

Non-uniqueness result for a hybrid inverse problem

Guillaume Bal and Kui Ren

ABSTRACT. Hybrid inverse problems aim to combine two imaging modalities, one that displays large contrast and one that gives high resolution. Mathematically, quantitative reconstructions in such hybrid problems involve reconstructing coefficients in a partial differential equation (PDE) from point-wise functionals of the coefficients and the PDE solution. There are many settings in which such inverse problems are shown to be well posed in the sense that the reconstruction of the coefficients is unique and stable for an appropriate functional setting. In this paper, we obtain an example where uniqueness fails to hold. Such a problem appears as a simplified model in acousto-optics, a hybrid medical imaging modality, and is related to the inverse medium problem where uniqueness results were obtained. Here, we show that two different solutions satisfying the same measurements can be reconstructed. The result is similar in spirit to the Ambrosetti-Prodi non-uniqueness result in the analysis of semi-linear equations. Numerical simulations confirm the theoretical predictions.

1. Introduction

Optical tomography consists of sending photons into tissues to probe their optical properties. Electrical impedance tomography consists of applying currents to probe their electrical properties. Both imaging techniques are very useful because of the large optical and electrical contrast displayed between healthy and non-healthy tissues. However, these imaging techniques suffer from very low resolution capabilities because the operators mapping the properties of interest to the available measurements are extremely smoothing [1, 17]. At the same time, the sound speed of such soft tissues displays small contrasts so that acoustic waves can propagate in a fairly homogeneous medium and display high resolutions (of order $\frac{\lambda}{2}$ where λ is the smallest observable wavelength in the measurements).

Several medical imaging modalities, which we will call hybrid modalities, allow us to physically couple the large contrast modality with the high resolution modality. A list of such modalities includes acousto-electric tomography, acousto-optic tomography (also known as ultrasound modulated optical tomography), magnetic

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resonance electrical impedance tomography, photo-acoustic tomography, thermo-acoustic tomography, transient elastography, as well as other modalities. We refer the reader to, e.g., [3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 18] for more details. Here, we focus on a model that appears in acousto-optic tomography [8].

The inverse problem of interest consists of reconstructing an absorption coefficient in an elliptic equation from measurements of the form $H(x) = \sigma(x)u^2(x)$ point-wise $x \in X$ in the domain of interest, where σ is absorption and u is the PDE solution; see (2.1) below. Such inverse problems find application in a simplified model for acousto-optics derived in [8]. In [7], it was shown that σ sufficiently small was uniquely reconstructed from such measurements. Moreover, in [18], it was shown that σ was uniquely determined by similar measurements when the sign of σ is changed in (2.1) in such a way that σ is now a non-negative potential rather than a non-negative absorption.

The non-uniqueness comes from the existence of singular points for functionals of the form $\phi(u) = u\Delta u$, where Δu is the Laplace operator. Such critical points arise when $\frac{\Delta u}{u}$ takes positive value. In the neighborhood of such singular points, $\phi(u)$ exhibits a fold that generalizes the behavior of the function $\phi(x) = x^2$ from \mathbb{R} to \mathbb{R} . A similar non-uniqueness result was obtained for a specific class of semilinear equations by Ambrosetti and Prodi in [2]. In the latter paper, specific equations are considered in which global non-uniqueness results are obtained. Because the singular points of ϕ do not form a connected (co-dimension 1) manifold in our setting, we were not able to deduce global non-uniqueness results. However, for the specific, quadratic, functionals $\phi(u)$ considered here, we obtain explicit, global, expressions for the two branches of solutions that emerge from any critical value of ϕ .

This example shows that uniqueness in hybrid inverse problems is not always guaranteed. In fact, in the vicinity of critical points of $\phi(u)$, small perturbations in the measurements may result in the inverse problem admitting no solution (think of perturbing x^2 by a small negative constant in the vicinity of $x = 0$).

The non-uniqueness results suppose that we acquire measurements of the form $H(x) = \sigma(x)u^2(x)$ for one specific, prescribed, boundary condition for the elliptic equation (2.1). It turns out that uniqueness of the reconstruction is restored when two well-chosen measurements are available as can be deduced from results obtained in [6].

The rest of the paper is as follows. Section 2 presents the inverse problem of interest, recasts it as a semilinear equation, and presents the main (local) result of non-uniqueness of this paper in Proposition 2.1 below. Once we have this non-uniqueness result, we show in section 3 how to construct explicitly two different global solutions that share the same measurements; see Proposition 3.1 below. As an application of results obtained in [6], we show in section 4 that two well-chosen boundary conditions allow us to uniquely characterize the absorption term σ as well as the diffusion coefficient in the elliptic equation when the latter is also unknown. Finally, we present in section 5 several numerical simulations that illustrate the non-uniqueness result in the presence of a single measurements and the uniqueness result in the presence of two measurements.

2. Local non-uniqueness result

In this paper we consider elliptic problems of the form

$$(2.1) \quad \begin{aligned} Pu &= \sigma u & \text{in } X \\ u &= g & \text{on } \partial X \end{aligned}$$

and assume that measurements of the form $H(x) = \sigma(x)u^2(x)$ are available. Here, P is a self-adjoint (with Dirichlet conditions), non-positive, elliptic operator, which for concreteness we will take of the form $Pu = \nabla \cdot D(x)\nabla u$ with $D(x)$ known, sufficiently smooth, and bounded above and below by positive constants. We assume $g > 0$ and $\sigma > 0$ so that by the maximum principle, $u > 0$ on X . We also assume enough regularity on ∂X and g so that $u \in C^{2,\beta}(\bar{X})$ for some $\beta > 0$ [11]. We want to show that H does not always uniquely determine σ .

We observe that

$$(2.2) \quad \begin{aligned} uPu &= H & \text{in } X \\ u &= g & \text{on } \partial X \end{aligned}$$

so that the inverse problem may be recast as a semilinear (after dividing by u) problem. The non-uniqueness result is an example of an Ambrosetti-Prodi result [2] and generalizes the simple observation that $x \rightarrow x^2$ admits 0, 1, or 2 (real-valued) solution(s) depending on the value of x^2 .

Let us define

$$(2.3) \quad \phi : C^{2,\beta}(\bar{X}) \rightarrow C^{0,\beta}(\bar{X}), \quad u \mapsto \phi(u) = uPu.$$

We need to find the singular points of ϕ and thus calculate its first-order Fréchet derivative:

$$(2.4) \quad \phi'(u)v = vPu + uPv.$$

The operator $\phi'(u)$ is not invertible when $\sigma := \frac{Pu}{u}$ is such that $P + \lambda\sigma$ admits $\lambda = 1$ as an eigenvalue. This implicitly defines the singular points u and critical values $\phi(u)$ of the functional ϕ . Note that if we replace $P = \Delta$ by $-\Delta$ as in [18], then ϕ does not admit any singular point and the hybrid inverse problem is then always well posed. We want to invert the functional $\phi(u)$ in the vicinity of a singular point $u_0 \in C^{2,\beta}(\bar{X})$. We assume that $\sigma_0 = \frac{Pu_0}{u_0} > 0$ on \bar{X} , that $u_0 > 0$ on \bar{X} and that $\phi'(u_0)$ has a kernel of dimension 1 and a range of co-dimension 1 as well. This implies the existence of unique (up to change of sign) functions v_0 and w_0 (assumed to be smooth) such that

$$(2.5) \quad \phi'(u_0)v_0 = 0, \quad (\phi'(u_0))^*w_0 = 0, \quad (v_0, v_0) = (v_0, w_0) = 1, \quad w_0 = \gamma \frac{v_0}{u_0},$$

with $(u, v) = \int_X uv dx$. Here, $(\phi'(u_0))^*$ is the adjoint operator to $\phi'(u_0)$ and w_0 is easily found to verify the above expression with a properly chosen normalizing constant γ .

Because of the simple, quadratic, expression for $\phi(u)$, we find that

$$(2.6) \quad \phi(u) = \phi(u_0) + \phi'(u_0)(u - u_0) + \phi(u - u_0).$$

Our objective is to invert the above equation for u in the vicinity of the singular point u_0 . We briefly present the derivation of such results and refer the reader to

[2] for the details. Let us define the projector on the range of $\phi'(u_0)$:

$$(2.7) \quad \pi u = u - (u, w_0)v_0,$$

and decompose

$$(2.8) \quad u = u_0 + \alpha v_0 + p, \quad \alpha = (u - u_0, w_0), \quad p := u - u_0 - \alpha v_0 = \pi(u - u_0).$$

We need to assume that

$$(2.9) \quad (v_0 P v_0, w_0) = \gamma \int_X \frac{v_0^2}{u_0} P v_0 dx = -\gamma \int_X \frac{\sigma v_0^3}{u_0} dx \neq 0.$$

Such a constant is certainly non 0 when v_0 is the non-negative eigenvector associated to $\lambda = 1$, the smallest eigenvalue of $(-P)^{-1}\sigma$. In the sequel, we assume that (2.9) holds.

Some algebra shows that

$$\phi(u) - \phi(u_0) = \phi'(u_0)p + \phi(u - u_0) = \phi'(u_0)p + \alpha^2 v_0 P v_0 + \alpha(p P v_0 + v_0 P p) + p P p.$$

We aim to reconstruct (α, p) from the above quadratic equation. We find the two equations:

$$\begin{aligned} \phi'(u_0)p + \pi\phi(\alpha v_0 + p) &= \pi[\phi(u) - \phi(u_0)] := h \\ (v_0 P v_0, w_0)\alpha^2 + (p P v_0 + v_0 P p, w_0)\alpha + (p P p, w_0) &= (\phi(u) - \phi(u_0), w_0) := s, \end{aligned}$$

where we have decomposed $\phi(u) - \phi(u_0) = s v_0 + h$ with $(h, w_0) = 0$. We recall that $\phi'(u_0)$ is invertible on the range of π and that $(v_0 P v_0, w_0) \neq 0$. Therefore, for $(h, s) = 0$, we find that the only solution is $(p, \alpha) = 0$. For h small, the first equation provides a solution $p(h, \alpha) = (\phi'(u_0))^{-1}h + \delta p(h, \alpha)$ with p small and δp of lower order. Plugging this into the equation for α , we obtain an approximately quadratic expression for α . Depending on s , this quadratic equation will have 0, 1, or 2 solutions. Following the proofs of [2, Theorems 2.7 & 2.11], we can prove the following result:

PROPOSITION 2.1. Let W be the set of singular points of ϕ and U be a sufficiently small neighborhood of the singular point $u_0 \in W$. Then $\phi(W \cap U)$ is a co-dimension 1 manifold in $C^{0,\beta}(\bar{X})$. If $\phi(u) \in \phi(W \cap U)$, then there is a unique $u \in C^{2,\beta}(\bar{X})$ solution of (2.6). Then on either side of $\phi(W \cap U)$, we have either zero or two solutions $u \in C^{2,\beta}(\bar{X})$ of (2.6).

3. Global non-uniqueness result

Because of the specific structure of $\phi(u)$, we now obtain an explicit expression for the two solutions in (2.6) in the vicinity of a critical point of ϕ . Moreover, these two branches extend to provide non-local non-uniqueness results as we shall see. This does not provide a global results as in [2], which is obtained for semilinear equations with hypotheses that are not satisfied by (2.2) (because $\frac{1}{u}$ cannot be written as $f(u)$ for a function $t \mapsto f(t)$ such that $f'(t)$ is equal to the first eigenvalue of the Laplace operator with Dirichlet conditions for only one value of t ; see [2, Theorem 3.1]).

Moreover, we want to show that the solutions obtained in (2.6) indeed allow us to construct two different absorption coefficients σ that are non-negative, equal on the boundary ∂X , and such that the measurements $H(x) = \sigma u^2(x)$ agree on X . In other words, we want to ensure that the non-uniqueness results for the semi-linear

equation (2.2) does translate into a non-uniqueness result for the hybrid inverse problem.

Let us first define

$$\mathcal{S}_\lambda = \left\{ \sigma \geq 0, 1 \in \text{Sp} \left((-P)^{-1} \frac{\sigma}{\lambda} \right) \right\} = \left\{ \sigma \geq 0, \lambda \in \text{Sp} \left((-P)^{-1} \sigma \right) \right\}.$$

We have seen that critical points of $\phi(u) = uPu$ corresponded to solutions $\sigma := \frac{Pu}{u} \in \mathcal{S}_1$.

Let us assume that we are given $0 < \sigma \in \mathcal{S}_\lambda$ with $\lambda < 1$ close to 1. Then $\frac{1}{\lambda}\sigma \in \mathcal{S}_1$ and we define \tilde{u}_0 the solution of (2.1) with σ replaced by $\frac{\sigma}{\lambda}$. Note that (2.1) is an elliptic problem for all $\sigma > 0$. It is $P + \sigma$ that may not be invertible.

We thus have constructed a singular point $\tilde{u}_0 \in W$ and we can apply Proposition 2.1 to obtain the reconstruction of two solutions $u_{\pm\delta}$ (what δ means will be apparent shortly) such that

$$u_{\pm\delta}Pu_{\pm\delta} = H_\delta := \sigma u^2.$$

We know that the above problem admits two local solutions with either u_δ or $u_{-\delta}$ being equal to u . Indeed, $\sigma \in \mathcal{S}_\lambda$ with $\lambda < 1$ so that $u \notin W$ and since u is a solution, there are exactly two solutions locally.

It turns out that the parameter δ may be chosen arbitrarily between $(0, \delta_0)$ to provide two solutions to (2.2). The construction goes as follows. Let $u_{\pm\delta}$ be defined as above and let

$$u_0 = \frac{1}{2}(u_\delta + u_{-\delta}), \quad \psi = \frac{1}{2\delta}(u_\delta - u_{-\delta}) \neq 0.$$

Here, δ is a constant chosen so that $(\psi, \psi) = 1$. Note that $u_0 > 0$ and $u_0 = g$ on ∂X . Moreover, we verify that

$$u_0P\psi + \psi Pu_0 = \frac{1}{4\delta}(u_\delta Pu_\delta - u_{-\delta} Pu_{-\delta}) = 0.$$

This shows that u_0 is a critical point of ϕ since $\phi'(u_0)\psi = 0$ with $\psi \neq 0$. Let us define

$$(3.1) \quad \sigma_\delta := \frac{Pu_\delta}{u_\delta} = \frac{H_\delta}{u_\delta^2}, \quad \sigma_0 := \frac{Pu_0}{u_0} = \frac{\sigma_\delta u_\delta + \sigma_{-\delta} u_{-\delta}}{u_\delta + u_{-\delta}}.$$

We know that the original σ we started with equals either σ_δ or $\sigma_{-\delta}$. When λ is sufficiently close to 1, then by continuity, we deduce that both $\sigma_{\pm\delta} > 0$ and that $\sigma_0 > 0$ on \bar{X} .

We are therefore in the presence of a pair (σ_0, u_0) such that

$$(3.2) \quad Pu_0 = \sigma_0 u_0, \quad X, \quad u_0 = g, \quad \partial X, \quad \sigma_0 > 0.$$

Moreover, u_0 is a singular point of $\phi(u)$ with $\phi'(u_0)\psi = 0$ and $\psi \neq 0$. When σ_0 is constructed in the vicinity of $\sigma \in \mathcal{S}_\lambda$ with $\lambda < 1$, we know that ψ can be chosen with a given sign, say $\psi(x) > 0$ on X while $\psi = 0$ on ∂X .

The construction of u_0 above was based on the availability of $\sigma \in \mathcal{S}_\lambda$. More generally, we can assume that u_0 is an arbitrary singular point of $\phi(u)$ associated with $\phi'(u_0)\psi = 0$, such that $u = g$ on ∂X and with an absorption coefficient $\sigma_0 := \frac{Pu_0}{u_0} > 0$ on \bar{X} . Then define

$$(3.3) \quad u_\delta := u_0 + \delta\psi, \quad X, \quad \delta \in (-\delta_0, \delta_0).$$

Define as well

$$(3.4) \quad \sigma_\delta := \frac{Pu_\delta}{u_\delta} = \sigma_0 \frac{u_0 - \delta\psi}{u_0 + \delta\psi}, \quad H_\delta := \sigma_\delta u_\delta^2 = \sigma_0 u_\delta u_{-\delta} = \sigma_0 (u_0^2 - \delta^2 \psi^2).$$

We choose δ_0 such that $\sigma_\delta > 0$ a.e. on X for all $\delta \in (-\delta_0, \delta_0)$. We have obtained the following result:

PROPOSITION 3.1. Let u_0 be a singular point and $H_0 = \phi(u_0)$ a critical value of ϕ as above and let ψ be the normalized solution of $\phi'(u_0)\psi = 0$. Let u_δ, σ_δ , and H_δ be defined as in (3.3)-(3.4) for $\delta \in (-\delta_0, \delta_0)$ for δ_0 sufficiently small. Then we verify that

$$\sigma_\delta \neq \sigma_{-\delta}, \quad \sigma_\delta > 0, \quad H_\delta = H_{-\delta}, \quad Pu_\delta = \sigma_\delta u_\delta \text{ in } X, \quad u_\delta = g \text{ on } \partial X.$$

This shows the non-uniqueness of the reconstruction of σ from knowledge of $H = \sigma u^2$. Moreover we verify that $\sigma_{\pm\delta}$ agree on ∂X so that this boundary information cannot be used to distinguish between σ_δ and $\sigma_{-\delta}$.

In this example, we observe that $\phi(u_\delta) = H_\delta = H_0 - \delta^2 \sigma \psi^2 < H_0 = \phi(u_0)$ when $\delta \neq 0$. This shows that $\phi(u)$ is invertible in the vicinity of the critical value H_0 provided that $\phi(u)$ is smaller than H_0 and that no solutions exist, at least locally, when $\phi(u) > H_0$. This shows that for exact measurements in the vicinity of a critical value, small amounts of noise may push available measurements to values H where the semi-linear equation $\phi(u) = H$ admits no solution.

4. Uniqueness result with two measurements

In the preceding sections, only one measurement $H(x)$, corresponding to a prescribed boundary condition g on ∂X , is available. The non-uniqueness actually disappears when two measurements corresponding to two well-chosen boundary conditions are available. In fact, we can show more precisely that both the absorption coefficient σ and the diffusion coefficient D can be reconstructed for two well chosen measurements. Let us recast (2.1) as

$$(4.1) \quad \begin{aligned} -\nabla \cdot D(x)\nabla u_j + \sigma(x)u_j &= 0 && \text{in } X \\ u_j &= g_j && \text{on } \partial X \end{aligned}$$

for $j = 1, 2$ with measurements

$$H_j(x) = \sigma(x)u_j^2(x) \quad \text{or equivalently} \quad \sqrt{H_j(x)} = \sqrt{\sigma(x)}u_j(x).$$

The theory in [6] shows that for well chosen pairs of boundary conditions (g_1, g_2) , we can reconstruct (μ, q) with

$$(4.2) \quad \mu = \frac{\sqrt{D}}{\sqrt{\sigma}}, \quad (\Delta + q)\sqrt{D} + \frac{\sigma}{\sqrt{D}} = 0.$$

We refer the reader to [6] for an explicit definition of well-chosen boundary conditions. So we have access to $\mu^2 = \frac{D}{\sigma}$ and we can recast the equation for \sqrt{D} as

$$(4.3) \quad (\Delta + q + \frac{1}{\mu^2})\sqrt{D} = 0.$$

We verify that [6]

$$(-\nabla \cdot D\nabla + \sigma) \cdot = (-\sqrt{D}(\Delta + q)\sqrt{D}) \cdot .$$

Let us assume that $\sqrt{D}\tau$ is another solution of (4.3) with $\tau = 1$ on ∂X . Then, using the above equality, we find the equation for τ :

$$\left(-\nabla \cdot D\nabla + \sigma - \frac{D}{\mu^2}\right)\tau = -\nabla \cdot D\nabla\tau = 0 \text{ in } X, \quad \tau = 1 \text{ on } \partial X.$$

The only solution is $\tau = 1$. This proves that (D, σ) is uniquely determined by (H_1, H_2) . The results in [6] show that the reconstruction of μ is Hölder stable with respect to errors in the measurements (H_1, H_2) . The above uniqueness result for τ and D may be modified to yield a stability result for D as well, and hence for σ . We are therefore in the setting of a hybrid inverse problem combining a large contrast (in the optical coefficients σ and D) with a high resolution (exemplified by the Hölder stability result).

5. Numerical verifications

We present here some numerical verifications of the non-uniqueness and uniqueness theories that have been developed in the previous sections. We limited ourselves to the dimension $d \leq 2$ case even though the theories also hold in more physical three-dimensional spaces.

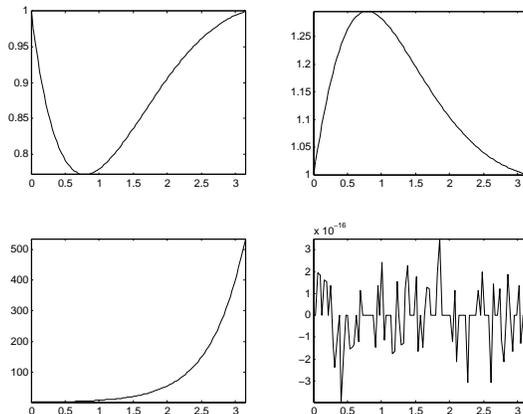


FIGURE 1. Verification of non-uniqueness in one-dimensional case. Top row: two absorption coefficients σ_δ (left) and $\sigma_{-\delta}$ (right). Bottom row: The interior data H_δ constructed with σ_δ (left) and the normalized difference $\frac{H_\delta - H_{-\delta}}{H_\delta}$.

In the first set of numerical simulations, we briefly verify the proposed construction leading to two absorption coefficients with the same measurements. We construct two absorption coefficients $\sigma_\delta \neq \sigma_{-\delta}$ that lead to the same interior data $H_\delta = H_{-\delta}$. For concreteness, we display in Fig. 1 a one-dimensional construction in the interval $(0, \pi)$ and in Fig. 2 a two-dimensional construction in the disc with radius 2 and centered at $\mathbf{x}_0 = (2, 2)$, $X = \{\mathbf{x} : |\mathbf{x} - \mathbf{x}_0| < 2\}$. As can be seen from both plots, the data generated with the two very different absorption coefficients are identical (with relative difference $< 10^{-10}$ in both cases).

In the second numerical simulation, we give an example of the non-uniqueness result by reconstructing both absorption coefficients from knowledge of the data

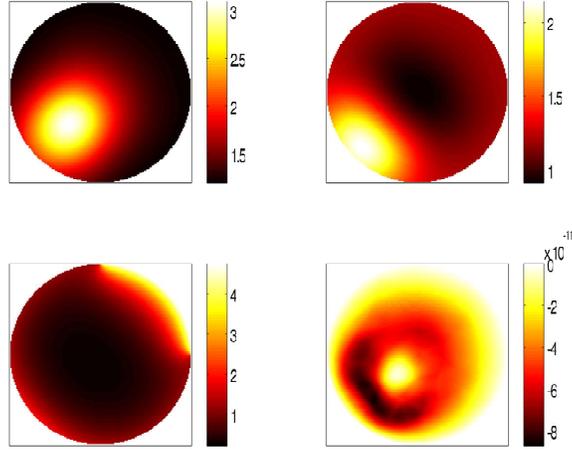
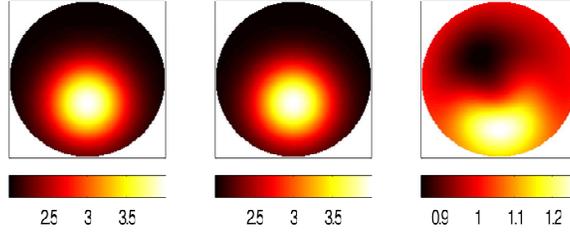


FIGURE 2. Same as Fig. 1 but in the two-dimensional case.

associated with one of the absorption coefficients. Assuming that we are given an absorption coefficient σ , which could be either σ_δ or $\sigma_{-\delta}$, we intend to recover both $\sigma_{\pm\delta}$ following the local constructions in section 2. We first calculate the largest eigenvalue of $(-P)^{-1}\sigma$ with $P = \Delta$ and rescale σ so that $\frac{\sigma}{\lambda} \in \mathcal{S}_1$. This provides us with a singular point u_0 as in section 2. We then use the projection algorithm and the equation for α to obtain the two solutions $\sigma_{\pm\delta}$ described in Proposition 3.1. We show in Fig. 3 the results of the reconstruction in the disc X defined above. The σ we start from in this simulation is such that 0.95 is the lowest eigenvalue of $(-\Delta)^{-1}\sigma$.

FIGURE 3. From left to right: original absorption coefficient σ , reconstructed σ_δ and reconstructed $\sigma_{-\delta}$.

The non-uniqueness results that we have seen above are all constructed when only one set of interior data is available. It turns out that with two sets of interior data, we can uniquely reconstruct two coefficients as presented in Section 4. We now present two typical reconstructions of the diffusion and absorption coefficients in Fig. 4 (for smooth coefficients) and Fig. 5 (for discontinuous coefficients) that are very similar in spirit to the reconstructions obtained in a similar setting in [6]. In both reconstructions, the synthetic data have been randomly perturbed by 5% multiplicative noise obtained by the algorithm $\tilde{H} = H * (1 + \frac{5}{100} \text{rand})$ with rand a random field with values in $[-1, 1]$. The reconstructions are done with a slightly

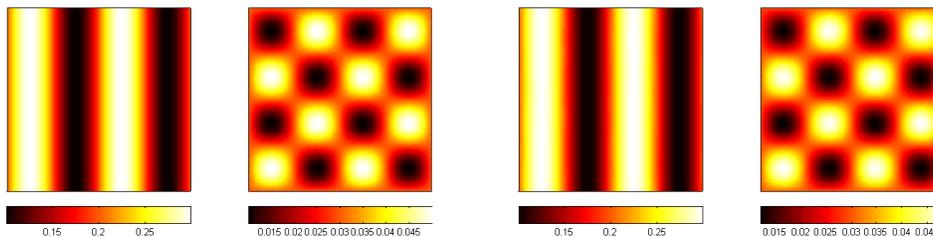


FIGURE 4. Reconstructions of the diffusion and absorption coefficients with two sets of interior data. From Left to right: true diffusion (left) and absorption (right) coefficients; reconstructed diffusion (left) and absorption (right) coefficients.

modified version of the vector field method implemented in [6]. The relative L^2 error in the reconstructions are 0.1% (diffusion) and 0.1% (absorption) in Fig. 4, and 0.2% (diffusion) and 0.1% (absorption) for the case in Fig. 5 respectively. This

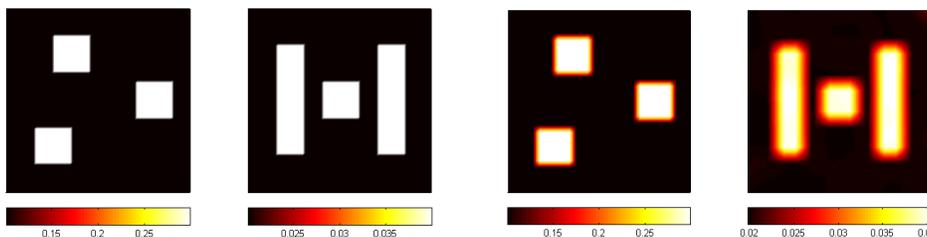


FIGURE 5. Same as in Fig. 4 except that the coefficients are discontinuous.

shows that the reconstructions are very accurate with a much higher resolution than what can be achieved in inverse problems for elliptic problems of the form (2.1) or (4.1) with boundary measurements (measurements of the form of a Dirichlet-to-Neumann map).

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DEPARTMENT OF APPLIED PHYSICS AND APPLIED MATHEMATICS, COLUMBIA UNIVERSITY,
NEW YORK, NY 10027

E-mail address: `gb2030@columbia.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS AT AUSTIN, AUSTIN, TX 78712

E-mail address: `ren@math.utexas.edu`