels of input model discrepancies in terms of KL divergence for $M/M/1$ baseline system

According to Proposition 5.1, the worst-case change of measure that gives rise to the values of the first order coefficients in Table 1 satisfies $L(x) \propto e^{g^L(x)/\alpha^*}$, with α^* being the Lagrange multiplier, when η is small. It is not possible to compute this change of measure exactly. What we can do, however, is to test our bounds from Theorem 3.2 against some parametric models. Consider for instance $s = 1$. The solid curves in Figure 2 plot the upper and lower bounds using only the first order approximations in Theorem 3.2 (the surrounding dashed curves are the 95% confidence bands for the bounds, and the dashed horizontal line is the baseline performance measure). For comparison, we simulate the performance measures using six different sequences of parametric models: the first two are kept as exponential distribution, with increasing and decreasing service rate starting from 1; the next two sequences are gamma distributions, one with the shape parameter varying from 1 and rate parameter kept at 1, whereas the other with rate parameter varying too so that the mean of the distribution is kept at 1; the last two sequences are mixtures of exponentials, one having two mixture components with rate 1 and 2 respectively and weight of the first component decreases from 1, whereas the other one having three components with the weights varying in a way that keeps the mean at 1.

As we can see, the first order bounds in Figure 2 appear to contain all the performance measures for η up to 0.005. It is expected that a second order correction would further improve the accuracy of the bounds. One side observation is that the sequences with the service times kept at mean equaling 1 are much closer to the baseline than the others.

The same methodology as above can be easily adapted to test other types of performance measures and models. For example, Table 2 and Figure 3 carry out the same assessment scheme for the service time of a non-Markovian $G/G/s$ queue with gamma arrivals and uniform service times. Here we consider a deviation from the uniform distribution of the service time. In this scenario, we see from Table 2 that both the performance measures themselves and the magnitudes of first order coefficients are smaller than those in the $M/M/s$ case. Nonetheless the relative impacts are relatively similar.

In real applications, the magnitude of η is chosen to represent the statistical uncertainty of the input model. Section 4.2 in [18] for instance provides some discussion on the choice based on past data. There are also studies on nonparametric estimation of KL divergence; see, for example, [6] for a review of older works, [40], and more recently [36] and [27].

Finally, we explain in more detail our estimation procedure for $\sqrt{2Var_0(g(X))}$. Note first that our

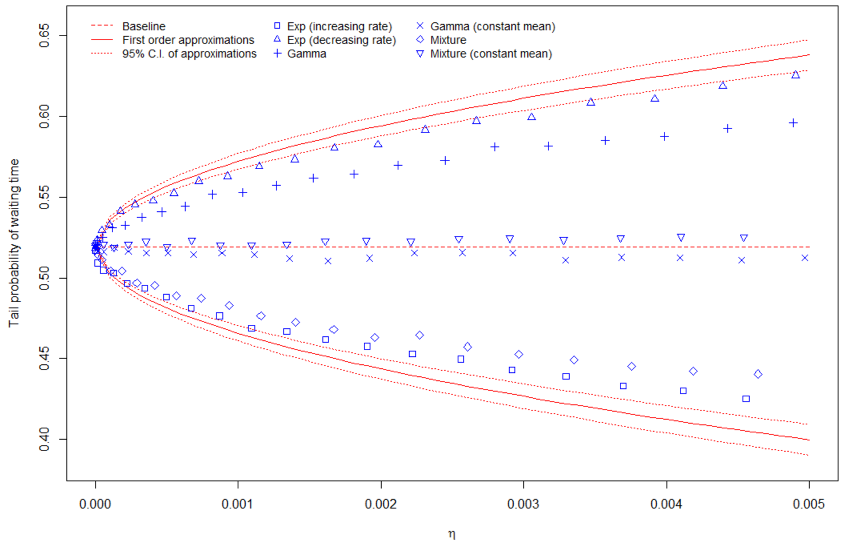

Figure 2: Comparison of the first order approximations of the worst-case performance measures against parametric models

Number of servers	Baseline performance measure Mean	C.I.	First order coefficient Mean	C.I.	Relative impact
1	0.316	$(0.315, 0.317)$	0.802	$(0.737, 0.867)$	2.535
2	0.119	$(0.119, 0.120)$	0.567	$(0.531, 0.602)$	4.741
3	0.047	$(0.047, 0.048)$	0.320	$(0.298, 0.341)$	6.770
4	0.019	$(0.019, 0.019)$	0.173	$(0.160, 0.185)$	9.146
5	0.008	$(0.008, 0.008)$	0.091	$(0.079, 0.104)$	11.879

Table 2: Simulation results for the performance measures and the first order coefficients in Theorem 3.2 for the tail probability of waiting time of the 100-th customer in $G/G/s$ systems with different server capacities

performance measure of interest depends on both the interarrival and service times, but the interarrival distribution is assumed known and so the cost function can be regarded as $E_0[h(\mathbf{X}_T, \mathbf{Y}_T)|\mathbf{X}_T]$, where $\mathbf{X}_T$ denotes the sequence of service times and $\mathbf{Y}_T$ as the interarrival times (see discussion point 4 in Section 3.4). Second, note also that Assumption 3.3 is easily satisfied since h as the indicator function is bounded. Moreover, Assumption 3.4 is trivially verified by our computation demonstrating that $g(X)$ is not a constant. Hence the assumptions in Theorem 3.2 are valid.

Now, it is easy to see that

$$g(x) = E_0\left[\sum_{t=1}^{T} h(\mathbf{X}_T^{(t)}, \mathbf{Y}_T^{(t)})\middle| X = x\right]$$

where $\mathbf{X}_T^{(t)} = (X_1^{(t)}, \ldots, X_{t-1}^{(t)}, X, X_{t+1}^{(t)}, \ldots, X_T^{(t)})$ and $\mathbf{Y}_T^{(t)} = (Y_1^{(t)}, \ldots, Y_T^{(t)})$, with $X_s^{(t)}$ and $Y_s^{(t)}$ being i.i.d. copies from the interarrival time and service time distributions respectively. Therefore $Var_0(g(X))$ is in the form of the variance of a conditional expectation, for which we can adopt an unbiased estimator from [46]. This estimator takes the following form. For convenience, denote $H := \sum_{t=1}^{T} h(\mathbf{X}_T^{(t)}, \mathbf{Y}_T^{(t)})$. To compute $Var_0(E[H|X])$, we carry out a nested simulation by first simulating $X_k, k = 1, \ldots, K$, and then given each X_k, simulating $H_{kj}, j = 1, \ldots, n$. Then an unbiased estimator is

$$\widehat{\sigma_M^2} = \frac{1}{K-1}\sum_{k=1}^{K}(\bar{H}_k - \bar{\bar{H}})^2 - \frac{1}{n}\widehat{\sigma_\epsilon^2} \tag{43}$$

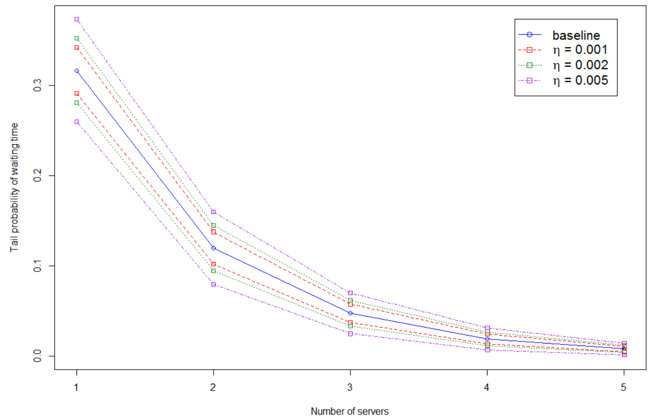

Figure 3: Upper and lower first order approximations of the worst-case performance measures under different levels of input model discrepancies in terms of KL divergence for $G/G/1$ baseline system

where

$$\widehat{\sigma_\epsilon^2} = \frac{1}{K(n-1)}\sum_{k=1}^K\sum_{j=1}^n (H_{kj} - \bar{H}_k)^2, \quad \bar{H}_k = \frac{1}{n}\sum_{j=1}^n H_{kj} \text{ and } \bar{\bar{H}} = \frac{1}{K}\sum_{k=1}^K \bar{H}_k.$$

To obtain a consistent point estimate and confidence interval for $\sqrt{2Var_0(g(X))}$, we use the delta method (see, for example, §III in [3]). The overall sampling strategy is as follows:

(i) Repeat the following N times:
  (a) Simulate K samples of X, say $X_k = x_k, k = 1, \dots, K$.
  (b) For each realized x_k, simulate n samples of H given $X = x_k$.
  (c) Calculate $\widehat{\sigma_M^2}$ using (43).

(ii) The above procedure generates N estimators $\widehat{\sigma_M^2}$. Call them $Z_l, l = 1, \dots, N$. The final point estimator is $\sqrt{2\bar{Z}}$, where $\bar{Z} = (1/N)\sum_{l=1}^N Z_l$, and the $1-\alpha$ confidence interval is $\sqrt{2} \times (\sqrt{\bar{Z}} \pm (\sigma/(2\sqrt{\bar{Z}}))t_{1-\alpha/2}/\sqrt{N})$ where $\sigma^2 = 1/(N-1)\sum_{n=1}^N (Z_n - \bar{Z})^2$ and $t_{1-\alpha/2}$ is the $1-\alpha/2$ percentile of the t-distribution with $N-1$ degree of freedom.

This gives a consistent point estimate for $\sqrt{2Var_0(g(X))}$ and an asymptotically valid confidence interval. In our implementation we choose $K = 100$, $n = 50$ and $N = 20$.

9. Proofs Proof of Proposition 3.1. We guess the solution (9) by applying Euler-Lagrange equation and informally differentiate the integrand with respect to L. We will then verify rigorously that this candidate solution is indeed optimal.

Relaxing the constraint $E_0[L] = 1$ in (8), the objective becomes

$$E_0[h(X)L - \alpha L \log L + \lambda L - \lambda]$$

where $\lambda \in \mathbb{R}$ is the Lagrange multiplier. Treating $E_0[\cdot]$ as an integral, Euler-Lagrange equation implies that the derivative with respect to L is

$$h(X) - \alpha \log L - \alpha + \lambda = 0$$

which gives

$$\log L = \frac{h(X)}{\alpha} + \frac{\lambda - \alpha}{\alpha}$$

or that $L = \lambda' e^{h(X)/\alpha}$ for some $\lambda' > 0$. With the constraint that $E_0[L] = 1$, a candidate solution is

$$L^* = \frac{e^{h(X)/\alpha}}{E_0[e^{h(X)/\alpha}]}. \tag{44}$$

To verify (44) formally, the following convexity argument will suffice. First, note that the objective value of (8) evaluated at L^* given by (44) is

$$E_0[h(X)L^* - \alpha L^* \log L^*] = E_0\left[h(X)L^* - \alpha L^*\left(\frac{h(X)}{\alpha} - \log E_0[e^{h(X)/\alpha}]\right)\right]$$
$$= \alpha \log E_0[e^{h(X)/\alpha}]. \tag{45}$$

Our goal is to show that

$$\alpha \log E_0[e^{h(X)/\alpha}] \geq E_0[h(X)L - \alpha L \log L]$$

for all $L \in \mathcal{L}$. Rearranging terms, this means we need

$$E_0[e^{h(X)/\alpha}] \geq e^{E_0[h(X)L - \alpha L \log L]/\alpha}. \tag{46}$$

To prove (46), observe that, for any likelihood ratio L,

$$E_0[e^{h(X)/\alpha}] = E_0[LL^{-1}e^{h(X)/\alpha}] = E_0[Le^{h(X)/\alpha - \log L}] \geq e^{E_0[h(X)L/\alpha - L\log L]}$$

by using the convexity of the function $e^{\cdot}$ and Jensen's inequality over the expectation $E_0[L\ \cdot]$ in the last inequality. Note that equality holds if and only if $h(X)/\alpha - \log L$ is degenerate, i.e. $h(X)/\alpha - \log L =$ *constant*, which reduces to L^*. Hence L^* is the unique optimal solution for (8).

In conclusion, when $1/\alpha \in \mathcal{D}^+ := \{\theta \in \mathbb{R}^+ \setminus \{0\} : \psi(\theta) < \infty\}$ where $\psi(\theta) = \log E_0[e^{\theta h(X)}]$, the optimal solution of (8) is given by (9), with the optimal value $\alpha \log E_0[e^{h(X)/\alpha}]$. □

Proof of Lemma 5.1. For any $\eta \leq N$, we want to show that $E_0[L\log L] \leq \eta$ and $L \in \mathcal{L}$ together imply $L \in \mathcal{L}(M)$ for some large $M > 0$. Note that $\Lambda(X)$ has exponential moment, since Holder's inequality implies

$$E_0[e^{\theta\Lambda(X)}] = E_0[e^{\theta\sum_{t=1}^T \Lambda_t(X)}] \leq \prod_{t=1}^T (E_0[e^{T\theta\Lambda_t(X)}])^{1/T} < \infty \tag{47}$$

when θ is small enough. Hence, for any $L \in \mathcal{L}$ that satisfies $E_0[L\log L] \leq \eta$, we have

$$E_0[e^{\theta\Lambda(X)}] = E_0[LL^{-1}e^{\theta\Lambda(X)}] = E_0[Le^{\theta\Lambda(X) - \log L}] < \infty$$

for small enough θ, by (47). Jensen's inequality implies that

$$e^{\theta E_0[\Lambda(X)L] - E_0[L\log L]} \leq E_0[Le^{\theta\Lambda(X) - \log L}] = E_0[e^{\theta\Lambda(X)}] < \infty.$$

Since $E_0[L\log L] \leq \eta \leq N$, we have $E_0[\Lambda(X)L] \leq M$ for some constant $M > 0$. So $L \in \mathcal{L}(M)$. This concludes the lemma. □

9.1 Proofs in Section 5.1 Proof of Lemma 5.2. We prove the statement point-by-point regarding the operator $\mathcal{K}$. For convenience, denote $S_h(X, \mathbf{X}_{T-1}) = S_h(X, X_1, X_2, \ldots, X_{T-1})$, where S_h is defined in (34), and $\underline{L}_{T-1} = \prod_{t=1}^{T-1} L_t(X_t)$ and $\underline{L}'_{T-1} = \prod_{t=1}^{T-1} L'_t(X_t)$. $X, X_1, \ldots, X_{T-1}$ are i.i.d. random variables with distribution P_0. Also, denote $\beta = 1/\alpha > 0$, so $\beta \to 0$ is equivalent to $\alpha \to \infty$. In this proof we let $C > 0$ be a constant that can be different every time it shows up.

<u>Well-definedness and closedness</u>: Recall the definition of K in (35), which can be written as

$$K(\mathbf{L})(x) = \frac{e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\underline{L}_{T-1}|X=x]/(T-1)!}}{E_0[e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\underline{L}_{T-1}|X]/(T-1)!}]}$$

for any $\mathbf{L} = (L_1, L_2, \ldots, L_{T-1}) \in \mathcal{L}(M)^{T-1}$. We shall show that, for any $\mathbf{L} \in \mathcal{L}(M)^{T-1}$, we have $0 < E_0[e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\underline{L}_{T-1}|X]/(T-1)!}] < \infty$ and that $K(\mathbf{L}) \in \mathcal{L}(M)$. This will imply that, starting from any $L_1, L_2, \ldots, L_{T-1} \in \mathcal{L}(M)$, we get a well-defined operator K and that $\tilde{L}_1, \tilde{L}_2, \ldots, \tilde{L}_{T-1}$ defined in (36) all remain in $\mathcal{L}(M)$. We then conclude that $\mathcal{K}$ is both well-defined and closed in $\mathcal{L}(M)^{T-1}$ by the definition in (37).

Now suppose $L_1, L_2, \ldots, L_{T-1} \in \mathcal{L}(M)$. Since $S_h(X, \mathbf{X}_{T-1}) \leq (T-1)!\left(\Lambda(X) + \sum_{t=1}^{T-1}\Lambda(X_t)\right)$ by definition, we have

$$\begin{aligned} E_0[e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\underline{L}_{T-1}|X]/(T-1)!}] &\leq E_0[e^{\beta(\Lambda(X) + \sum_{t=1}^{T-1} E_0[\Lambda(X_t)L_t(X_t)])}] \\ &= E_0[e^{\beta\Lambda(X)}]e^{\beta\sum_{t=1}^{T-1} E_0[\Lambda(X)L_t(X)]} \\ &\leq E_0[e^{\beta\Lambda(X)}]e^{\beta(T-1)M} \\ &< \infty. \end{aligned} \tag{48}$$

This also implies that $e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\underline{L}_{T-1}|X]/(T-1)!} < \infty$ a.s.. Similarly,

$$E_0[e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\underline{L}_{T-1}|X]/(T-1)!}] \geq E_0[e^{-\beta\Lambda(X)}]e^{-\beta(T-1)M} > 0. \tag{49}$$

Hence K is well-defined. To show closedness, consider

$$\begin{aligned} E_0[\Lambda(X)K(\mathbf{L}_{T-1})(X)] &= E_0\left[\Lambda(X)\frac{e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\underline{L}_{T-1}|X]/(T-1)!}}{E_0[e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\underline{L}_{T-1}|X]/(T-1)!}]}\right] \\ &\leq \frac{E_0[\Lambda(X)e^{\beta\Lambda(X)}]e^{2\beta(T-1)M}}{E_0[e^{-\beta\Lambda(X)}]}. \end{aligned} \tag{50}$$

Since $E_0[\Lambda(X)e^{\beta\Lambda(X)}] \to E_0[\Lambda(X)]$ and $E_0[e^{-\beta\Lambda(X)}] \to 1$ as $\beta \to 0$, (50) is bounded by M for small enough β, if we choose $M > E_0[\Lambda(X)]$. Hence K is closed in $\mathcal{L}(M)$.

By recursing using (36), we get that $\mathcal{K}$ is well-defined, and that for any $\mathbf{L} = (L_1, \ldots, L_{T-1}) \in \mathcal{L}(M)^{T-1}$, we have $\max_{t=1,\ldots,T-1} E_0[\Lambda(X)\tilde{L}_t(X)] \leq M$, and so $\mathcal{K}$ is closed in $\mathcal{L}(M)^{T-1}$.

<u>Contraction</u>: Consider, for any $\mathbf{L} = (L_1, \ldots, L_{T-1}), \mathbf{L}' = (L'_1, \ldots, L'_{T-1}) \in \mathcal{L}(M)^{T-1}$,

$$\begin{aligned} & E_0[(1+\Lambda(X))|K(\mathbf{L})(X) - K(\mathbf{L}')(X)|] \\ = {} & E_0\left[(1+\Lambda(X))\left|\frac{e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\underline{L}_{T-1}|X]/(T-1)!}}{E_0[e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\underline{L}_{T-1}|X]/(T-1)!}]} - \frac{e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\underline{L}'_{T-1}|X]/(T-1)!}}{E_0[e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\underline{L}'_{T-1}|X]/(T-1)!}]}\right|\right] \\ = {} & E_0\Bigg[(1+\Lambda(X))\Bigg|\frac{1}{\xi_2}(e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\underline{L}_{T-1}|X]/(T-1)!} - e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\underline{L}'_{T-1}|X]/(T-1)!}) \\ & - \frac{\xi_1}{\xi_2^2}(E_0[e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\underline{L}_{T-1}|X]/(T-1)!}] - E_0[e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\underline{L}'_{T-1}|X]/(T-1)!}])\Bigg|\Bigg] \end{aligned} \tag{51}$$

by using mean value theorem, where (ξ_1, ξ_2) lies in the line segment between $(e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\underline{L}_{T-1}|X]/(T-1)!}, E_0[e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\underline{L}_{T-1}|X]/(T-1)!}])$ and $(e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\underline{L}'_{T-1}|X]/(T-1)!}, E_0[e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\underline{L}'_{T-1}|X]/(T-1)!}])$. By (49), we have $\xi_2 > 1 - \epsilon$ for some small $\epsilon > 0$, when β is small enough. Moreover, $\xi_1 \leq e^{\beta(\Lambda(X)+(T-1)M)}$. Hence, (51) is less than or equal to

$$\begin{aligned}
&E_0\Bigg[(1+\Lambda(X))\Bigg(\sup\left|\frac{1}{\xi_2}\right|\left|e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\underline{L}_{T-1}|X]/(T-1)!}-e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\underline{L}'_{T-1}|X]/(T-1)!}\right|\\
&+\sup\left|\frac{\xi_1}{\xi_2^2}\right|\left|E_0[e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\underline{L}_{T-1}|X]/(T-1)!}]-E_0[e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\underline{L}'_{T-1}|X]/(T-1)!}]\right|\Bigg)\Bigg]\\
\leq\ &\frac{1}{1-\epsilon}E_0\left[(1+\Lambda(X))\left|e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\underline{L}_{T-1}|X]/(T-1)!}-e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\underline{L}'_{T-1}|X]/(T-1)!}\right|\right]\\
&+\frac{e^{\beta(T-1)M}}{(1-\epsilon)^2}E_0\Big[(1+\Lambda(X))e^{\beta\Lambda(X)}\Big|E_0[e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\underline{L}_{T-1}|X]/(T-1)!}]\\
&-E_0[e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\underline{L}'_{T-1}|X]/(T-1)!}]\Big|\Big]\\
\leq\ &CE_0\left[(1+\Lambda(X))(e^{\beta\Lambda(X)}+1)\left|e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\underline{L}_{T-1}|X]/(T-1)!}-e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\underline{L}'_{T-1}|X]/(T-1)!}\right|\right]\\
\leq\ &C\beta E_0\left[(1+\Lambda(X))\frac{e^{2\beta\Lambda(X)}}{(T-1)!}|E_0[S_h(X,\mathbf{X}_{T-1})\underline{L}_{T-1}|X]-E_0[S_h(X,\mathbf{X}_{T-1})\underline{L}'_{T-1}|X]|\right]\\
&\text{ by mean value theorem again}\\
\leq\ &C\beta E_0\left[(1+\Lambda(X))\frac{e^{2\beta\Lambda(X)}}{(T-1)!}|S_h(X,\mathbf{X}_{T-1})||\underline{L}_{T-1}-\underline{L}'_{T-1}|\right]\\
\leq\ &C\beta E_0\left[(1+\Lambda(X))e^{2\beta\Lambda(X)}\left(\Lambda(X)+\sum_{t=1}^{T-1}\Lambda(X_t)\right)|\underline{L}_{T-1}-\underline{L}'_{T-1}|\right]\\
=\ &C\beta\Bigg(E_0[(1+\Lambda(X))^2e^{2\beta\Lambda(X)}]E_0|\underline{L}_{T-1}-\underline{L}'_{T-1}|+E_0[(1+\Lambda(X))e^{2\beta\Lambda(X)}]\\
&E_0\left[\sum_{t=1}^{T-1}\Lambda(X_t)|\underline{L}_{T-1}-\underline{L}'_{T-1}|\right]\Bigg) \qquad (52)
\end{aligned}$$

when β is small enough (or α is large enough). Now note that

$$|\underline{L}_{T-1}-\underline{L}'_{T-1}|\leq\sum_{s=1}^{T-1}\dot{\underline{L}}_{T-1}^{s}|L_s(X_s)-L'_s(X_s)|$$

where each $\dot{\underline{L}}_{T-1}^{s}$ is a product of either one of $L(X_r)$ or $L'(X_r)$ for $r=1,\dots,T-1,r\neq s$. Hence (52) is less than or equal to

$$\begin{aligned}
&C\beta\Bigg(E_0[(1+\Lambda(X))^2e^{2\beta\Lambda(X)}]\sum_{s=1}^{T-1}E_0|L_s(X)-L'_s(X)|+E_0[(1+\Lambda(X))e^{2\beta\Lambda(X)}]\\
&\sum_{t=1}^{T-1}\Bigg(E_0[\Lambda(X)|L_t(X)-L'_t(X)|]+E_0[\Lambda(X)L(X)]\sum_{\substack{s=1,\dots,T-1\\ s\neq t}}E_0|L_s(X)-L'_s(X)|\Bigg)\Bigg). \qquad (53)
\end{aligned}$$

Now for convenience denote $y_t=E_0[(1+\Lambda(X))|L_t(X)-L'_t(X)|]$ and $\tilde{y}_t=E_0[(1+\Lambda(X))|\tilde{L}_t(X)-\tilde{L}'_t(X)|]$, where $\tilde{L}_t$ is defined in (36). Also denote $\mathbf{y}=(y_1,\dots,y_{T-1})'$ and $\tilde{\mathbf{y}}=(\tilde{y}_1,\dots,\tilde{y}_{T-1})'$, where $'$ denotes transpose. Then (53) gives $\tilde{y}_1\leq a\mathbf{1}'\mathbf{y}$ for some $a:=a(\beta)=O(\beta)$ as $\beta\to0$, and $\mathbf{1}$ denotes the $(T-1)$-dimensional vector of constant 1. Then, by iterating the same argument as above, (36) implies that

$$\begin{aligned}
\tilde{y}_2&\leq a\mathbf{1}'(\tilde{y}_1,y_2,\dots,y_{T-1})'\\
\tilde{y}_3&\leq a\mathbf{1}'(\tilde{y}_1,\tilde{y}_2,y_3,\dots,y_{T-1})'\\
&\ \vdots\\
\tilde{y}_{T-1}&\leq a\mathbf{1}'(\tilde{y}_1,\dots,\tilde{y}_{T-2},y_{T-1})'.
\end{aligned}$$

Hence

$$\max_{t=1,\dots,T-1}\tilde{y}_t\leq\max_{t=1,\dots,T-1}\sum_{s=1}^{T}A_{ts}y_s$$

where $A_{ts} := A_{ts}(\beta)$ are constants that go to 0 as $\beta \to 0$. Therefore, when β is small enough, $d(\mathcal{K}(\mathbf{L}_{T-1}), \mathcal{K}(\mathbf{L}'_{T-1})) \leq wd(\mathbf{L}_{T-1}, \mathbf{L}'_{T-1})$ for some $0 < w < 1$, and $\mathcal{K}$ is a contraction. By Banach fixed point theorem, there exists a unique fixed point $\mathbf{L}^*$. Moreover, as a consequence, starting from any initial value $\mathbf{L}^{(1)} = \mathbf{L} \in \mathcal{L}(M)^{T-1}$, the recursion $\mathbf{L}^{(k+1)} = \mathcal{K}(\mathbf{L}^{(k)})$ satisfies $\mathbf{L}^{(k)} \xrightarrow{d} \mathbf{L}^*$ where $\mathbf{L}^*$ is the fixed point of $\mathcal{K}$.

Identical Components: It remains to show that all components of $\mathbf{L}^*$ are the same. Denote $\mathbf{L}^* = (L_1^*, \ldots, L_{T-1}^*)$. By definition $\mathcal{K}(\mathbf{L}^*) = \mathbf{L}^*$. So, using (36), we have

$$\begin{aligned}
\tilde{L}_1 &= K(L_1^*, \ldots, L_{T-1}^*) = L_1^* \\
\tilde{L}_2 &= K(\tilde{L}_1, L_2^*, \ldots, L_{T-1}^*) = K(L_1^*, L_2^*, \ldots, L_{T-1}^*) = L_2^* \\
\tilde{L}_3 &= K(\tilde{L}_1, \tilde{L}_2, L_3^*, \ldots, L_{T-1}^*) = K(L_1^*, L_2^*, L_3^*, \ldots, L_{T-1}^*) = L_3^* \\
&\vdots \\
\tilde{L}_{T-1} &= K(\tilde{L}_1, \ldots, \tilde{L}_{T-2}, L_{T-1}^*) = K(L_1^*, \ldots, L_T^*, L_{T-1}^*) = L_{T-1}^*.
\end{aligned}$$

Hence $L_1^* = L_2^* = \cdots = L_{T-1}^* = \tilde{L}_1 = \cdots = \tilde{L}_{T-1} = K(L_1^*, \ldots, L_{T-1}^*)$. This concludes the lemma.

□

Proof of Corollary 5.1. By Lemma 5.2, $\mathcal{K}$ has a fixed point in $\mathcal{L}(M)^{T-1}$ that has all equal components. Since convergence in $\mathcal{L}(M)^{T-1}$ implies convergence in each component in $\mathcal{L}(M)$ (in the $\|\cdot\|_\Lambda$-norm), and that by construction $K(L_1, L_2, \ldots, L_{T-1}) = K(L_{T-1}, L_1, L_2, \ldots, L_{T-2})$ for any $L_1, L_2, \ldots, L_{T-1} \in \mathcal{L}(M)$, the result follows. □

Proof of Lemma 5.3. Consider

$$\begin{aligned}
& E_0\left[S_h(\mathbf{X}_T)\prod_{t=1}^T L^{(k+t-1)}(X_t)\right] - \alpha(T-1)!\sum_{t=1}^T E_0[L^{(k+t-1)}\log L^{(k+t-1)}] \\
= \;& E_0\left[E_0\left[S_h(\mathbf{X}_T)\prod_{t=2}^T L^{(k+t-1)}(X_t)\middle|X_1\right]L^{(k)}(X_1)\right] - \alpha(T-1)!E_0[L^{(k)}\log L^{(k)}] \\
& -\alpha(T-1)!\sum_{t=2}^T E_0[L^{(k+t-1)}\log L^{(k+t-1)}] \\
\leq \;& E_0\left[E_0\left[S_h(\mathbf{X}_T)\prod_{t=2}^T L^{(k+t-1)}(X_t)\middle|X_1\right]L^{(k+T)}(X_1)\right] - \alpha(T-1)!E_0[L^{(k+T)}\log L^{(k+T)}] \\
& -\alpha(T-1)!\sum_{t=2}^T E_0[L^{(k+t-1)}\log L^{(k+t-1)}] \\
= \;& E_0\left[S_h(\mathbf{X}_T)\prod_{t=1}^T L^{(k+t)}(X_t)\right] - \alpha(T-1)!\sum_{t=1}^T E_0[L^{(k+t)}\log L^{(k+t)}].
\end{aligned}$$

The inequality holds because $L^{(k+T)} = K(L^{(k+1)}, \ldots, L^{(k+T-1)})$, which, by Proposition 3.1 and the definition of K, maximizes the objective $E_0\left[E_0\left[S_h(\mathbf{X}_T)\prod_{t=2}^T L^{(k+t-1)}(X_t)\middle|X_1\right]L(X_1)\right] - \alpha(T-1)!E_0[L\log L]$ over L. The last equality can be seen by the invariance of S_h over permutations of its arguments, and relabeling X_2 by X_1, X_3 by X_2, up to X_T by X_{T-1} and X_1 by X_T. □

Proof of Lemma 5.4. We consider convergence of the first and the second terms of (39) separately.

For the first term, consider

$$\begin{aligned}
&\left|E_0\left[S_h(\mathbf{X}_T)\prod_{t=1}^T L^{(k+t-1)}(X_t)\right]-E_0\left[S_h(\mathbf{X}_T)\prod_{t=1}^T L^*(X_t)\right]\right|\\
\leq\ & E_0\left[|S_h(\mathbf{X}_T)|\left|\prod_{t=1}^T L^{(k+t-1)}(X_t)-\prod_{t=1}^T L^*(X_t)\right|\right]\\
\leq\ & (T-1)!E_0\left[\sum_{t=1}^T\Lambda(X_t)\sum_{s=1}^T \dot{\underline{L}}_T^s|L^{(k+s-1)}(X_s)-L^*(X_s)|\right]\\
&\text{where each } \dot{\underline{L}}_T^s \text{ is product of either one of } L^{(k+r-1)}(X_r) \text{ or } L^*(X_r) \text{ for } r=1,\dots,T,\ r\neq s\\
\leq\ & C\sum_{s=1}^T E_0[(1+\Lambda(X))|L^{(k+s-1)}(X)-L^*(X)|] \text{ for some constant } C>0\\
\to\ & 0
\end{aligned} \tag{54}$$

as $k\to\infty$, since $L^{(k)}\to L^*$ in $\|\cdot\|_\Lambda$-norm by Corollary 5.1.

We now consider the second term in (39). By the recursion of K, we have, for $k\geq 1$,

$$\begin{aligned}
&|E_0[L^{(k+T-1)}\log L^{(k+T-1)}]-E_0[L^*\log L^*]|\\
=\ & \Bigg|\left(\frac{\beta}{(T-1)!}E_0\left[S_h(\mathbf{X}_T)\prod_{t=1}^T L^{(k+t-1)}(X_t)\right]-\log E_0[e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\prod_{t=1}^{T-1}L^{(k+t-1)}(X_t)|X]/(T-1)!}]\right)\\
&-\left(\frac{\beta}{(T-1)!}E_0\left[S_h(\mathbf{X}_T)\prod_{t=1}^T L^*(X_t)\right]-\log E_0[e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\prod_{t=1}^{T-1}L^*(X_t)|X]/(T-1)!}]\right)\Bigg|\\
=\ & \frac{\beta}{(T-1)!}\left|E_0\left[S_h(\mathbf{X}_T)\prod_{t=1}^T L^{(k+t-1)}(X_t)\right]-E_0\left[S_h(\mathbf{X}_T)\prod_{t=1}^T L^*(X_t)\right]\right|\\
&+|\log E_0[e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\prod_{t=1}^{T-1}L^{(k+t-1)}(X_t)|X]/(T-1)!}]-\log E_0[e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\prod_{t=1}^{T-1}L^*(X_t)|X]/(T-1)!}]|
\end{aligned} \tag{55}$$

The first term in (55) converges to 0 by the same argument as in (54). For the second term, we can write, by mean value theorem, that

$$\begin{aligned}
&|\log E_0[e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\prod_{t=1}^{T-1}L^{(k+t-1)}(X_t)|X]/(T-1)!}]-\log E_0[e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\prod_{t=1}^{T-1}L^*(X_t)|X]/(T-1)!}]|\\
=\ & \frac{1}{\xi_1}|E_0[e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\prod_{t=1}^{T-1}L^{(k+t-1)}(X_t)|X]/(T-1)!}]-E_0[e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\prod_{t=1}^{T-1}L^*(X_t)|X]/(T-1)!}]|
\end{aligned} \tag{56}$$

where ξ_1 lies between $E_0[e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\prod_{t=1}^{T-1}L^{(k+t-1)}(X_t)|X]/(T-1)!}]$ and $E_0[e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\prod_{t=1}^{T-1}L^*(X_t)|X]/(T-1)!}]$, and hence $\xi_1\geq E_0[e^{-\beta\Lambda(X)}]e^{-\beta E_0[\Lambda(X)L(X)]}\geq 1-\epsilon$ for some small $\epsilon>0$, when β is small enough, by a similar argument as in the proof of Lemma 5.2. Moreover,

$$\begin{aligned}
&|E_0[e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\prod_{t=1}^{T-1}L^{(k+t-1)}(X_t)|X]/(T-1)!}]-E_0[e^{\beta E_0[S_h(X,\mathbf{X}_{T-1})\prod_{t=1}^{T-1}L^*(X_t)|X]/(T-1)!}]|\\
\leq\ & \beta E_0\left[e^{\beta\xi_2}|S_h(X,\mathbf{X}_{T-1})|\left|\prod_{t=1}^{T-1}L^{(k+t-1)}(X_t)-\prod_{t=1}^{T-1}L^*(X_t)\right|\right]
\end{aligned}$$

for some ξ_2 lying between $E_0[S_h(X,\mathbf{X}_{T-1})\prod_{t=1}^{T-1}L^{(k+t-1)}(X_t)|X]/(T-1)!$ and $E_0[S_h(X,\mathbf{X}_{T-1})\prod_{t=1}^{T-1}L^*(X_t)|X]/(T-1)!$. Hence, much like the argument in proving the contraction property in Lemma 5.2, we have $\xi_2\leq\Lambda(X)+(T-1)M$ and (56) is less than or equal to

$$C\beta\max_{t=1,\dots,T-1}E_0[(1+\Lambda(X))|L^{(k+t-1)}(X)-L^*(X)|]\to 0$$

as $k\to\infty$ for some $C>0$. This concludes the lemma. □

9.2 Proof of Theorem 3.2 For convenience, denote $\beta=1/\alpha^*>0$, so β is small when α^* is large. Also let X be a generic random variable with distribution P_0. Then from (30) we have

$$L^*(x)=\frac{e^{\beta g^{L^*}(x)}}{E_0[e^{\beta g^{L^*}(X)}]} \tag{57}$$

where $g^{L^*}(x) = \sum_{t=1}^T g_t^{L^*}(x) = \sum_{t=1}^T E_0[h(\mathbf{X}_T)\prod_{\substack{1\leq r\leq T\\ r\neq t}} L^*(X_r)|X_t = x]$. Also recall that

$$g(x) = \sum_{t=1}^T g_t(x) = \sum_{t=1}^T E_0[h(\mathbf{X}_T)|X_t = x]$$

as defined in (14), so that $E_0[g(X)] = TE_0[h(\mathbf{X}_T)]$. Furthermore, let us denote, for any $p \geq 1$, $\bar{O}(\beta^p) := \bar{O}(\beta^p; x)$ as a deterministic function in x such that $E_0[h(\mathbf{X}_T)^q \bar{O}(\beta^p; X_t)] = O(\beta^p)$ for any $q \geq 1$ and $t = 1, \ldots, T$, when $\beta \to 0$. Finally, we also let $\psi_L(\beta) := \log E_0[e^{\beta g^L(X)}]$ for convenience.

We first give a quadratic approximation of L^* as $\beta \to 0$ (equivalently $\alpha^* \to \infty$). Then we find the relation between β and η, which verifies the optimality condition given in Theorem A.1. After that we expand the objective value in terms of β, and hence η, to conclude Theorem 3.2.

Asymptotic expansion of L^*: We shall obtain a quadratic approximation of L^* by first getting a first order approximation of L^* and then iterating via the quantity g^{L^*} to get to the second order. Note that as the logarithmic moment generating function of $g^{L^*}(X)$,

$$\begin{aligned}\psi_{L^*}(\beta) &= \log E_0[e^{\beta g^{L^*}(X)}] \\ &= \beta E_0[g^{L^*}(X)] + \frac{\beta^2}{2}\kappa_2(g^{L^*}(X)) + \frac{\beta^3}{3!}\kappa_3(g^{L^*}(X)) + O(\beta^4) \qquad (58)\end{aligned}$$

where $\kappa_2(g^{L^*}(X)) := E_0[(g^{L^*}(X) - E_0[g^{L^*}(X)])^2]$ and $\kappa_3(g^{L^*}(X)) := [(g^{L^*}(X) - E_0[g^{L^*}(X)])^3]$. Using (57) and (58), and the finiteness of the exponential moment of $g^{L^*}(X)$ guaranteed by a calculation similar to (48), we have

$$\begin{aligned}L^*(x) &= \frac{e^{\beta g^{L^*}(x)}}{E_0[e^{\beta g^{L^*}(X)}]} = e^{\beta g^{L^*}(x) - \psi_{L^*}(\beta)} \\ &= 1 + \beta(g^{L^*}(x) - E_0[g^{L^*}(X)]) + \bar{O}(\beta^2). \qquad (59)\end{aligned}$$

But notice that

$$\begin{aligned}g^{L^*}(x) &= \sum_{t=1}^T E_0\left[h(\mathbf{X}_T)\prod_{\substack{1\leq r\leq T\\ r\neq t}} L^*(X_r)\middle|X_t = x\right] \\ &= \sum_{t=1}^T E_0\left[h(\mathbf{X}_T)\prod_{\substack{1\leq r\leq T\\ r\neq t}} (1 + \bar{O}(\beta; X_r))\middle|X_t = x\right] \\ &= \sum_{t=1}^T E_0[h(\mathbf{X}_T)|X_t = x] + \bar{O}(\beta) \\ &= g(x) + \bar{O}(\beta)\end{aligned}$$

and hence $E_0[g^{L^*}(X)] = E_0[g(X)] + O(\beta)$. Consequently, from (59) we have

$$L^*(x) = 1 + \beta(g(x) - E_0[g(X)]) + \bar{O}(\beta^2). \qquad (60)$$

This gives a first order approximation of L^*. Using (60), we strengthen our approximation of g^{L^*} to get

$$\begin{aligned}g^{L^*}(x) &= \sum_{t=1}^T E_0\left[h(\mathbf{X}_T)\prod_{\substack{1\leq r\leq T\\ r\neq t}} (1 + \beta(g(X_r) - E_0[g(X)]) + \bar{O}(\beta^2))\middle|X_t = x\right] \\ &= g(x) + \beta\sum_{t=1}^T \sum_{\substack{1\leq r\leq T\\ r\neq t}} E_0[h(\mathbf{X}_T)(g(X_r) - E_0[g(X)])|X_t = x] + \bar{O}(\beta^2) \\ &= g(x) + \beta W(x) + \bar{O}(\beta^2) \qquad (61)\end{aligned}$$

where we define $W(x) := \sum_{t=1}^T \sum_{\substack{1\leq r\leq T\\ r\neq t}} E_0[h(\mathbf{X}_T)(g(X_r)-E_0[g(X)])|X_t=x]$. With (61), and using (58) again, we then strengthen the approximation of L^* to get

$$\begin{aligned}
L^*(x) &= e^{\beta g^{L^*}(x)-\psi_{L^*}(\beta)} = e^{\beta(g^{L^*}(x)-E_0[g^{L^*}(X)])-\frac{\beta^2}{2}E_0[(g^{L^*}(X)-E_0[g^{L^*}(X)])^2]+\bar{O}(\beta^3)}\\
&= 1+\beta(g^{L^*}(x)-E_0[g^{L^*}(X)])+\frac{\beta^2}{2}[(g^{L^*}(x)-E_0[g^{L^*}(X)])^2-E_0[(g^{L^*}(X)-E_0[g^{L^*}(X)])^2]]\\
&\quad +\bar{O}(\beta^3)\\
&= 1+\beta(g(x)-E_0[g(X)])+\beta^2\Big[W(x)-E_0[W(X)]+\frac{1}{2}((g(x)-E_0[g(X)])^2\\
&\quad -E_0[(g(X)-E_0[g(X)])^2])\Big]+\bar{O}(\beta^3)\\
&= 1+\beta(g(x)-E_0[g(X)])+\beta^2V(x)+\bar{O}(\beta^3)
\end{aligned}\tag{62}$$

where we define $V(x) := W(x)-E_0[W(X)]+\frac{1}{2}((g(x)-E_0[g(X)])^2-E_0[(g(X)-E_0[g(X)])^2])$.

Relation between β and η: By substituting L^* depicted in (57) into $\eta=E_0[L^*\log L^*]$, we have

$$\eta = E_0[L^*\log L^*] = \beta E_0[g^{L^*}(X)L^*(X)]-\log E_0[e^{\beta g^{L^*}(X)}] = \beta T E_0[h(\mathbf{X}_T)\underline{L}_T^*]-\psi_{L^*}(\beta) \tag{63}$$

Using (58), we can write (63) as

$$\begin{aligned}
&\beta T E_0[h(\mathbf{X}_T)\underline{L}_T^*]-\beta E_0[g^{L^*}(X)]-\frac{\beta^2}{2}\kappa_2(g^{L^*}(X))-\frac{\beta^3}{3!}\kappa_3(g^{L^*}(X))+O(\beta^4)\\
=\ &\beta\sum_{t=1}^T E_0\left[h(\mathbf{X}_T)\prod_{\substack{1\leq r\leq T\\ r\neq t}} L^*(X_r)(L^*(X_t)-1)\right]-\frac{\beta^2}{2}\kappa_2(g^{L^*}(X))-\frac{\beta^3}{3!}\kappa_3(g^{L^*}(X))\\
&+O(\beta^4).
\end{aligned}\tag{64}$$

We analyze (64) term by term. For the first term, using (62), we have

$$\begin{aligned}
&\sum_{t=1}^T E_0\left[h(\mathbf{X}_T)\prod_{\substack{1\leq r\leq T\\ r\neq t}} L^*(X_r)(L^*(X_t)-1)\right]\\
=\ &\sum_{t=1}^T E_0\Bigg[h(\mathbf{X}_T)\prod_{\substack{1\leq r\leq T\\ r\neq t}}\big(1+\beta(g(X_r)-E_0[g(X)])+\beta^2V(X_r)+\bar{O}(\beta^3)\big)\\
&\cdot\big(\beta(g(X_t)-E_0[g(X)])+\beta^2V(X_t)+\bar{O}(\beta^3)\big)\Bigg]\\
=\ &\beta\sum_{t=1}^T E_0[h(\mathbf{X}_T)(g(X_t)-E_0[g(X)])]+\beta^2\Bigg[\sum_{t=1}^T\sum_{\substack{1\leq r\leq T\\ r\neq t}} E_0[h(\mathbf{X}_T)(g(X_r)-E_0[g(X)])(g(X_t)-E_0[g(X)])]\\
&+\sum_{t=1}^T E_0[h(\mathbf{X}_T)V(X_t)]\Bigg]+O(\beta^3)\\
=\ &\beta Var_0(g(X))+\beta^2[\nu+E_0[g(X)V(X)]]+O(\beta^3)
\end{aligned}\tag{65}$$

where ν is defined in (16). The last equality follows since

$$\begin{aligned}
&\sum_{t=1}^T E_0[h(\mathbf{X}_T)(g(X_t)-E_0[g(X)])] = \sum_{t=1}^T E_0[E_0[h(\mathbf{X}_T)|X_t](g(X_t)-E_0[g(X)])]\\
=\ &\sum_{t=1}^T E_0[g_t(X)(g(X)-E_0[g(X)])] = E_0[g(X)(g(X)-E_0[g(X)])] = Var_0(g(X)),
\end{aligned}$$

$$\begin{aligned}
&\sum_{t=1}^{T}\sum_{\substack{1\leq r\leq T\\ r\neq t}} E_0[h(\mathbf{X}_T)(g(X_r)-E_0[g(X)])(g(X_t)-E_0[g(X)])]\\
=&\sum_{t=1}^{T}\sum_{\substack{1\leq r\leq T\\ r\neq t}} E_0[E_0[h(\mathbf{X}_T)|X_r,X_t](g(X_r)-E_0[g(X)])(g(X_t)-E_0[g(X)])]\\
=&\ E_0[G(X,Y)(g(X)-E_0[g(X)])(g(Y)-E_0[g(Y)])]=\nu
\end{aligned}$$

where $G(X,Y)$ is defined in (17), and

$$\sum_{t=1}^{T} E_0[h(\mathbf{X}_T)V(X_t)]=\sum_{t=1}^{T} E_0[E_0[h(\mathbf{X}_T)|X_t]V(X_t)]=E_0[g(X)V(X)].$$

For the second term in (64), by using (61), we have

$$\begin{aligned}
\kappa_2(g^{L^*}(X))&=E_0[(g^{L^*}(X)-E_0[g^{L^*}(X)])^2]\\
&=E_0[((g(X)-E_0[g(X)])+\beta(W(X)-E_0[W(X)])+\bar{O}(\beta^2;X))^2]\\
&=Var_0(g(X))+2\beta E_0[(g(X)-E_0[g(X)])(W(X)-E_0[W(X)])]+O(\beta^2). \qquad (66)
\end{aligned}$$

Now notice that $W(x)$ can be written as

$$\begin{aligned}
W(x)&=\sum_{t=1}^{T}\sum_{\substack{1\leq r\leq T\\ r\neq t}} E_0[h(\mathbf{X}_T)(g(X_r)-E_0[g(X)])|X_t=x]\\
&=\sum_{t=1}^{T}\sum_{\substack{1\leq r\leq T\\ r\neq t}} E_0[E_0[h(\mathbf{X}_T)|X_r,X_t](g(X_r)-E_0[g(X)])|X_t=x]\\
&=E_0[G(X,Y)(g(Y)-E_0[g(Y)])|X=x]
\end{aligned}$$

where $G(X,Y)$ is defined in (17). Hence

$$\begin{aligned}
&E_0[(g(X)-E_0[g(X)])(W(X)-E_0[W(X)])]=E_0[(g(X)-E_0[g(X)])W(X)]\\
=&\ E_0[(g(X)-E_0[g(X)])G(X,Y)(g(Y)-E_0[g(Y)])]=\nu.
\end{aligned}$$

Consequently, (66) becomes

$$Var_0(g(X))+2\beta\nu+O(\beta^2). \qquad (67)$$

Finally, for the third term in (64), we have

$$\kappa_3(g^{L^*}(X))=E_0[(g(X)-E_0[g(X)])^3]+O(\beta)=\kappa_3(g(X))+O(\beta). \qquad (68)$$

Combining (65), (67) and (68), we have

$$\begin{aligned}
\eta&=\beta^2 Var_0(g(X))+\beta^3[\nu+E_0[g(X)V(X)]]-\frac{\beta^2}{2}Var_0(g(X))-\beta^3\nu-\frac{\beta^3}{6}\kappa_3(g(X))+O(\beta^4)\\
&=\frac{\beta^2}{2}Var_0(g(X))+\beta^3\left[E_0[g(X)V(X)]-\frac{1}{6}\kappa_3(g(X))\right]+O(\beta^4). \qquad (69)
\end{aligned}$$

Under Assumption 3.4, and by routinely checking that the term $O(\beta^4)$ in (69) above is continuous in β, we can invert (69) to get

$$\begin{aligned}
\beta&=\sqrt{\frac{2\eta}{Var_0(g(X))}}\left(1+\frac{2\beta(E_0[g(X)V(X)]-(1/6)\kappa_3(g(X)))}{Var_0(g(X))}+O(\beta^2)\right)^{-1/2}\\
&=\sqrt{\frac{2\eta}{Var_0(g(X))}}-\frac{1}{2}\sqrt{\frac{2\eta}{Var_0(g(X))}}\ \frac{2\beta(E_0[g(X)V(X)]-(1/6)\kappa_3(g(X)))}{Var_0(g(X))}+O(\eta^{1/2}\beta^2)\\
&=\sqrt{\frac{2\eta}{Var_0(g(X))}}-\frac{2\eta(E_0[g(X)V(X)]-(1/6)\kappa_3(g(X)))}{(Var_0(g(X)))^2}+O(\eta^{3/2}). \qquad (70)
\end{aligned}$$

This in particular verifies the condition in Theorem A.1, i.e. for any small η, there exists a large enough $\alpha^*>0$ and a corresponding L^* that satisfies (83). This L^* is an optimal solution to (26).

Relation between the objective value and β, and hence η: Using (62) again, the optimal objective value in (26) can be written as

$$\begin{aligned}
&E_0[h(\mathbf{X}_T)\underline{L}_T^*]\\
=\;& E_0\left[h(\mathbf{X}_T)\prod_{t=1}^T(1+\beta(g(X_t)-E_0[g(X)])+\beta^2V(X_t)+\bar{O}(\beta^3;X_t))\right]\\
=\;& E_0[h(\mathbf{X}_T)]+\beta\sum_{t=1}^T E_0[h(\mathbf{X}_T)(g(X_t)-E_0[g(X)])]\\
&+\beta^2\left[\sum_{t=1}^T\sum_{\substack{1\leq r\leq T\\ r<t}}E_0[h(\mathbf{X}_T)(g(X_r)-E_0[g(X)])(g(X_t)-E_0[g(X)])]+\sum_{t=1}^T E_0[h(\mathbf{X}_T)V(X_t)]\right]+O(\beta^3)\\
=\;& E_0[h(\mathbf{X}_T)]+\beta Var_0(g(X))+\beta^2\left[\frac{\nu}{2}+E_0[g(X)V(X)]\right]+O(\beta^3)
\end{aligned}\tag{71}$$

where the last equality follows from similar argument in (65). Finally, substituting (70) into (71) gives

$$\begin{aligned}
&E_0[h(\mathbf{X}_T)]+\sqrt{2Var_0(g(X))\eta}+\frac{2\eta}{Var_0(g(X))}\left[-E_0[g(X)V(X)]+\frac{1}{6}\kappa_3(g(X))+\frac{\nu}{2}+E_0[g(X)V(X)]\right]\\
&+O(\eta^{3/2})\\
=\;& E_0[h(\mathbf{X}_T)]+\sqrt{2Var_0(g(X))\eta}+\frac{\eta}{Var_0(g(X))}\left[\frac{1}{3}\kappa_3(g(X))+\nu\right]+O(\eta^{3/2})
\end{aligned}$$

which coincides with Theorem 3.2.

9.3 Proof of Proposition 3.3 under Assumption 3.6 Our goal here is to obtain an analog of Proposition 5.1 for the random time horizon setting under Assumption 3.6. Once this is established, the asymptotic expansion will follow the same argument as the proof of Theorem 3.2.

We use a truncation argument. First let us focus on the finite horizon setting, i.e. cost function is $h(\mathbf{X}_T)$. We begin by observing that the operator $\bar{\mathcal{K}}:\mathcal{L}(M)\to\mathcal{L}(M)$ defined as

$$\bar{\mathcal{K}}(L)(x)=\frac{e^{g^L(x)/\alpha}}{E_0[e^{g^L(X)/\alpha}]}\tag{72}$$

where $g^L(x)$ is defined in (30), possesses similar contraction properties as the operator $\mathcal{K}$ in (37) in the following sense:

Lemma 9.1 *With Assumption 3.3 on the cost function $h(\mathbf{X}_T)$, for sufficiently large α, the operator $\bar{\mathcal{K}}:\mathcal{L}(M)\to\mathcal{L}(M)$ is well-defined, closed, and a strict contraction in $\mathcal{L}(M)$ under the metric induced by $\|\cdot\|_\Lambda$. Hence there exists a unique fixed point $L^*\in\mathcal{L}(M)$ that satisfies $\bar{\mathcal{K}}(L)=L$. Moreover, L^* is equal to each identical component of the fixed point of $\mathcal{K}$ defined in* (37).

Proof. We shall utilize our result on the operator $\mathcal{K}$ in Lemma 5.2. It is easy to check that given $L\in\mathcal{L}(M)$, $\bar{\mathcal{K}}$ acted on L has the same effect as the mapping K, defined in (35), acted on $(L,\ldots,L)\in\mathcal{L}(M)^{T-1}$. In the proof of Lemma 5.2 we have already shown that $K(L_1,\ldots,L_{T-1})$ for any $(L_1,\ldots,L_{T-1})\in\mathcal{L}(M)$ is well-defined, closed, and a strict contraction under $\|\cdot\|_\Lambda$, when α is large enough (or $\beta=1/\alpha$ is small enough in that proof). These properties are inherited immediately to the operator $\bar{\mathcal{K}}$.

Next, note that (30) is the fixed point equation associated with $\bar{\mathcal{K}}$. Moreover, we have already shown in Proposition 5.1 that the same equation governs the fixed point of $\mathcal{K}$, in the sense that the $T-1$ components of its fixed point are all identical and satisfy (30). By the uniqueness property of fixed points, we conclude that the fixed point of $\bar{\mathcal{K}}$ coincides with each identical component of the fixed point of $\mathcal{K}$. □

Now consider a cost function $h(\mathbf{X}_\tau)$ with a random time τ that satisfies Assumption 3.6. Again let $\beta = 1/\alpha > 0$ for convenience. We introduce a sequence of truncated random time $\tau \wedge T$, and define $\tilde{\mathcal{K}}_T : \mathcal{L} \to \mathcal{L}$ and $\tilde{\mathcal{K}} : \mathcal{L} \to \mathcal{L}$ as

$$\tilde{\mathcal{K}}_T(L)(x) := \frac{e^{\beta \tilde{g}^{L,T}(x)}}{E_0[e^{\beta \tilde{g}^{L,T}(X)}]}$$

and

$$\tilde{\mathcal{K}}(L)(x) := \frac{e^{\beta \tilde{g}^{L}(x)}}{E_0[e^{\beta \tilde{g}^{L}(X)}]}$$

where

$$\tilde{g}^{L,T}(x) := \sum_{t=1}^{T} E_0[h(\mathbf{X}_{\tau\wedge T}) \underline{L}^t_{\tau\wedge T}; \tau \wedge T \geq t | X_t = x]$$

and

$$\tilde{g}^{L}(x) := \sum_{t=1}^{T} E_0[h(\mathbf{X}_{\tau}) \underline{L}^t_{\tau}; \tau \geq t | X_t = x].$$

Here $\underline{L}^t_s = \prod_{r=1,\ldots,s, r\neq t} L(X_r)$. In other words, $\tilde{\mathcal{K}}_T$ is the map identical to $\tilde{\mathcal{K}}$ except that τ is replaced by $\tau \wedge T$.

We first need the following proposition:

PROPOSITION 9.1 *Suppose Assumption 3.6 holds. For $\beta \leq \epsilon$ for some small $\epsilon > 0$, both $\tilde{\mathcal{K}}_T$, for any $T \geq 1$, and $\tilde{\mathcal{K}}$ are well-defined, closed and strict contractions with the same Lipschitz constant on the space $\mathcal{L}$ equipped with the metric induced by the $\mathcal{L}_1$-norm $\|L - L'\|_1 := E_0|L - L'|$.*

PROOF. We first consider the map $\tilde{\mathcal{K}}$. Recall that by Assumption 3.6 we have $|h(\mathbf{X}_\tau)| \leq C$ for some constant $C > 0$. Consider

$$\begin{aligned}
e^{\beta \sum_{t=1}^\infty E_0[h(\mathbf{X}_\tau)\underline{L}^t_\tau; \tau \geq t | X_t = x]} &\leq e^{C\beta \sum_{t=1}^\infty E_0[\underline{L}^t_\tau; \tau \geq t | X_t = x]} \\
&= e^{C\beta \sum_{t=1}^\infty E_0[\underline{L}^t_\tau; \tau \geq t]} \quad \text{since } \underline{L}^t_\tau I(\tau \geq t) \text{ is independent of } X_t \\
&= e^{C\beta \sum_{t=1}^\infty P_0(\tau \geq t)} \quad \text{since } \tau \text{ is independent of } \{X_t\}_{t\geq 1} \\
&= e^{C\beta E_0 \tau} \\
&< \infty \quad \text{by Assumption 3.6.}
\end{aligned} \tag{73}$$

Similarly,

$$e^{\beta \sum_{t=1}^\infty E_0[h(\mathbf{X}_\tau)\underline{L}^t_\tau; \tau \geq t | X_t = x]} \geq e^{-C\beta E_0 \tau} > 0. \tag{74}$$

Therefore $\tilde{\mathcal{K}}$ is well-defined and also closed in $\mathcal{L}$. To prove that $\tilde{\mathcal{K}}$ is a contraction, consider, for any $L, L' \in \mathcal{L}$,

$$\begin{aligned}
E_0|\tilde{\mathcal{K}}(L) - \tilde{\mathcal{K}}(L')| &= E_0 \left| \frac{e^{\beta \tilde{g}^L(X)}}{E_0[e^{\beta \tilde{g}^L(X)}]} - \frac{e^{\beta \tilde{g}^{L'}(X)}}{E_0[e^{\beta \tilde{g}^{L'}(X)}]} \right| \\
&\leq E_0 \left[\sup \left| \frac{1}{\xi_2} \right| |e^{\beta \tilde{g}^L(X)} - e^{\beta \tilde{g}^{L'}(X)}| + \sup \left| \frac{\xi_1}{\xi_2^2} \right| |E_0[e^{\beta \tilde{g}^L(X)}] - E_0[e^{\beta \tilde{g}^{L'}(X)}]| \right]
\end{aligned} \tag{75}$$

by mean value theorem, where (ξ_1, ξ_2) lies in the line segment between $(e^{\beta \tilde{g}^L(X)}, E_0[e^{\beta \tilde{g}^L(X)}])$ and $(e^{\beta \tilde{g}^{L'}(X)}, E_0[e^{\beta \tilde{g}^{L'}(X)}])$. By (73) and (74), we have $\xi_1 \leq e^{C\beta E_0 \tau}$ and $\xi_2 \geq e^{-C\beta E_0\tau}$ a.s.. So (75) is less than or equal to

$$2e^{3C\beta E_0\tau} E_0 |e^{\beta \tilde{g}^L(X)} - e^{\beta \tilde{g}^{L'}(X)}| \leq 2e^{3C\beta E_0 \tau} \beta E_0 |\xi| |\tilde{g}^L(X) - \tilde{g}^{L'}(X)| \tag{76}$$

by mean value theorem again, where ξ lies between $e^{\beta \tilde{g}^L(X)}$ and $e^{\beta \tilde{g}^{L'}(X)}$ and hence $\xi \leq e^{C\beta E_0\tau}$ a.s.. Therefore (76) is further bounded by

$$2e^{4C\beta E_0\tau} \beta \sum_{t=1}^\infty E_0[|\underline{L}^t_\tau - \underline{L'}^t_\tau|; \tau \geq t]. \tag{77}$$

Conditioning on τ, $|\underline{L}_\tau^t - \underline{L}_\tau'^{\,t}| \leq \sum_{s=1,\ldots,\tau,s\neq t} \dot{\underline{L}}_\tau^{t,s}|L_s - L_s'|$ where $\dot{\underline{L}}_\tau^{t,s}$ is a product of either one $L(X_r)$ or $L'(X_r)$ for $r=1,\ldots,\tau, r\neq t,s$. Since τ is independent of $\{X_t\}_{t\geq 1}$ and $\{X_t\}_{t\geq 1}$ are i.i.d. under P_0, we have $E_0[|\underline{L}_\tau^t - \underline{L}_\tau'^{\,t}||\tau] \leq (\tau-1)E_0|L-L'|$. Consequently, (77) is bounded by

$$2e^{4C\beta E_0\tau}\beta\sum_{t=1}^\infty E_0[\tau-1;\tau\geq t]E_0|L-L'| = 2e^{4C\beta E_0\tau}\beta E_0[\tau(\tau-1)]E_0|L-L'| \leq wE_0|L-L'|$$

for some $w<1$ when β is small enough, since $E_0\tau^2<\infty$ by Assumption 3.6.

Finally, we note that the above arguments all hold with τ replaced by $\tau\wedge T$, in the same range of β and with the same Lipschitz constant w. This concludes the proposition. □

Next we show that $\tilde{\mathcal{K}}_T \to \tilde{\mathcal{K}}$ pointwise on $\mathcal{L}$:

Proposition 9.2 *Suppose Assumption 3.6 is in hold. For any $L\in\mathcal{L}$, we have $\tilde{\mathcal{K}}_T(L)\to\tilde{\mathcal{K}}(L)$ in $\|\cdot\|_1$, uniformly on $\beta\leq\epsilon$ for some small $\epsilon>0$.*

Proof. Consider

$$\begin{aligned} E_0|\tilde{\mathcal{K}}_T(L)-\tilde{\mathcal{K}}(L)| &= E_0\left|\frac{e^{\beta\tilde{g}^{L,T}(X)}}{E_0[e^{\beta\tilde{g}^{L,T}(X)}]} - \frac{e^{\beta\tilde{g}^{L}(X)}}{E_0[e^{\beta\tilde{g}^{L}(X)}]}\right| \\ &\leq 2e^{4C\beta E_0\tau}\beta\sum_{t=1}^\infty E_0|h(\mathbf{X}_{\tau\wedge T})\underline{L}_{\tau\wedge T}^t I(\tau\wedge T\geq t) - h(\mathbf{X}_\tau)\underline{L}_\tau^t I(\tau\geq t)| \qquad (78) \end{aligned}$$

by an argument similar to (77). Now consider

$$\begin{aligned} &\sum_{t=1}^\infty E_0|h(\mathbf{X}_{\tau\wedge T})\underline{L}_{\tau\wedge T}^t I(\tau\wedge T\geq t) - h(\mathbf{X}_\tau)\underline{L}_\tau^t I(\tau\geq t)| \\ = &\sum_{t=1}^\infty E_0[|h(\mathbf{X}_\tau)\underline{L}_\tau^t I(\tau\geq t) - h(\mathbf{X}_\tau)\underline{L}_\tau^t I(\tau\geq t)|;\tau<T] \\ &+\sum_{t=1}^\infty E_0[|h(\mathbf{X}_T)\underline{L}_T^t I(T\geq t) - h(\mathbf{X}_\tau)\underline{L}_\tau^t I(\tau\geq t)|;\tau\geq T] \\ = &\sum_{t=1}^\infty E_0[|h(\mathbf{X}_T)\underline{L}_T^t I(T\geq t) - h(\mathbf{X}_\tau)\underline{L}_\tau^t I(\tau\geq t)|;\tau\geq T] \\ = &\sum_{t=1}^T E_0[|h(\mathbf{X}_T)\underline{L}_T^t - h(\mathbf{X}_\tau)\underline{L}_\tau^t|;\tau\geq T] + \sum_{t=T+1}^\infty E_0[|h(\mathbf{X}_\tau)|\underline{L}_\tau^t;\tau\geq t] \\ \leq &\ 2CTP_0(\tau\geq T) + C\sum_{t=T+1}^\infty P_0(\tau\geq t) \ \text{ for some constant } C>0 \\ \to &\ 0 \end{aligned}$$

as $T\to\infty$, since $E_0\tau<\infty$. Hence (78) converges to 0 uniformly over $\beta\leq\epsilon$ for some small $\epsilon>0$. This concludes the proposition. □

By a simple argument on the continuity of fixed points (see, for example, Theorem 1.2 in [8]), Proposition 9.2 implies the following convergence result:

Corollary 9.1 *Suppose Assumption 3.6 holds. For small enough β, and letting $L^{(T)}$ and L^* be the fixed points of $\tilde{\mathcal{K}}_T$ and $\tilde{\mathcal{K}}$ respectively, we have $L^{(T)}\xrightarrow{\mathcal{L}_1}L^*$.*

Finally, we show that L^* is the optimal solution to the Lagrangian relaxation of (40):

Proposition 9.3 *Under Assumption 3.6, the fixed point L^* of the operator $\tilde{\mathcal{K}}$ maximizes*

$$E_0[h(\mathbf{X}_\tau)\underline{L}_\tau] - \alpha E_0[L\log L]$$

when α is large enough.

Proof of Proposition 9.3. In the proof we let $C > 0$ be a constant, not necessarily the same every time it shows up. To begin, we use the fact that for any fixed T, $L^{(T)}$ is the optimal solution to $E_0[h(\mathbf{X}_{\tau\wedge T})\underline{L}_{\tau\wedge T}] - \alpha E_0[L\log L]$, as a direct consequence of Proposition 5.1. Hence we have the inequality

$$E_0[h(\mathbf{X}_{\tau\wedge T})\underline{L}_{\tau\wedge T}] - \alpha E_0[L\log L] \leq E_0[h(\mathbf{X}_{\tau\wedge T})\underline{L}^{(T)}_{\tau\wedge T}] - \alpha E_0[L^{(T)}\log L^{(T)}] \tag{79}$$

for any $L \in \mathcal{L}$ (since h is bounded we can merely replace $\mathcal{L}(M)$ by $\mathcal{L}$, i.e. putting $M = \infty$, in Proposition 5.1). Here (79) holds for any $T \geq 1$ for α uniformly large (the uniformity can be verified using Proposition 9.1 and repeating the argument in the proof of Lemma 5.4, noting that $\tau\wedge T \leq \tau$ a.s.). Our main argument consists of letting $T \to \infty$ on both sides of (79).

We first show that, for any $L \in \mathcal{L}$, the first term on the left hand side of (79) converges to $E_0[h(\mathbf{X}_\tau)\underline{L}_\tau]$. Consider

$$\begin{aligned}
& |E_0[h(\mathbf{X}_{\tau\wedge T})\underline{L}_{\tau\wedge T}] - E_0[h(\mathbf{X}_\tau)\underline{L}_\tau]| \\
= & \left|\sum_{t=1}^{\infty} E_0[h(\mathbf{X}_t)\underline{L}_t]P_0(\tau\wedge T = t) - \sum_{t=1}^{\infty} E_0[h(\mathbf{X}_t)\underline{L}_t]P_0(\tau = t)\right| \\
\leq & C\sum_{t=1}^{T-1}|P_0(\tau\wedge T = t) - P_0(\tau = t)| + C|P_0(\tau \geq T) - P_0(\tau = T)| + C\sum_{t=T+1}^{\infty} P_0(\tau = t) \\
= & CP_0(\tau > T) + CP_0(\tau \geq T+1) \\
\to & 0
\end{aligned}$$

as $T \to \infty$, since $E_0\tau < \infty$. Hence the left hand side of (79) converges to $E_0[h(\mathbf{X}_\tau)\underline{L}_\tau] - \alpha E_0[L\log L]$ for any $L \in \mathcal{L}$. Now consider the right hand side. For the first term, consider

$$\begin{aligned}
& |E_0[h(\mathbf{X}_{\tau\wedge T})\underline{L}^{(T)}_{\tau\wedge T}] - E_0[h(\mathbf{X}_\tau)\underline{L}^*_\tau]| \\
= & \left|\sum_{t=1}^{\infty} E_0[h(\mathbf{X}_t)\underline{L}^{(T)}_t]P_0(\tau\wedge T = t) - \sum_{t=1}^{\infty} E_0[h(\mathbf{X}_t)\underline{L}^*_t]P_0(\tau = t)\right| \\
\leq & C\sum_{t=1}^{T-1} E_0|\underline{L}^{(T)}_t - \underline{L}^*_t|P_0(\tau = t) + 2C(P_0(\tau \geq T) + P(\tau = T)) + C\sum_{t=T+1}^{\infty} P_0(\tau = t) \\
\leq & C\sum_{t=1}^{T} tP_0(\tau = t)E_0|L^{(T)} - L^*| + 2C(P_0(\tau \geq T) + P(\tau = T)) + CP_0(\tau \geq T+1) \\
& \text{by the argument following (77)} \\
= & CE_0[\tau;\tau \leq T]E_0|L^{(T)} - L^*| + 2C(P_0(\tau \geq T) + P(\tau = T)) + CP_0(\tau \geq T+1) \\
\to & 0.
\end{aligned} \tag{80}$$

Moreover, for the second term in (79), write

$$E_0[L^{(T)}\log L^{(T)}] = \beta E_0[h(\mathbf{X}_{\tau\wedge T})\underline{L}^{(T)}_{\tau\wedge T}] - \log E_0[e^{\beta\tilde{g}^{L^{(T)},T}(X)}]$$

and

$$E_0[L^*\log L^*] = \beta E_0[h(\mathbf{X}_\tau)\underline{L}^*_\tau] - \log E_0[e^{\beta\tilde{g}^{L^*}(X)}]$$

by the definition of the fixed points for $\tilde{\mathcal{K}}_T$ and $\tilde{\mathcal{K}}$. To prove that $E_0[L^{(T)}\log L^{(T)}] \to E_0[L^*\log L^*]$, we have to show that $E_0[h(\mathbf{X}_{\tau\wedge T})\underline{L}^{(T)}_{\tau\wedge T}] \to E_0[h(\mathbf{X}_\tau)\underline{L}^*_\tau]$, which is achieved by (80), and that $\log E_0[e^{\beta\tilde{g}^{L^{(T)},T}(X)}] \to \log E_0[e^{\beta\tilde{g}^{L^*}(X)}]$, which we will show as follows. Consider

$$\begin{aligned}
& |\log E_0[e^{\beta\tilde{g}^{L^{(T)},T}(X)}] - \log E_0[e^{\beta\tilde{g}^{L^*}(X)}]| \\
\leq & e^{C\beta E_0\tau}|E_0[e^{\beta\tilde{g}^{L^{(T)},T}(X)}] - E_0[e^{\beta\tilde{g}^{L^*}(X)}]| \quad \text{by mean value theorem and the bound in (74)} \\
\leq & e^{2C\beta E_0\tau}\beta\sum_{t=1}^{\infty} E_0|h(\mathbf{X}_{\tau\wedge T})\underline{L}^{(T)}_{\tau\wedge T}{}^{t} I(\tau\wedge T \geq t) - h(\mathbf{X}_\tau)\underline{L}^{*}_\tau{}^{t} I(\tau \geq t)|
\end{aligned} \tag{81}$$

where ${\underline{L}^{(T)}_{\tau\wedge T}}^t = \prod_{\substack{s=1,\ldots,\tau\wedge T\\ s\neq t}} L(X_s)^{(T)}$ and ${\underline{L}^{*}_{\tau\wedge T}}^t = \prod_{\substack{s=1,\ldots,\tau\wedge T\\ s\neq t}} L(X_s)^{*}$, by arguments similar to (75)-(77). Now

$$\begin{aligned}
&\sum_{t=1}^\infty E_0|h(\mathbf{X}_{\tau\wedge T}){\underline{L}^{(T)}_{\tau\wedge T}}^t I(\tau\wedge T\geq t)-h(\mathbf{X}_\tau){\underline{L}^{*}_{\tau}}^t I(\tau\geq t)|\\
=\ &\sum_{t=1}^\infty E_0[|h(\mathbf{X}_{\tau}){\underline{L}^{(T)}_{\tau}}^t I(\tau\geq t)-h(\mathbf{X}_\tau){\underline{L}^{*}_{\tau}}^t I(\tau\geq t)|;\tau<T]\\
&+\sum_{t=1}^\infty E_0[|h(\mathbf{X}_{T}){\underline{L}^{(T)}_{T}}^t I(T\geq t)-h(\mathbf{X}_\tau){\underline{L}^{*}_{\tau}}^t I(\tau\geq t)|;\tau\geq T].
\end{aligned}\tag{82}$$

Note that the first term in (82) is bounded by

$$\begin{aligned}
C\sum_{t=1}^\infty E_0[|{\underline{L}^{(T)}_{\tau}}^t-{\underline{L}^{*}_{\tau}}^t|;\tau\geq t,\tau<T] &\leq C\sum_{t=1}^{T-1}E_0[\tau-1;t\leq\tau<T]E_0|L^{(T)}-L^*|\\
&\qquad\text{by the argument following (77)}\\
&= CE_0[\tau(\tau-1);\tau<T]E_0|L^{(T)}-L^*|\\
&\to 0
\end{aligned}$$

since $E_0\tau^2<\infty$. The second term in (82) can be written as

$$\begin{aligned}
&\sum_{t=1}^T E_0[|h(\mathbf{X}_T){\underline{L}^{(T)}_{T}}^t-h(\mathbf{X}_\tau){\underline{L}^{*}_{\tau}}^t I(\tau\geq t)|;\tau\geq T]+\sum_{t=T+1}^\infty E_0[|h(\mathbf{X}_\tau)|{\underline{L}^{*}_{\tau}}^t;\tau\geq t]\\
\leq\ & 2CTP_0(\tau\geq T)+C\sum_{t=T+1}^\infty P_0(\tau\geq t)\\
\to\ & 0
\end{aligned}$$

since $E_0\tau<\infty$. We therefore prove that (81) converges to 0 and the right hand side of (79) converges to $E_0[h(\mathbf{X}_\tau)\underline{L}^*_\tau]-\alpha E_0[L^*\log L^*]$. This concludes the proposition. □

Appendix A. Sufficiency Theorem

THEOREM A.1 (A.K.A. CHAPTER 8, THEOREM 1 IN [37]) *Consider $\phi(\cdot):\mathcal{L}\to\mathbb{R}$ and $\mathcal{C}$ a subset of $\mathcal{L}$. Suppose there is an α^*, with $\alpha^*\geq 0$, and an $L^*\in\mathcal{C}$ such that*

$$\phi(L^*)-\alpha^*E_0[L^*\log L^*]\geq\phi(L)-\alpha^*E_0[L\log L]\tag{83}$$

for all $L\in\mathcal{C}$. Then L^* solves

$$\begin{array}{ll}
\max & \phi(L)\\
\textit{subject to} & E_0[L\log L]\leq E_0[L^*\log L^*]\\
& L\in\mathcal{C}.
\end{array}$$

For the proof for Theorem 3.1, $\mathcal{C}$ is chosen as $\mathcal{L}$ and $\phi(L)=E_0[h(X)L]$. For Theorem 3.2, $\mathcal{C}$ is chosen as $\mathcal{L}(M)$ and $\phi(L)=E_0[h(\mathbf{X}_T)\underline{L}_T]$, and for Theorem 3.3 under Assumption 3.6, $\mathcal{C}$ as $\mathcal{L}$ and $\phi(L)=E_0[h(\mathbf{X}_\tau)\underline{L}_\tau]$.

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