Unique determination of absorption coefficients in a semilinear transport equation

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Abstract

Motivated by applications in quantitative photoacoustic imaging, we study inverse problems to a semilinear radiative transport equation (RTE) where we intend to reconstruct absorption coefficients in the equation from single and multiple internal data sets. We derive uniqueness and stability results for the inverse transport problem in the absence of scattering (in which case we also derive some explicit reconstruction methods) and in the presence of known scattering.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^d$ ($d \geq 2$) be a domain with boundary $\partial\Omega$, and $\mathbb{S}^{d-1}$ the unit sphere in $\mathbb{R}^d$. We define the phase space $X := \Omega \times \mathbb{S}^{d-1}$ and the incoming boundary of the phase space $\Gamma_- := \{(\mathbf{x}, \mathbf{v}) \mid (\mathbf{x}, \mathbf{v}) \in \partial\Omega \times \mathbb{S}^{d-1} \text{ s.t. } -\boldsymbol{\nu}(\mathbf{x}) \cdot \mathbf{v} > 0\}$, $\boldsymbol{\nu}(\mathbf{x})$ being the unit outer normal vector at $\mathbf{x} \in \partial\Omega$. We are interested in the semilinear radiative transport equation:

$$\begin{aligned} \mathbf{v} \cdot \nabla u + (\sigma_a + \sigma_s) u(\mathbf{x}, \mathbf{v}) + \sigma_b \langle u \rangle u(\mathbf{x}, \mathbf{v}) &= \sigma_s(\mathbf{x}) K u(\mathbf{x}, \mathbf{v}), && \text{in } X \\ u(\mathbf{x}, \mathbf{v}) &= g(\mathbf{x}, \mathbf{v}), && \text{on } \Gamma_- \end{aligned} \tag{1}$$

where $\langle u \rangle$ denotes the average of $u(\mathbf{x}, \mathbf{v})$ over the variable $\mathbf{v}$, that is,

$$\langle u \rangle := \int_{\mathbb{S}^{d-1}} u(\mathbf{x}, \mathbf{v}) d\mathbf{v}\,, \tag{2}$$

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with $d\mathbf{v}$ being the *normalized* surface measure on $\mathbb{S}^{d-1}$. The linear operator K is defined through the relation

$$Ku(\mathbf{x},\mathbf{v}) := \int_{\mathbb{S}^{d-1}} \Theta(\mathbf{v},\mathbf{v}')u(\mathbf{x},\mathbf{v}')d\mathbf{v}', \tag{3}$$

with the kernel $\Theta(\mathbf{v},\mathbf{v}')$ being symmetric and satisfying the normalization conditions

$$\int_{\mathbb{S}^{d-1}} \Theta(\mathbf{v},\mathbf{v}')d\mathbf{v}' = \int_{\mathbb{S}^{d-1}} \Theta(\mathbf{v},\mathbf{v}')d\mathbf{v} = 1.$$

Transport equations such as (1) often appear in the literature as the mathematical models to describe radiative transfer processes in heterogeneous media. We are interested in the application of this equation in modeling the propagation of near infra-red photons in biological tissues [3, 4, 30]. In such a case, $u(\mathbf{x},\mathbf{v})$ denotes the density of the photons at position $\mathbf{x}$ traveling in direction $\mathbf{v}$. The coefficients $\sigma_a(\mathbf{x})$ and $\sigma_s(\mathbf{x})$ are the usual single-photon absorption and scattering coefficients respectively, and the kernel $\Theta(\mathbf{v},\mathbf{v}')$ describes the probability of photons traveling in direction $\mathbf{v}'$ getting scattered into direction $\mathbf{v}$. The coefficient $\sigma_b(\mathbf{x})$ is called the two-photon absorption coefficient. It is used to model the two-photon absorption process, that is, the phenomenon that an electron transfers to an excited state after simultaneously absorbing two photons whose total energy exceed the electronic energy band gap. Such two-photon absorption process can also be viewed as a regular physical absorption process whose effective absorption strength, $\sigma_b\langle u\rangle$, depends on the local density of the photons. We refer interested readers to [8, 32] and references therein for more details on the modeling of two-photon absorption in diffusive media.

In the rest of this paper, we study an inverse problem to the transport model (1) where we intend to reconstruct the absorption coefficients σ_a and σ_b from internal data of the form

$$H(\mathbf{x}) := \sigma_a(\mathbf{x})\langle u\rangle(\mathbf{x}) + \sigma_b(\mathbf{x})\langle u\rangle^2(\mathbf{x}), \quad \mathbf{x}\in\bar{\Omega}\,. \tag{4}$$

Such inverse problems originate from applications in quantitative photoacoustic imaging where internal data (4) can be obtained from photoacoustic measurements; see [6, 29] and references therein for recent developments in the field. In the diffusive regime, that is, when the transport model (1) is replaced with its diffusion approximation, it has been shown in [8, 32] that one can reconstruct all three coefficients $(\sigma_a,\sigma_b,\sigma_s)$ from a finite set of internal data of the form (4). The objective of this work is to show that one can reconstruct uniquely (σ_a,σ_b) in the semilinear transport equation (1) from two sets of internal data.

The inverse problem we described above is closely related to an inverse problem to the linear transport equation:

$$\begin{aligned} \mathbf{v}\cdot\nabla u + (\Sigma_a+\sigma_s)u(\mathbf{x},\mathbf{v}) &= \sigma_s(\mathbf{x})Ku(\mathbf{x},\mathbf{v}), && \text{in } X\\ u(\mathbf{x},\mathbf{v}) &= g(\mathbf{x},\mathbf{v}), && \text{on } \Gamma_- \end{aligned} \tag{5}$$

with data of the form

$$H(\mathbf{x}) := \Sigma_a(\mathbf{x})\langle u\rangle(\mathbf{x}), \quad \mathbf{x}\in\bar{\Omega}\,. \tag{6}$$

In fact, the semilinear transport equation (1) can be viewed as the linear transport equation (5) whose absorption coefficient Σ_a depends on the density $\langle u\rangle$ in a linear manner: $\Sigma_a = \sigma_a + \sigma_b\langle u\rangle$.

Inverse problems to the radiative transport equation have been studied extensively in the past two decades; see for instance [4] for a recent review on the topic. Most of existing analytical and computational results are on the linear transport equation (5). These include, but not limited to, problems where boundary data encoded in the map $u_{|\Gamma_-} \mapsto u_{|\Gamma_+}$ and alike are available [5, 7, 9, 12, 11, 13, 16, 18, 24, 26, 31, 35, 36, 38, 41, 43], as well as problems where internal data of the type (6) [6, 29, 33, 40, 20, 39, 34] and alike [14, 28] are available. Existing results, either analytical or computational, on nonlinear transport models such as (1) are very limited; see [10, 27, 25] for some related results.

2 The forward problem

We start by establishing the well-posedness theory for the semilinear transport equation (1). To setup the analysis, we denote by $L^p(X)$ (resp. $L^p(\Omega)$) the space of real-valued functions whose p-th power are Lebesgue integrable on X (resp. Ω), and $\mathcal{H}^p(X)$ the space of $L^p(X)$ functions whose derivative in direction $\mathbf{v}$ is in $L^p(X)$, i.e. $\mathcal{H}^p(X) = \{f(\mathbf{x}, \mathbf{v}) : f \in L^p(X) \text{ and } \mathbf{v} \cdot \nabla f \in L^p(X)\}$. We denote by $L^p(\Gamma_-)$ the space of functions that are traces of $\mathcal{H}^p(X)$ functions on Γ_- under the norm $\|f\|_{L^p(\Gamma_-)} = (\int_{\partial\Omega} \int_{\mathbb{S}^{d-1}_{\mathbf{x}-}} |\mathbf{n}(\mathbf{x}) \cdot \mathbf{v}||f|^p d\mathbf{v} d\gamma)^{1/p}$, $d\gamma$ being the surface measure on $\partial\Omega$ and $\mathbb{S}^{d-1}_{\mathbf{x}-} = \{\mathbf{v} : \mathbf{v} \in \mathbb{S}^{d-1} \text{ s.t. } -\boldsymbol{\nu}(\mathbf{x}) \cdot \mathbf{v} > 0\}$. It is well-known [2, 15] that both $\mathcal{H}^p(X)$ and $L^p(\Gamma_-)$ are well-defined.

For a given set Y, we introduce the space of bounded functions on Y:

$$\mathcal{F}_{\underline{f}}^{\overline{f}}(Y) := \{f \in L^\infty(Y) \mid \exists \underline{f}, \overline{f} \text{ s.t. } 0 < \underline{f} \le f \le \overline{f} < +\infty \ a.e.\}.$$

Unless stated otherwise, we make the following assumptions on the domain Ω, the scattering coefficient and the scattering phase function throughout the paper:

($\mathcal{A}$). (i) the domain Ω is bounded, convex and smooth; (ii) the scattering coefficient $\sigma_s(\mathbf{x}) \in \mathcal{F}_{\underline{\sigma}_s}^{\overline{\sigma}_s}(\Omega)$ for some constants $\underline{\sigma}_s$ and $\overline{\sigma}_s$; and (iii) the scattering phase function $\Theta(\mathbf{v}, \mathbf{v}') \in \mathcal{F}_{\underline{\theta}}^{\overline{\theta}}(\mathbb{S}^{d-1} \times \mathbb{S}^{d-1})$ for some $\underline{\theta}$ and $\overline{\theta}$.

With the convention that the surface measure $d\mathbf{v}$ on $\mathbb{S}^{d-1}$ is normalized, we observe that this assumption means that $\underline{\theta} \le 1$ while $\overline{\theta} \ge 1$.

For any point $(\mathbf{x}, \mathbf{v}) \in X$, we use $\tau_-(\mathbf{x}, \mathbf{v})$ to denote the distance a particle starting from $\mathbf{x}$ and traveling in the direction $-\mathbf{v}$ has to travel to reach the boundary of the domain. That is:

$$\tau_-(\mathbf{x}, \mathbf{v}) := \sup\{s \in \mathbb{R} \mid \mathbf{x} - s\mathbf{v} \in \Omega\}. \tag{7}$$

Note that due to the assumption that Ω is convex, $\tau_-(\mathbf{x}, \mathbf{v})$ is uniquely determined for any $(\mathbf{x}, \mathbf{v}) \in X$.

The following simple result on the linear transport equation (5) turns out to be useful.

Lemma 2.1. *Let $g \in L^\infty(\Gamma_-)$ be given such that $\underline{g} := \inf_{\Gamma_-} g > 0$ and u be the unique solution to (5) with $(\Sigma_a, \sigma_s, \Theta)$. Assume that $\Sigma_a \in \mathcal{F}_{\underline{\Sigma}_a}^{\overline{\Sigma}_a}(\Omega)$ for some $\underline{\Sigma}_a$ and $\overline{\Sigma}_a$. Then, under the assumptions in ($\mathcal{A}$), there exists some constant $\mathfrak{c} > 0$ such that $u \ge \mathfrak{c} > 0$.*

Proof. We first observe that with all the assumptions made, we have that $u \geq 0$ from the standard transport theory [15]. Let $\widetilde{u}$ be the solution to

$$\begin{aligned} \mathbf{v} \cdot \nabla \widetilde{u}(\mathbf{x}, \mathbf{v}) + (\Sigma_a + \sigma_s)\widetilde{u}(\mathbf{x}, \mathbf{v}) &= 0, \quad && \text{in } X \\ \widetilde{u}(\mathbf{x}, \mathbf{v}) &= g(\mathbf{x}, \mathbf{v}), \quad && \text{on } \Gamma_- . \end{aligned}$$

Then $\widetilde{u}$ can be found analytically as

$$\widetilde{u}(\mathbf{x}, \mathbf{v}) = g(\mathbf{x} - \tau_-(\mathbf{x}, \mathbf{v})\mathbf{v}, \mathbf{v}) \exp(-\int_0^{\tau_-(\mathbf{x},\mathbf{v})} (\Sigma_a + \sigma_s)(\mathbf{x} - t\mathbf{v}) dt),$$

where $\tau_-(\mathbf{x}, \mathbf{v})$ has been defined in (7). This expression implies, together with the facts that Ω is bounded and $g \geq \underline{g}$, that $\widetilde{u}(\mathbf{x}, \mathbf{v}) \geq \mathfrak{c}'$ for some $\mathfrak{c}' > 0$.

We then check that $\phi := u - \widetilde{u}$ solves the linear transport equation

$$\begin{aligned} \mathbf{v} \cdot \nabla \phi + (\Sigma_a + \sigma_s)\phi &= \sigma_s(\mathbf{x}) K u, \quad && \text{in } X \\ \phi(\mathbf{x}, \mathbf{v}) &= 0, \quad && \text{on } \Gamma_- . \end{aligned}$$

Using the fact that $u \geq 0$ (and therefore $\sigma_s K u \geq 0$), we conclude that this transport equation has a solution $\phi \geq 0$. Therefore, $u \geq \widetilde{u}$. The final result then follows. □

We are now ready study solutoin properties of the semilinear transport equation (1). For physical reasons, we are only interested in *non-negative* solutions. We consider two types of incoming boundary sources.

2.1 General bounded sources

For a given set of functions $(\sigma_a, \sigma_b, \sigma_s, \Theta)$, let $g(\mathbf{x}, \mathbf{v}) \in L^\infty(\Gamma_-)$ be the boundary source for (1). We denote by $\underline{g} := \inf_{(\mathbf{x},\mathbf{v})\in\Gamma_-} g(\mathbf{x}, \mathbf{v})$ and $\overline{g} := \sup_{(\mathbf{x},\mathbf{v})\in\Gamma_-} g(\mathbf{x}, \mathbf{v})$. We assume that

$$\underline{g} > 0, \quad \text{and,} \quad \overline{g} \leq \begin{cases} \inf\limits_{\Omega} \dfrac{\sigma_a}{\sigma_b}, & \text{when } \sigma_s \equiv 0, \\ \max\left(\inf\limits_{\Omega} \dfrac{\sigma_a}{\sigma_b},\ 2\underline{\theta} \inf\limits_{\Omega} \dfrac{\sigma_s(\mathbf{x})}{\sigma_b(\mathbf{x})} \right), & \text{when } \sigma_s \neq 0. \end{cases} \tag{8}$$

with $\underline{\theta}$ being the constant introduced in Assumption $(\mathcal{A})$. We can show that a non-negative solution to (1) with such a g exists and is unique. Our main strategy of proof is to analyze the fixed-point iteration: $k \geq 1$

$$\begin{aligned} \mathbf{v} \cdot \nabla u^k + (\sigma_a + \sigma_s) u^k(\mathbf{x}, \mathbf{v}) + \sigma_b \langle u^{k-1} \rangle u^k(\mathbf{x}, \mathbf{v}) &= \sigma_s(\mathbf{x}) K u^k(\mathbf{x}, \mathbf{v}), \quad && \text{in } X \\ u^k(\mathbf{x}, \mathbf{v}) &= g(\mathbf{x}, \mathbf{v}), \quad && \text{on } \Gamma_- \end{aligned} \tag{9}$$

using Kellogg's uniqueness theory [23] for the Schauder Fixed-Point Theorem together with the averaging lemma [19]. For the convenience of the readers, we recalled both results in the Appendix A.

We now prove the existence and uniqueness of non-negative solutions to (1).

Theorem 2.2. *For any $\sigma_a \in \mathcal{F}_{\underline{\sigma}_a}^{\overline{\sigma}_a}$ and $\sigma_b \in \mathcal{F}_{\underline{\sigma}_b}^{\overline{\sigma}_b}$, let $g \in L^\infty(\Gamma_-)$ be given as in (8). Then, under the assumption $(\mathcal{A})$, the transport equation (1) has a unique bounded solution $u \in L^\infty(X)$ that is non-negative: $u(\mathbf{x}, \mathbf{v}) \geq 0$.*

Proof. We first show the existence of non-negative solution by showing that a fixed point exist for the iteration (9). We introduce a set of bounded functions:

$$\mathcal{M} = \{m \in L^2(\Omega) \mid 0 \leq m(\mathbf{x}) \leq \overline{g} \ a.e.\} .$$

It is clear that $\mathcal{M}$ is convex, bounded and closed under the L^2 topology.

For any given function $m(\mathbf{x}) \in \mathcal{M}$, we introduce a linear transport equation:

$$\begin{aligned} \mathbf{v} \cdot \nabla u + (\sigma_a + \sigma_s) u(\mathbf{x}, \mathbf{v}) + \sigma_b(\mathbf{x}) m(\mathbf{x}) u(\mathbf{x}, \mathbf{v}) &= \sigma_s(\mathbf{x}) K u(\mathbf{x}, \mathbf{v}), && \text{in } X \\ u(\mathbf{x}, \mathbf{v}) &= g(\mathbf{x}, \mathbf{v}), && \text{on } \Gamma_- . \end{aligned} \tag{10}$$

This is simply the semilinear transport equation (1) with $\langle u \rangle$ replaced by $m(\mathbf{x})$. Under the assumptions we have made, we conclude from the standard transport theory [2, 15] that this linear transport equation has a unique solution $u(\mathbf{x}, \mathbf{v}) \in L^\infty(X)$. Moreover, u satisfies $0 \leq u(\mathbf{x}, \mathbf{v}) \leq \overline{g}$. This means also that $0 \leq \langle u \rangle \leq \overline{g}$.

Therefore, the operator $\mathcal{C} : m \mapsto \langle u \rangle$, defined through the relation

$$\mathcal{C}(m) := \langle u \rangle \tag{11}$$

with u being the solution to (10), maps $\mathcal{M}$ into a subset of it, that is, $\mathcal{C}(\mathcal{M}) \subseteq \mathcal{M}$. Meanwhile, we can also verify that $\mathcal{C}$ is a continuous operator on $\mathcal{M}$. To see that, let u and $\tilde{u}$ be the solutions of (10) with m and $\tilde{m}$ respectively. Then $w = \tilde{u} - u$ solves

$$\begin{aligned} \mathbf{v} \cdot \nabla w + (\sigma_a + \sigma_s) w + \sigma_b(\mathbf{x}) m(\mathbf{x}) w &= \sigma_s(\mathbf{x}) K w - (m - \tilde{m}) \tilde{u}, && \text{in } X \\ w(\mathbf{x}, \mathbf{v}) &= 0, && \text{on } \Gamma_- . \end{aligned}$$

With the assumptions we have, especially the fact that $m \geq 0$, this transport equation admits a unique solution that satisfies the stability bound

$$\|w\|_{\mathcal{H}^1(X)} := \|u - \tilde{u}\|_{\mathcal{H}^1(X)} \leq \tilde{\mathfrak{c}} \|(m - \tilde{m}) \tilde{u}\|_{L^2(X)} \leq \mathfrak{c} \|(m - \tilde{m})\|_{L^2(\Omega)}$$

for some constants $\mathfrak{c}, \tilde{\mathfrak{c}} > 0$. The last inequality comes from the fact that $\tilde{u} \in L^\infty(X)$. Using this bound, together with the averaging lemma [19], that is, Theorem A.1, and the fact that $w_{|\Gamma_-} := u_{|\Gamma_-} - \tilde{u}_{|\Gamma_-} = 0$, we conclude that $\langle w \rangle := \langle u \rangle - \langle \tilde{u} \rangle \in W^{1/2,2}(\Omega)$ and

$$\|\mathcal{C}(m) - \mathcal{C}(\tilde{m})\|_{W^{1/2,2}(\Omega)} := \|\langle u \rangle - \langle \tilde{u} \rangle\|_{W^{1/2,2}(\Omega)} \leq \|u - \tilde{u}\|_{\mathcal{H}^1(X)} \leq \mathfrak{c} \|(m - \tilde{m})\|_{L^2(\Omega)}.$$

This bound, combined with the Kondrachov embedding theorem [1], leads to the fact that the operator $\mathcal{M}$ is a continuous compact operator from $\mathcal{M}$ to itself. The Schauder Fixed-Point Theorem [17, 42] then implies that exists a fixed point $m^* \in \mathcal{M}$ that $\mathcal{C}(m^*) = m^*$. Therefore, there exists a bounded non-negative solution to the transport equation (1).

We now use Kellogg's theory [23], that is, Theorem A.2, to show uniqueness of the above fixed point. To verify that the fixed point cannot live on $\partial\mathcal{M}$, we observe that since $\sigma_a > 0$, the solution operator of (10) is a strict contraction even when $m \equiv 0$. Therefore $\langle u \rangle < \overline{g}$. Meanwhile, Lemma 2.1 implies that $\langle u \rangle > 0$. Therefore, $\mathcal{C}$ maps $\mathcal{M}$ into its interior $\mathcal{M}^\circ$. This shows that the fixed point of $\mathcal{C}$ cannot live on $\partial\mathcal{M}$.

The remaining task is to show that the Frechét derivative of $\mathcal{C}$ does not have 1 as its eigenvalue in $\mathcal{M}$. Let u be the solution to (10) with function $m(\mathbf{x})$, $\delta m(\mathbf{x})$ a perturbation of m such that $m + \delta m \in \mathcal{M}$, and ϕ the solution to

$$\begin{array}{rcll} \mathbf{v}\cdot\nabla\phi + (\sigma_a+\sigma_s)\phi(\mathbf{x},\mathbf{v}) + \sigma_b m\phi(\mathbf{x},\mathbf{v}) &=& \sigma_s K\phi - \sigma_b \delta m(\mathbf{x}) u(\mathbf{x},\mathbf{v}), & \text{in } X \\ \phi(\mathbf{x},\mathbf{v}) &=& 0, & \text{on } \Gamma_- . \end{array} \tag{12}$$

Then it is straightforward to verify that the Frechét derivative of $\mathcal{C}$ at m in the direction δm is given as $\mathcal{C}'[m](\delta m) = \langle\phi\rangle$. Assume now that $\mathcal{C}'[m]$ indeed has 1 as its eigenvalue and let $\langle\phi\rangle \not\equiv 0$ be the corresponding eigenfunction, i.e., $\mathcal{C}'[m](\langle\phi\rangle) = \langle\phi\rangle$. Then the transport equation (12) is equivalent to

$$\begin{array}{rcll} \mathbf{v}\cdot\nabla\phi + (\sigma_a+\sigma_s)\phi(\mathbf{x},\mathbf{v}) + \sigma_b m\phi(\mathbf{x},\mathbf{v}) &=& \sigma_s K\phi - \sigma_b \langle\phi\rangle u(\mathbf{x},\mathbf{v}), & \text{in } X\,, \\ \phi(\mathbf{x},\mathbf{v}) &=& 0, & \text{on } \Gamma_- . \end{array}$$

Let $\sigma_t(\mathbf{x}) := \sigma_a(\mathbf{x}) + \sigma_s(\mathbf{x}) + \sigma_b(\mathbf{x})m(\mathbf{x})$ and $R(\mathbf{x},\mathbf{v}) := \sigma_s K\phi - \sigma_b\langle\phi\rangle u(\mathbf{x},\mathbf{v})$. By the standard method of characteristics, it is straightforward to check that ϕ satisfies

$$\begin{aligned} |\phi(\mathbf{x},\mathbf{v})| &= \left| \int_0^{\tau_-(\mathbf{x},\mathbf{v})} \exp\left[-\int_0^\ell \sigma_t(\mathbf{x}-s\mathbf{v})ds \right] R(\mathbf{x}-\ell\mathbf{v},\mathbf{v}) d\ell \right| \\ &= \left| \int_0^{\tau_-(\mathbf{x},\mathbf{v})} \sigma_t(\mathbf{x}-\ell\mathbf{v}) \exp\left[-\int_0^\ell \sigma_t(\mathbf{x}-s\mathbf{v})ds \right] \frac{R(\mathbf{x}-\ell\mathbf{v},\mathbf{v})}{\sigma_t(\mathbf{x}-\ell\mathbf{v})} d\ell \right| \\ &\le \int_0^{\tau_-(\mathbf{x},\mathbf{v})} \sigma_t(\mathbf{x}-\ell\mathbf{v}) \exp\left[-\int_0^\ell \sigma_t(\mathbf{x}-s\mathbf{v})ds \right] \sup_{\ell\in(0,\tau_-(\mathbf{x},\mathbf{v}))} \left| \frac{R(\mathbf{x}-\ell\mathbf{v},\mathbf{v})}{\sigma_t(\mathbf{x}-\ell\mathbf{v})} \right| d\ell \\ &\le \left(1 - \exp\left[-\int_0^{\tau_-(\mathbf{x},\mathbf{v})} \sigma_t(\mathbf{x}-s\mathbf{v})ds \right] \right) \sup_{(\mathbf{y},\mathbf{v})\in X} \frac{|R(\mathbf{y},\mathbf{v})|}{\sigma_t(\mathbf{y})} \\ &\le \beta \sup_{(\mathbf{y},\mathbf{v})\in X} \frac{|R(\mathbf{y},\mathbf{v})|}{\sigma_t(\mathbf{y})}, \qquad \text{for some } \beta < 1\,. \end{aligned} \tag{13}$$

When $\sigma_s \equiv 0$, we have

$$|R(\mathbf{x},\mathbf{v})| = \sigma_b u(\mathbf{x},\mathbf{v}) |\langle\phi\rangle| \le \sigma_b\, u(\mathbf{x},\mathbf{v}) \sup_X |\phi|.$$

This, together with (13), gives that

$$|\phi(\mathbf{x},\mathbf{v})| \le \beta \sup_X \frac{\sigma_b u(\mathbf{x},\mathbf{v})}{\sigma_t} \sup_X |\phi| \le \beta\overline{g} \sup_\Omega \frac{\sigma_b}{\sigma_t} \sup_X |\phi|. \tag{14}$$

When $\overline{g} \le \inf_\Omega \dfrac{\sigma_a}{\sigma_b}$, we have that $|\phi(\mathbf{x},\mathbf{v})| \le \beta \sup_X |\phi|$, $\forall\,(\mathbf{x},\mathbf{v}) \in X$. Therefore $\phi \equiv 0$.

When $\sigma_s \not\equiv 0$ satisfies the assumption $(\mathcal{A})$, we have

$$|R(\mathbf{x},\mathbf{v})| \le |\sigma_s K\phi| + |\sigma_b u(\mathbf{x},\mathbf{v})\langle\phi\rangle| \le \Big(\sigma_s + \sigma_b u(\mathbf{x},\mathbf{v})\Big) \sup_X |\phi|.$$

This, together with (13), gives that

$$|\phi(\mathbf{x},\mathbf{v})| \le \sup_X \frac{\sigma_s + \sigma_b u(\mathbf{x},\mathbf{v})}{\sigma_t} \sup_X |\phi| \le \sup_\Omega \frac{\sigma_s + \sigma_b \overline{g}}{\sigma_t} \sup_X |\phi|. \tag{15}$$

Therefore, when $\overline{g} \le \inf_\Omega \frac{\sigma_a}{\sigma_b}$, we have that $|\phi(\mathbf{x},\mathbf{v})| \le \beta \sup_X |\phi|$, $\forall\,(\mathbf{x},\mathbf{v}) \in X$, for some $\beta < 1$. Therefore $\phi \equiv 0$.

Meanwhile, we can also have

$$\begin{aligned}|R(\mathbf{y},\mathbf{v})| = |\sigma_s K\phi - \underline{\theta}\sigma_s\langle\phi\rangle + \underline{\theta}\sigma_s\langle\phi\rangle - \sigma_b u\langle\phi\rangle| &\le |\sigma_s K\phi - \underline{\theta}\sigma_s\langle\phi\rangle| + |\underline{\theta}\sigma_s\langle\phi\rangle - \sigma_b u\langle\phi\rangle| \\ &\le ((1-\underline{\theta})\sigma_s + |\sigma_s(\mathbf{y})\underline{\theta} - \sigma_b(\mathbf{y})u(\mathbf{y},\mathbf{v})|) \sup |\phi|.\end{aligned}$$

This, together with (13), gives that

$$|\phi(\mathbf{x},\mathbf{v})| \le \sup_{(\mathbf{y},\mathbf{v})\in X} \frac{((1-\underline{\theta})\sigma_s + |\sigma_s(\mathbf{y})\underline{\theta} - \sigma_b(\mathbf{y})u(\mathbf{y},\mathbf{v})|)}{\sigma_t} \sup |\phi|. \tag{16}$$

Therefore, when $\overline{g} \le 2\underline{\theta} \inf_\Omega \dfrac{\sigma_s(\mathbf{x})}{\sigma_b(\mathbf{x})}$, we have that $|\phi(\mathbf{x},\mathbf{v})| \le \sup_\Omega \frac{\sigma_s}{\sigma_t} \sup_X |\phi|$, $\forall\,(\mathbf{x},\mathbf{v}) \in X$. Therefore $\phi \equiv 0$.

We have thus shown that 1 is not an eigenvalue of $\mathcal{C}'[m]$ in $\mathcal{M}$. Therefore, the fixed-point of $\mathcal{C}$ in $\mathcal{M}$ is unique. This concludes the proof. □

Remark 2.3. The above theory requires the smallness of the boundary source $g(\mathbf{x},\mathbf{v})$ (as a sufficient condition) for the solution to the transport equation (1) to be unique. This type of smallness assumptions is common for nonlinear problems. Note that in the diffusive limit when $\sigma_s \to +\infty$, the ratio $\sigma_s/\sigma_b \to +\infty$. This means that the smallness requirement is not necessary anymore in the diffusive regime. This is exactly what happened in [32] where it is shown that the diffusion approximation of (1) has a unique non-negative solution for any given bounded non-negative boundary source.

The following fact about non-negative solutions to the transport equation (1) can be proved using the same ideas of Lemma 2.1.

Corollary 2.4. *For any $\sigma_a \in \mathcal{F}_{\underline{\sigma}_a}^{\overline{\sigma}_a}$ and $\sigma_b \in \mathcal{F}_{\underline{\sigma}_b}^{\overline{\sigma}_b}$, let $g \in L^\infty(\Gamma_-)$ be given as in (8) and u be the corresponding unique non-negative solution to (1). Then, under the assumption $(\mathcal{A})$, $u \ge \mathfrak{c}$ for some $\mathfrak{c} > 0$.*

Proof. This result can be seen from two comparisons between solutions. Let w be the solution to the linear transport equation

$$\begin{aligned}\mathbf{v}\cdot\nabla w(\mathbf{x},\mathbf{v}) + (\sigma_a + \sigma_s)w(\mathbf{x},\mathbf{v}) + \sigma_b \overline{g} w(\mathbf{x},\mathbf{v}) &= \sigma_s K u, \quad \text{in } X \\ w(\mathbf{x},\mathbf{v}) &= g(\mathbf{x},\mathbf{v}), \quad \text{on } \Gamma_-.\end{aligned} \tag{17}$$

Using the fact that $u \geq 0$, we conclude that $\sigma_s K u \geq 0$, and therefore $w \geq 0$. Let $\phi := u - w$. Then ϕ solves

$$\begin{array}{rcll} \mathbf{v}\cdot\nabla\phi + (\sigma_a+\sigma_s)\phi + \sigma_b\langle u\rangle\phi &=& \sigma_b(\overline{g}-\langle u\rangle)w, & \text{in } \ X \\ \phi(\mathbf{x},\mathbf{v}) &=& 0, & \text{on } \ \Gamma_-\,. \end{array} \tag{18}$$

The right-hand-side of the equation is clearly non-negative (since $\overline{g} \geq \langle u\rangle$ and $w \geq 0$). Therefore $\phi \geq 0$. This implies that $u \geq w$.

Next, let $\widetilde{w}$ be the solution to (17) with the right-hand-side removed, that is, $\widetilde{w}$ solves

$$\begin{array}{rcll} \mathbf{v}\cdot\nabla\widetilde{w}(\mathbf{x},\mathbf{v}) + (\sigma_a+\sigma_s)\widetilde{w}(\mathbf{x},\mathbf{v}) + \sigma_b\overline{g}\widetilde{w}(\mathbf{x},\mathbf{v}) &=& 0, & \text{in } \ X \\ w(\mathbf{x},\mathbf{v}) &=& g(\mathbf{x},\mathbf{v}), & \text{on } \ \Gamma_- \end{array}$$

Then, $\widetilde{w}$ can be written as

$$\widetilde{w}(\mathbf{x},\mathbf{v}) = g(\mathbf{x}-\tau_-(\mathbf{x},\mathbf{v})\mathbf{v},\mathbf{v})\exp\Big(-\int_0^{\tau_-(\mathbf{x},\mathbf{v})}(\sigma_a+\sigma_s+\sigma_b\overline{g})(\mathbf{x}-t\mathbf{v})dt\Big).$$

We therefore have that $\widetilde{w} \geq \mathfrak{c} := \underline{g}e^{-(\overline{\sigma}_a+\overline{\sigma}_s+\overline{g}\overline{\sigma}_b)\mathrm{diam}(\Omega)} > 0$. Let $\widetilde{\phi} := w - \widetilde{w}$, then $\widetilde{\phi}$ solves

$$\begin{array}{rcll} \mathbf{v}\cdot\nabla\widetilde{\phi} + (\sigma_a+\sigma_s)\widetilde{\phi} + \sigma_b\overline{g}\widetilde{\phi} &=& \sigma_s K u, & \text{in } \ X \\ \widetilde{\phi}(\mathbf{x},\mathbf{v}) &=& 0, & \text{on } \ \Gamma_-\,. \end{array}$$

Non-negativity of $\sigma_a K u$ then implies that $\widetilde{\phi} \geq 0$. This gives that $w \geq \widetilde{w}$. We are now able to conclude that $u \geq w \geq \widetilde{w} \geq \mathfrak{c} > 0$. □

2.2 Collimated sources

We now consider the transport equation (1) with collimated illumination sources of the form:

$$g(\mathbf{x},\mathbf{v}) = \mathfrak{g}(\mathbf{x})\delta(\mathbf{v}-\mathbf{v}'), \quad \mathbf{v}' \in \mathbb{S}^{d-1}_{\mathbf{x}-}, \tag{19}$$

where $\mathfrak{g}(\mathbf{x}) \geq 0$ on $\partial\Omega$. This is a type of illumination strategies that is practically important.

By analyzing again the fixed-point iteration (9), we can establish the following existence and uniqueness of non-negative solutions to (1) with this new boundary source.

Theorem 2.5. *For any $\sigma_a \in \mathcal{F}^{\overline{\sigma}_a}_{\underline{\sigma}_a}$, $\sigma_b \in \mathcal{F}^{\overline{\sigma}_b}_{\underline{\sigma}_b}$, and (σ_s, Θ) satisfying the assumptions in $(\mathcal{A})$, let $\mu := \sup_{\mathbf{x}\in\Omega} \dfrac{\sigma_s}{\sigma_a+\sigma_s}$ and $\kappa := \sup_{\mathbf{x}\in\Omega} \dfrac{\sigma_b}{\sigma_a+\sigma_s}$. Assume that $\overline{\mathfrak{g}} := \sup_{\mathbf{x}\in\partial\Omega} \mathfrak{g}(\mathbf{x})$, μ and κ satisfy the condition*

$$\big(1 + \big[\mu^2\overline{\theta}/(1-\mu) + \mu\big] + \big[\mu\overline{\theta}/(1-\mu)^2\big]\big)\,\kappa\overline{\mathfrak{g}} < 1.$$

Then the transport equation (1) with boundary source (19) has a unique solution u such that $0 \leq \langle u\rangle \leq \overline{\mathfrak{g}}$.

Proof. For any $m(\mathbf{x}) \geq 0$, let u be the solution to the following linear transport equation

$$\begin{aligned} \mathbf{v}\cdot\nabla u + (\sigma_a+\sigma_s)u(\mathbf{x},\mathbf{v}) + \sigma_b(\mathbf{x})m(\mathbf{x})u(\mathbf{x},\mathbf{v}) &= \sigma_s(\mathbf{x})Ku(\mathbf{x},\mathbf{v}), \quad \text{in } X \\ u(\mathbf{x},\mathbf{v}) &= \mathfrak{g}(\mathbf{x})\delta(\mathbf{v}-\mathbf{v}'), \quad \text{on } \Gamma_- . \end{aligned} \tag{20}$$

We then define an operator $\mathcal{C} : m \mapsto \langle u \rangle$ as in (11), and introduce the following set of functions

$$\mathcal{M} = \{m \in L^\infty(\Omega) \mid 0 \leq m(\mathbf{x}) \leq \mathcal{C}(0)\}, \tag{21}$$

where $\mathcal{C}(0)$ is the angularly averaged solution to (20) with $m = 0$. It follows from linear transport theory that $\mathcal{C}(\mathcal{M}) \subseteq \mathcal{M}$.

Let u_1 and u_2 be the solutions to (20) with $m = m_1 \in \mathcal{M}$ and $m = m_2 \in \mathcal{M}$ respectively. Following the same notation as before, we define $\sigma_{t,i} := \sigma_a + \sigma_s + m_i\sigma_b$, $i = 1, 2$. We can then write the solutions u_i $(1 \leq i \leq 2)$ as $u_i = u_{i,b} + u_{i,s}$ with

$$\begin{aligned} u_{i,b}(\mathbf{x},\mathbf{v}) &= \mathfrak{g}(\mathbf{x}-\tau_-(\mathbf{x},\mathbf{v})\mathbf{v})\delta(\mathbf{v}-\mathbf{v}')\exp\left(-\int_0^{\tau_-(\mathbf{x},\mathbf{v})}\sigma_{t,i}(\mathbf{x}-s\mathbf{v})ds\right), \\ u_{i,s}(\mathbf{x},\mathbf{v}) &= \int_0^{\tau_-(\mathbf{x},\mathbf{v})}\exp\left(-\int_0^l \sigma_{t,i}(\mathbf{x}-s\mathbf{v})ds\right)(\sigma_s K u_i)(\mathbf{x}-l\mathbf{v},\mathbf{v})dl\,. \end{aligned} \tag{22}$$

Following the definition of the operator K in (3), we have that

$$Ku_{i,b}(\mathbf{x}',\mathbf{v}) = \Theta(\mathbf{v},\mathbf{v}')\mathfrak{g}(\mathbf{x}'-\tau_-(\mathbf{x}',\mathbf{v}')\mathbf{v}')\exp\left(-\int_0^{\tau_-(\mathbf{x}',\mathbf{v}')}\sigma_{t,i}(\mathbf{x}'-s\mathbf{v}')ds\right) \leq \bar{\mathfrak{g}}\bar{\theta}.$$

where $\mathbf{x}' := \mathbf{x} - l\mathbf{v}$, $l \in (0, \tau_-(\mathbf{x},\mathbf{v}))$. Meanwhile, using the same procedure as in (13), we have that,

$$\begin{aligned} \|u_{i,s}\|_{L^\infty(X)} &\leq \left(1-\exp\left[-\int_0^{\tau_-(\mathbf{x},\mathbf{v})}\sigma_{t,i}(\mathbf{x}-s\mathbf{v})ds\right]\right)\sup_{\mathbf{x}\in\Omega}\left|\frac{\sigma_s(\mathbf{x})}{\sigma_{t,i}(\mathbf{x})}\right|(\|u_{i,s}\|_{L^\infty(X)}+\bar{\mathfrak{g}}\bar{\theta}) \\ &\leq \mu\|u_{i,s}\|_{L^\infty(X)} + \mu\bar{\mathfrak{g}}\bar{\theta}\,. \end{aligned}$$

This implies that

$$\|u_{i,s}\|_{L^\infty(X)} \leq \mu\bar{\mathfrak{g}}\bar{\theta}/(1-\mu). \tag{23}$$

Let us now verify that $\phi := u_1 - u_2$ solves

$$\begin{aligned} \mathbf{v}\cdot\nabla\phi + (\sigma_a+\sigma_s+\sigma_b m_1)\phi &= \sigma_s K\phi + u_2\sigma_b(m_2-m_1), \quad \text{in } X \\ \phi(\mathbf{x},\mathbf{v}) &= 0, \quad \text{on } \Gamma_- . \end{aligned}$$

In the same manner, we write $\phi = \phi_b + \phi_s$ with ϕ_b and ϕ_s given as

$$\begin{aligned} \phi_b(\mathbf{x},\mathbf{v}) &= \int_0^{\tau_-(\mathbf{x},\mathbf{v})}\exp\left(-\int_0^l\sigma_{t,1}(\mathbf{x}-s\mathbf{v})ds\right)(\sigma_s K\phi_b + u_{2,b}\sigma_b(m_2-m_1))(\mathbf{x}-l\mathbf{v},\mathbf{v})dl\,, \\ \phi_s(\mathbf{x},\mathbf{v}) &= \int_0^{\tau_-(\mathbf{x},\mathbf{v})}\exp\left(-\int_0^l\sigma_{t,1}(\mathbf{x}-s\mathbf{v})ds\right)(\sigma_s K\phi_s + u_{2,s}\sigma_b(m_2-m_1))(\mathbf{x}-l\mathbf{v},\mathbf{v})dl\,. \end{aligned}$$

Use the representation of $u_{2,b}$, we obtain ϕ_b in the following form,

$$\phi_b(\mathbf{x}, \mathbf{v}) = \phi_{b,b}(\mathbf{x})\delta(\mathbf{v} - \mathbf{v}') + \phi_{b,s}(\mathbf{x}, \mathbf{v})\,,$$

where $|\phi_{b,b}| \le \kappa\bar{\mathfrak{g}}\|m_1 - m_2\|_{L^\infty(\Omega)}$ with $\kappa = \sup_{\mathbf{x}\in\Omega} |\frac{\sigma_b}{\sigma_a+\sigma_s}|$, and

$$\begin{aligned}\phi_{b,s}(\mathbf{x}, \mathbf{v}) = &\int_0^{\tau_-(\mathbf{x},\mathbf{v})} \exp\left(-\int_0^l \sigma_{t,1}(\mathbf{x} - s\mathbf{v})ds\right) \sigma_s K\phi_{b,s}(\mathbf{x} - l\mathbf{v}, \mathbf{v})dl \\ &+ \Theta(\mathbf{v}, \mathbf{v}') \int_0^{\tau_-(\mathbf{x},\mathbf{v})} \exp\left(-\int_0^l \sigma_{t,1}(\mathbf{x} - s\mathbf{v})ds\right) \sigma_s\phi_{b,b}(\mathbf{x} - l\mathbf{v})dl\,.\end{aligned} \tag{24}$$

Note that the second term on the right-hand-side is bounded by $\bar{\theta}\bar{\mathfrak{g}}\kappa\mu\|m_1 - m_2\|_{L^\infty(\Omega)}$. Therefore, we have

$$\|\phi_{b,s}\|_{L^\infty(X)} \le \bar{\theta}\bar{\mathfrak{g}}\kappa\mu\|m_1 - m_2\|_{L^\infty(\Omega)}/(1 - \mu)\,. \tag{25}$$

Integrating (24) over $\mathbb{S}^{d-1}$, and then using (25), we have,

$$|\langle\phi_{b,s}\rangle| \le \mu\|\phi_{b,s}\|_{L^\infty(X)} + \mu\kappa\bar{\mathfrak{g}}\|m_1 - m_2\|_{L^\infty(\Omega)} \le \left[\mu^2\bar{\theta}/(1 - \mu) + \mu\right] \kappa\bar{\mathfrak{g}}\|m_1 - m_2\|_{L^\infty(\Omega)}\,. \tag{26}$$

In a similar manner, we can estimate, using (23),

$$\|\phi_s\|_{L^\infty(X)} \le \kappa\|u_{2,s}\|_{L^\infty(X)}\|m_1 - m_2\|_{L^\infty(\Omega)}/(1 - \mu) \le [\mu\bar{\theta}/(1 - \mu)^2]\kappa\bar{\mathfrak{g}}\|m_1 - m_2\|_{L^\infty(\Omega)}\,. \tag{27}$$

The bounds in (26) and (27) now allow us to have

$$\begin{aligned}|\langle\phi\rangle| &\le |\langle\phi_b\rangle| + |\langle\phi_s\rangle| \\ &\le |\phi_{b,b}| + |\langle\phi_{b,s}\rangle| + \|\phi_s\|_{L^\infty(X)} \\ &\le \left(1 + \left[\mu^2\bar{\theta}/(1 - \mu) + \mu\right] + [\mu\bar{\theta}/(1 - \mu)^2]\right) \kappa\bar{\mathfrak{g}}\|m_1 - m_2\|_{L^\infty(\Omega)}.\end{aligned}$$

When the constant $(1 + \left[\mu^2\bar{\theta}/(1 - \mu) + \mu\right] + [\mu\bar{\theta}/(1 - \mu)^2])\kappa\bar{\mathfrak{g}} < 1$, the mapping $\mathcal{C}$ is a contraction in $L^\infty(\Omega)$ norm, the Banach Fixed-Point Theorem [42] implies that the solution is unique in $\mathcal{M}$. □

Remark 2.6. Unlike in the previous section, we are not able to use the Schauder Fixed-Point Theorem in this proof due to the lack of compactness of the map $\mathcal{C}$. One can make additional assumptions on the smoothness of all the coefficients involved as well as the physical domain Ω to recover such compactness. We did not pursue in this direction.

3 Inversion in non-scattering media

We start with the case of non-scattering media where $\sigma_s(\mathbf{x}) \equiv 0$. In this case, the original transport model (1) is simplified into a free transport equation which is essentially an ordinary differential equation parameterized by the angular variable $\mathbf{v}$. We can obtain an explicit method for the reconstructions with either collimated sources or point sources. Similar analysis for the linear transport equation can be found in [29]. Inversion in this setup with a general bounded source will be treated in the next section as a special case.

3.1 Inversion with collimated sources

With collimated sources, we can integrate the transport equation along direction $\mathbf{v}$ to get the following integral representation of the transport solution, when $\sigma_s \equiv 0$:

$$u(\mathbf{x},\mathbf{v}) = g(\mathbf{x}-\tau_-(\mathbf{x},\mathbf{v})\mathbf{v},\mathbf{v}) \exp\left(-\int_0^{\tau_-(\mathbf{x},\mathbf{v})} (\sigma_a(\mathbf{x}-s\mathbf{v}) + \sigma_b\langle u\rangle(\mathbf{x}-s\mathbf{v}))ds\right). \quad (28)$$

Let us assume that we have data generated from two collimated sources, $g_j(\mathbf{x},\mathbf{v}) = \mathfrak{g}_j(\mathbf{x})\delta(\mathbf{v}-\mathbf{v}')$ ($j=1,2$), focused in the same direction $\mathbf{v}' \in \mathbb{S}^{d-1}$ but with different strengths $\mathfrak{g}_1 \neq \mathfrak{g}_2$. Then the corresponding data are:

$$H_j(\mathbf{x}) = \sigma_a(\mathbf{x})\langle u_j\rangle(\mathbf{x}) + \sigma_b(\mathbf{x})\langle u_j\rangle^2(\mathbf{x}), \quad j=1,\ 2$$

with u_j satisfying

$$u_j(\mathbf{x},\mathbf{v}) = \mathfrak{g}_j(\mathbf{x}-\tau_-(\mathbf{x},\mathbf{v})\mathbf{v})\delta(\mathbf{v}-\mathbf{v}') \exp\left(-\int_0^{\tau_-(\mathbf{x},\mathbf{v})} (\sigma_a(\mathbf{x}-s\mathbf{v}) + \sigma_b\langle u_j\rangle(\mathbf{x}-s\mathbf{v}))ds\right).$$

We can integrate $u_j(\mathbf{x},\mathbf{v})$ over $\mathbf{v} \in \mathbb{S}^{d-1}$ to get,

$$\langle u_j\rangle(\mathbf{x}) = \mathfrak{g}_j(\mathbf{x}-\tau_-(\mathbf{x},\mathbf{v}'),\mathbf{v}') \exp\left(-\int_0^{\tau_-(\mathbf{x},\mathbf{v}')} (\sigma_a(\mathbf{x}-s\mathbf{v}') + \sigma_b\langle u_j\rangle(\mathbf{x}-s\mathbf{v}'))ds\right). \quad (29)$$

For any fixed $\mathbf{x} \in \Omega$, we introduce the notations $\mathbf{x}' := \mathbf{x}-\tau_-(\mathbf{x},\mathbf{v}')\mathbf{v}' \in \partial\Omega$, $\phi_j(s) := \langle u_j\rangle(\mathbf{x}'+s\mathbf{v}')$, $\widetilde{\sigma}_a(s) := \sigma_a(\mathbf{x}'+s\mathbf{v}')$ and $\widetilde{\sigma}_b(s) := \sigma_b(\mathbf{x}'+s\mathbf{v}')$. Then (29) is equivalent to:

$$\phi_j(t) = \mathfrak{g}_j(\mathbf{x}') \exp\left(-\int_0^t (\widetilde{\sigma}_a(s) + \widetilde{\sigma}_b(s)\phi_j(s))ds\right) \quad (30)$$

with $\phi_j(0) = \mathfrak{g}_j(\mathbf{x}')$. Taking the logarithm of both sides of (30) and then differentiate with respect to t, we obtain the following ODE for $\phi_j(t)$,

$$\phi_j'(t) = -(\widetilde{\sigma}_a(t) + \widetilde{\sigma}_b(t)\phi_j(t))\phi_j(t), \quad t \in [0, \tau_-(\mathbf{x},\mathbf{v}')]. \quad (31)$$

From the definition of the internal data (4), we notice that the right-hand-side of (31) is exactly $-H_j(\mathbf{x}'+t\mathbf{v}')$. We can therefore reconstruct $\phi_j(t)$ from the datum H_j as

$$\widetilde{\phi}_j(t) = \mathfrak{g}_j(\mathbf{x}') - \int_0^t H_j(\mathbf{x}'+s\mathbf{v}')ds. \quad (32)$$

Once we reconstructed $\{\widetilde{\phi}_j(t)\}_{j=1}^2$, we can reconstruct $\widetilde{\sigma}_a$ and $\widetilde{\sigma}_b$ from the data by solving the following system of equations at any $t \in [0, \tau_-(\mathbf{x},\mathbf{v}')]$:

$$\begin{array}{rcl} \widetilde{\sigma}_a(t)\widetilde{\phi}_1(t) + \widetilde{\sigma}_b(t)\widetilde{\phi}_1^2(t) &=& H_1(\mathbf{x}'+t\mathbf{v}'), \\ \widetilde{\sigma}_a(t)\widetilde{\phi}_2(t) + \widetilde{\sigma}_b(t)\widetilde{\phi}_2^2(t) &=& H_2(\mathbf{x}'+t\mathbf{v}'). \end{array} \quad (33)$$

This linear system, for the unknown coefficient pair (σ_a, σ_b), is uniquely invertible at $t \in [0, \tau_-(\mathbf{x}, \mathbf{v}')]$ if $\phi_1(t) \neq \phi_2(t)$.

The following result shows that if the data $\{H_j\}_{j=1}^2$ are consistent with the model, that is, if the data are generated from the model with the true coefficients, then we can select the illumination sources $\mathfrak{g}_1$ and $\mathfrak{g}_2$ to be sufficiently different to make the system (33) invertible.

Lemma 3.1. *If* $\mathfrak{g}_1 > \mathfrak{g}_2 > 0$*, then* $\phi_1(t) > \phi_2(t)$*,* $\forall t \in [0, \tau_-(\mathbf{x}, \mathbf{v}')]$*.*

Proof. From (30) and the non-negativity of transport solutions, we conclude that $\mathfrak{g}_j > 0$ implies $\mathfrak{g}_j \geq \phi_j(t) > 0$. We check that $z(t) := \phi_1(t) - \phi_2(t)$ satisfies

$$\frac{z'(t)}{z(t)} = -(\widetilde{\sigma}_a(t) + \widetilde{\sigma}_b(t)(\phi_1(t) + \phi_2(t)))\,. \tag{34}$$

This implies that

$$z(t) = z(0) \exp\left(- \int_0^t (\widetilde{\sigma}_a(s) + \widetilde{\sigma}_b(s)(\phi_1(s) + \phi_2(s)))ds \right). \tag{35}$$

We then conclude that $z(t) > 0$ using the assumption that $z(0) = \phi_1(0) - \phi_2(0) > 0$. □

To summarize, in order to reconstruct the coefficients σ_a and σ_b, we first reconstruct the solutions (32) from the data. We then solve the linear system (33) to reconstruct (σ_a, σ_b).

3.2 Inversion with point sources

An explicit reconstruction method can also be developed in the case when point sources are used to illuminate the media. Let $g_j(\mathbf{x}, \mathbf{v}) = \mathfrak{g}_j(\mathbf{v})\delta(\mathbf{x} - \mathbf{x}')$ $(j = 1, 2)$ with $\mathfrak{g}_1 \neq \mathfrak{g}_2$ positive constants. To be technically correct in the derivation below, we assume that σ_b vanishes in the vicinity of $\mathbf{x}' \in \partial\Omega$, that is, $\sigma_b \equiv 0$ in $B_\varepsilon(\mathbf{x}') \cap \Omega$ for some $\varepsilon > 0$. In applications, this can be done in a straightforward way by placing the illuminating point source a little away from the surface of the media (which, mathematically, is equivalent to extending the domain Ω to a slightly larger domain Ω' with $\sigma_b \equiv 0$ in $\Omega' \backslash \bar{\Omega}$). We can then integrate the transport equation along the direction of each ray out of the point source to have

$$\langle u_j \rangle(\mathbf{x}) = \mathfrak{g}_j(\mathbf{v})\, |\mathbf{n}(\mathbf{x}') \cdot \mathbf{v}|\, \frac{\exp\left(- \int_0^{\tau_-(\mathbf{x},\mathbf{v})} (\sigma_a + \sigma_b \langle u_j \rangle)(\mathbf{x} - s\mathbf{v}) ds \right)}{|\mathbf{x} - \mathbf{x}'|^{d-1}}, \tag{36}$$

where $\mathbf{v} = \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|}$. The parameterization of the line segment between $\mathbf{x}$ and $\mathbf{x}'$ is the same as before: $\{\mathbf{x}' + s\mathbf{v} \mid s \in (0, |\mathbf{x} - \mathbf{x}'|)\}$. Let $\phi_j(s) := \langle u_j \rangle(\mathbf{x}' + s\mathbf{v})$, $\widetilde{\sigma}_a(s) := \sigma_a(\mathbf{x}' + s\mathbf{v})$, $\widetilde{\sigma}_b(s) := \sigma_b(\mathbf{x}' + s\mathbf{v})$, and $\widetilde{H}_j(s) := H_j(\mathbf{x}' + s\mathbf{v})$, then we can write (36) as

$$\phi_j(t) = \mathfrak{g}_j(\mathbf{v})\, |\mathbf{n}(\mathbf{x}') \cdot \mathbf{v}|\, t^{1-d} \exp\left(- \int_0^t (\widetilde{\sigma}_a + \widetilde{\sigma}_b \phi_j)(s) ds \right).$$

Taking the derivative with respect to t, we obtain that

$$\phi_j'(t) = \frac{1-d}{t}\phi_j(t) - \big(\widetilde{\sigma}_a + \widetilde{\sigma}_b\phi_j\big)\phi_j(t). \tag{37}$$

We can then replace $\big(\widetilde{\sigma}_a + \widetilde{\sigma}_b\phi_j\big)\phi_j(t)$ in the equation with the data H_j and integrate the ODE, using the asymptotic behavior of $\phi_j(t)$ as $t \to 0$ from (36), to reconstruct the solution ϕ_j:

$$\widetilde{\phi}_j(t) = \frac{1}{t^{d-1}}\left(\mathfrak{g}_j(\mathbf{v})|\mathbf{n}(\mathbf{x}')\cdot\mathbf{v}| - \int_0^t \widetilde{H}_j(s)s^{d-1}ds\right). \tag{38}$$

The remaining task is to reconstruct $\widetilde{\sigma}_a$ and $\widetilde{\sigma}_b$ from the system of equations:

$$\widetilde{\sigma}_a(t)\phi_j(t) + \widetilde{\sigma}_b(t)\phi_j^2(t) = \widetilde{H}_j(t), \quad j = 1, 2. \tag{39}$$

Use the similar argument as in Lemma 3.1, it can be shown that $0 < \mathfrak{g}_1 < \mathfrak{g}_2$ is sufficient to ensure uniqueness of the inversion when the corresponding data are consistent with the model.

4 Inversion in media with known scattering

We now study the inverse problem of reconstructing the absorption coefficients σ_a and σ_b from data H in scattering media with the scattering coefficient σ_s assumed known.

4.1 Stability of inversion

We start with the inverse problem of reconstructing the absorption coefficient Σ_a in the linear transport equation (5) from internal data set of the form (6).

Let H be the internal datum (6) generated from the linear transport model (5) with the absorption coefficient $\Sigma_a \in \mathcal{F}_{\underline{\Sigma}_a}^{\overline{\Sigma}_a}(\Omega)$ and the boundary source g. For a given $\alpha > 0$, we define the set

$$\Pi_\alpha := \left\{(\Sigma_a, H, g) \mid \Sigma_a - \frac{\mathbf{v}\cdot\nabla\Sigma_a}{\Sigma_a} + \frac{\mathbf{v}\cdot\nabla H}{H} \ge \alpha > 0, \forall(\mathbf{v},\mathbf{x}) \in X\right\}.$$

Using the fact that $H = \Sigma_a\langle u\rangle$, u being the solution to (5) with coefficient Σ_a and source g, we see that Π_α is equivalent to

$$\Pi_\alpha' := \left\{(\Sigma_a, H, g) \mid \Sigma_a + \frac{\mathbf{v}\cdot\nabla\langle u\rangle}{\langle u\rangle} \ge \alpha > 0, \forall(\mathbf{v},\mathbf{x}) \in X\right\}.$$

We show next that we could stably reconstruct coefficients and data combinations (Σ_a, H, g) in the class of Π_α.

Theorem 4.1. *Let H and $\widetilde{H}$ be two data sets generated with coefficients Σ_a and $\widetilde{\Sigma}_a$ respectively from (5) in the form of (6) with boundary source g. Assume that there exists constants $\alpha > 0$ and $1 > \beta > 0$ such that:*

$(\mathcal{A}')$ *(i)* (Σ_a, H, g), $(\widetilde{\Sigma}_a, \widetilde{H}, g) \in \Pi_\alpha$, *and, (ii)* $\Sigma_{a|\partial\Omega}$ *is know and* $\overline{\Sigma}_a \| \frac{gH}{\Sigma_{a|\partial\Omega}} \|_{L^\infty(\Gamma_-)} \leq \beta$.

Then, under the assumtions in $(\mathcal{A})$*, the following stability holds for some constants* $\mathfrak{c}, \widetilde{\mathfrak{c}} > 0$*:*

$$\widetilde{\mathfrak{c}} \|H - \widetilde{H}\|_{L^2(\Omega)} \leq \|\Sigma_a - \widetilde{\Sigma}_a\|_{L^2(\Omega)} \leq \mathfrak{c} \|H - \widetilde{H}\|_{L^2(\Omega)}. \tag{40}$$

Proof. Let u and $\widetilde{u}$ be solutions to the transport equation (5) with coefficients Σ_a and $\tilde{\Sigma}_a$ respectively. By Lemma 2.1, we have that $\overline{g} \geq u, \widetilde{u} \geq \varepsilon > 0$ for some ε.

Let $w := u - \widetilde{u}$. Then we check that

$$H - \widetilde{H} = \widetilde{\Sigma}_a \langle w \rangle + (\Sigma_a - \widetilde{\Sigma}_a) \langle u \rangle.$$

This leads to the following equality:

$$\frac{u}{\langle u \rangle}(H - \widetilde{H}) = \widetilde{\Sigma}_a \frac{u}{\langle u \rangle} \langle w \rangle + (\Sigma_a - \widetilde{\Sigma}_a) u. \tag{41}$$

Therefore, we have that,

$$\| \frac{u}{\langle u \rangle}(H - \widetilde{H}) \|_{L^2(X)} \leq \| \widetilde{\Sigma}_a \frac{u}{\langle u \rangle} \langle w \rangle \|_{L^2(X)} + \| (\Sigma_a - \widetilde{\Sigma}_a) u \|_{L^2(X)}. \tag{42}$$

We also observe that $w \in L^\infty(X)$ solves the following transport equation:

$$\begin{aligned} \mathbf{v} \cdot \nabla w + (\widetilde{\Sigma}_a + \sigma_s) w &= \sigma_s K w(\mathbf{x}, \mathbf{v}) - (\Sigma_a - \widetilde{\Sigma}_a) u, && \text{in } X \\ w(\mathbf{x}, \mathbf{v}) &= 0, && \text{on } \Gamma_-. \end{aligned}$$

We therefore deduce, from the standard transport theory [15], that

$$\|w\|_{L^2(X)} \leq \|(\Sigma_a - \widetilde{\Sigma}_a) u\|_{L^2(X)}. \tag{43}$$

The left-hand-side of (40) then follows from (42) and (43), together with the boundedness of the coefficients and the corresponding solutions as well as the fact that $\|\langle w \rangle\|_{L^2(\Omega)} \leq \|w\|_{L^2(X)}$.

Meanwhile, (41) also implies that

$$\begin{aligned} \|(\Sigma_a - \widetilde{\Sigma}_a) u\|_{L^2(X)} &\leq \| \frac{u}{\langle u \rangle}(H - \widetilde{H}) \|_{L^2(X)} + \| \widetilde{\Sigma}_a \frac{u}{\langle u \rangle} \langle w \rangle \|_{L^2(X)} \\ &\leq \overline{\Sigma}_a \| \frac{u}{\langle u \rangle} \|_{L^\infty(X)} \Big(\| \frac{H - \widetilde{H}}{\widetilde{\Sigma}_a} \|_{L^2(\Omega)} + \|w\|_{L^2(X)} \Big) \\ &\leq \overline{\Sigma}_a \| \frac{u}{\langle u \rangle} \|_{L^\infty(X)} \Big(\| \frac{H - \widetilde{H}}{\widetilde{\Sigma}_a} \|_{L^2(\Omega)} + \|(\Sigma_a - \widetilde{\Sigma}_a) u\|_{L^2(X)} \Big), \end{aligned} \tag{44}$$

where the last step comes from (43).

Let $\phi := \frac{u}{\langle u \rangle}$. Then some simple algebra shows that ϕ solves the transport equation:

$$\begin{aligned} \mathbf{v} \cdot \nabla \phi + (\Sigma_a + \mathbf{v} \cdot \nabla \ln \langle u \rangle + \sigma_s) \phi &= \sigma_s K \phi, && \text{in } X \\ \phi(\mathbf{x}, \mathbf{v}) &= \frac{gH}{\Sigma_{a|\partial\Omega}}, && \text{on } \Gamma_- \end{aligned}$$

where the boundary condition comes from the assumption that $\Sigma_{a|\partial\Omega}$ is know (which implies that $\langle u\rangle_{|\partial\Omega} = \dfrac{H}{\Sigma_{a|\partial\Omega}}$). The first assumption in $(\mathcal{A}')$ means that $\Sigma_a + \mathbf{v}\cdot\nabla \ln\langle u\rangle \geq \alpha > 0$. Therefore, we can use the maximum principle, ensured by the assumption on the scattering kernel Θ in $(\mathcal{A})$, to conclude that

$$\|\phi\|_{L^\infty(X)} \leq \|\frac{gH}{\Sigma_{a|\partial\Omega}}\|_{L^\infty(\Gamma_-)}. \tag{45}$$

The bound in (44) then implies that

$$\|(\Sigma_a - \widetilde{\Sigma}_a)u\|_{L^2(X)} \leq \overline{\Sigma}_a\|\frac{gH}{\Sigma_{a|\partial\Omega}}\|_{L^\infty(\Gamma_-)}\Big(\|\frac{H-\widetilde{H}}{\widetilde{\Sigma}_a}\|_{L^2(\Omega)} + \|(\Sigma_a - \widetilde{\Sigma}_a)u\|_{L^2(X)}\Big).$$

This bound, together with the second assumption in $(\mathcal{A}')$, then implies that

$$\|(\Sigma_a - \widetilde{\Sigma}_a)u\|_{L^2(X)} \leq \frac{\beta}{1-\beta}\|\frac{H-\widetilde{H}}{\widetilde{\Sigma}_a}\|_{L^2(\Omega)}.$$

This gives the right-hand-side of the stability bound (40). □

The above theorem shows that, in appropriate settings, the absorption coefficient in the transport equation can be reconstructed stably with one interior datum H. This means that if we think of the term $\sigma_a + \sigma_b\langle u\rangle$ in the semilinear transport equation (1) as a single absorption coefficient, we can reconstruct this coefficient from a single data. This simple idea leads to a method to reconstruct σ_a and σ_b from two data sets. We now describe the method.

We will need the following result.

Lemma 4.2. *For a given set of $(\sigma_a, \sigma_b) \in \mathcal{F}_{\underline{\sigma}_a}^{\overline{\sigma}_a}(\Omega)\times\mathcal{F}_{\underline{\sigma}_b}^{\overline{\sigma}_b}(\Omega)$ and (σ_s, Θ) satisfying $(\mathcal{A})$, there exist two boundary sources g_1 and g_2 given as in (8) such that:*

$$|\langle u_1\rangle - \langle u_2\rangle| \geq \varepsilon, \quad \textit{for some } \varepsilon > 0$$

where u_1 and u_2 are solutions to (1) with g_1 and g_2 respectively.

Proof. Let g be such that

$$\gamma \min\Big(\frac{\underline{\sigma}_a}{\overline{\sigma}_b}, \ \underline{\theta}\frac{\underline{\sigma}_s(\mathbf{x})}{\overline{\sigma}_b(\mathbf{x})}\Big) \geq \overline{g}, \quad \text{and} \quad \underline{g} > 0, \text{ for some } 0 < \gamma < 1. \tag{46}$$

It is clear that g satisify (8). Let $g_1 \neq g_2$ be given as in (46). Following Corollary 2.4, we have $0 < \varepsilon' \leq u_1 \leq g_1$ and $0 < \varepsilon' \leq u_2 \leq g_2$ for some $\varepsilon' > 0$. Let $w := u_1 - u_2$. Then w solves

$$\begin{aligned} \mathbf{v}\cdot\nabla w(\mathbf{x},\mathbf{v}) + \Sigma w &= \int_{\mathbb{S}^{d-1}} \Sigma_s(\mathbf{x},\mathbf{v},\mathbf{v}')w, && \text{in } X \\ w(\mathbf{x},\mathbf{v}) &= g_1 - g_2, && \text{on } \Gamma_- . \end{aligned} \tag{47}$$

where $\Sigma(\mathbf{x},\mathbf{v}) := \sigma_a + \sigma_b \frac{\langle u_1\rangle + \langle u_2\rangle}{2} + \sigma_s$, $\Sigma_s(\mathbf{x},\mathbf{v},\mathbf{v}') := \sigma_s(\mathbf{x})\Theta(\mathbf{v},\mathbf{v}') - \sigma_b \frac{u_1+u_2}{2}$. With the assumptions in $(\mathcal{A})$ and the fact that g_1 and g_2 satisfying (46), we can verify that $\sigma_s\overline{\theta} \geq \Sigma_s(\mathbf{x},\mathbf{v},\mathbf{v}') \geq (1-\gamma)\underline{\theta}\underline{\sigma}_s > 0$ and $\Sigma - \int_{\mathbb{S}^{d-1}} \Sigma_s(\mathbf{x},\mathbf{v},\mathbf{v}')d\mathbf{v}' \geq (1-\gamma)\underline{\sigma}_a > 0$ (where γ is given in (46)). Therefore, the solution to (47) satisfies the maximum principle. By selecting $g_1 - g_2 \geq \varepsilon''$ for some $\varepsilon'' > 0$, we have that $w \geq \varepsilon$ for some $\varepsilon > 0$ using Lemma 2.1. □

Theorem 4.1 allows us to estimate the stability of reconstructing (σ_a, σ_b).

Corollary 4.3. *Let $(\sigma_a, \sigma_b) \in \mathcal{F}_{\underline{\sigma}_a}^{\overline{\sigma}_a}(\Omega) \times \mathcal{F}_{\underline{\sigma}_b}^{\overline{\sigma}_b}(\Omega)$ and $(\widetilde{\sigma}_a, \widetilde{\sigma}_b) \in \mathcal{F}_{\underline{\sigma}_a}^{\overline{\sigma}_a}(\Omega) \times \mathcal{F}_{\underline{\sigma}_b}^{\overline{\sigma}_b}(\Omega)$ be two sets of absorption coefficients, and $\mathbf{H} := (H_1, H_2)$ and $\widetilde{\mathbf{H}} := (\widetilde{H}_1, \widetilde{H}_2)$ the corresponding data generated with $\mathbf{g} = (g_1, g_2)$. Assume that $\mathbf{g}$ is selected as in Lemma 4.2, and $(\Sigma_a^j := \sigma_a + \sigma_b\langle u_j\rangle, H_j, g_j)$ and $(\widetilde{\Sigma}_a^j := \sigma_a + \widetilde{\sigma}_b\langle \widetilde{u}_j\rangle, \widetilde{H}_j, \widetilde{g}_j)$ $(j = 1, 2)$ satisfy $(\mathcal{A}')$. Then, under $(\mathcal{A})$, there exists constants $\mathfrak{c}, \widetilde{\mathfrak{c}} > 0$ such that*

$$\widetilde{\mathfrak{c}}\|\mathbf{H} - \widetilde{\mathbf{H}}\|_{L^2(\Omega)} \leq \left\| \begin{pmatrix} \sigma_a \\ \sigma_b \end{pmatrix} - \begin{pmatrix} \widetilde{\sigma}_a \\ \widetilde{\sigma}_b \end{pmatrix} \right\|_{L^2(\Omega)} \leq \mathfrak{c}\|\mathbf{H} - \widetilde{\mathbf{H}}\|_{L^2(\Omega)}. \tag{48}$$

Proof. The left inequality can be derived in the same manner as in Theorem 4.1. We define $w_j := u_j - \widetilde{u}_j$. Then some straightforward algebra leads us to the fact that

$$H_j - \widetilde{H}_j = \Big(\widetilde{\Sigma}_a^j + \widetilde{\sigma}_b\langle u_j\rangle\Big)\langle w_j\rangle + \Big[(\sigma_a - \widetilde{\sigma}_a) + (\sigma_b - \widetilde{\sigma}_b)\langle u_j\rangle\Big]\langle u_j\rangle.$$

With the boundedness of the coefficients as well as the solutions, we conclude that

$$\|H_j - \widetilde{H}_j\|_{L^2(\Omega)} \leq \mathfrak{c}_1'\|w_j\|_{L^2(\Omega)} + \mathfrak{c}_2'\|(\sigma_a - \widetilde{\sigma}_a) + (\sigma_b - \widetilde{\sigma}_b)\langle u_j\rangle\|_{L^2(\Omega)}. \tag{49}$$

The next step is to verify that w_j solves the linear transport equation:

$$\begin{aligned} \mathbf{v}\cdot\nabla w_j + (\widetilde{\Sigma}_a^j + \sigma_s)w_j &= \widetilde{K}w_j - \Big[(\sigma_a - \widetilde{\sigma}_a) + (\sigma_b - \widetilde{\sigma}_b)\langle u_j\rangle\Big]u_j, \quad \text{in } X \\ w_j(\mathbf{x},\mathbf{v}) &= 0, \quad \text{on } \Gamma_- \end{aligned}$$

with the scattering operator $\widetilde{K}$ defined as

$$\widetilde{K}w_j := \int_{\mathbb{S}^{d-1}} \Big[\sigma_s\Theta - \widetilde{\sigma}_b u_j\Big] w_j(\mathbf{x},\mathbf{v}')d\mathbf{v}'.$$

Following the same argument as in Lemma 4.2, this transport equation is uniquely invertible with a stability bound

$$\|w_j\|_{L^2(X)} \leq \mathfrak{c}_3'\|(\sigma_a - \widetilde{\sigma}_a) + (\sigma_b - \widetilde{\sigma}_b)\langle u_j\rangle\|_{L^2(X)}. \tag{50}$$

Therefore, we have, from (49) and (50), that

$$\|H_j - \widetilde{H}_j\|_{L^2(\Omega)} \leq \mathfrak{c}_4'\|(\sigma_a - \widetilde{\sigma}_a) + (\sigma_b - \widetilde{\sigma}_b)\langle u_j\rangle\|_{L^2(X)}. \tag{51}$$

With the selection of g_1 and g_2, we conclude from Lemma 4.2 that the matrix

$$P := \begin{pmatrix} 1 & \langle u_1 \rangle \\ 1 & \langle u_2 \rangle \end{pmatrix}$$

is invertible with a bounded inverse at every point $\mathbf{x} \in \Omega$. The left-hand-side of (48) then follows this fact and (51).

To get the second bound in (48), we notice that, by Theorem 4.1 (which requires the assumptions we have made), we have the bound

$$\|\Sigma_a^j - \widetilde{\Sigma}_a^j\|_{L^2(\Omega)} \le \mathfrak{c}' \|H_j - \widetilde{H}_j\|_{L^2(\Omega)}$$

for some constant $\mathfrak{c}' > 0$. This gives that

$$\left\| \begin{pmatrix} \Sigma_a^1 \\ \Sigma_a^2 \end{pmatrix} - \begin{pmatrix} \widetilde{\Sigma}_a^1 \\ \widetilde{\Sigma}_a^2 \end{pmatrix} \right\|_{L^2(\Omega)} \le \mathfrak{c}'' \|\mathbf{H} - \widetilde{\mathbf{H}}\|_{L^2(\Omega)} \tag{52}$$

for some constant $\mathfrak{c}'' > 0$. Meanwhile, we verify that

$$\begin{pmatrix} \Sigma_a^1 - \widetilde{\Sigma}_a^1 \\ \Sigma_a^2 - \widetilde{\Sigma}_a^2 \end{pmatrix} = \begin{pmatrix} 1 & \langle u_1 \rangle \\ 1 & \langle u_2 \rangle \end{pmatrix} \begin{pmatrix} \sigma_a - \widetilde{\sigma}_a \\ \sigma_b - \widetilde{\sigma}_b \end{pmatrix} - \begin{pmatrix} \frac{u_1}{\widetilde{\Sigma}_a^1}(\Sigma_a^1 - \widetilde{\Sigma}_a^1) \\ \frac{u_2}{\widetilde{\Sigma}_a^2}(\Sigma_a^2 - \widetilde{\Sigma}_a^2) \end{pmatrix} + \begin{pmatrix} \frac{\widetilde{\sigma}_b}{\widetilde{\Sigma}_a^1}(H_1 - \widetilde{H}_1) \\ \frac{\widetilde{\sigma}_b}{\widetilde{\Sigma}_a^2}(H_2 - \widetilde{H}_2) \end{pmatrix}.$$

This leads, using again the fact that the matrix P has a bounded inverse, to the bound

$$\left\| \begin{pmatrix} \sigma_a \\ \sigma_b \end{pmatrix} - \begin{pmatrix} \widetilde{\sigma}_a \\ \widetilde{\sigma}_b \end{pmatrix} \right\|_{L^2(\Omega)} \le \left\| \begin{pmatrix} \Sigma_a^1 \\ \Sigma_a^2 \end{pmatrix} - \begin{pmatrix} \widetilde{\Sigma}_a^1 \\ \widetilde{\Sigma}_a^2 \end{pmatrix} \right\|_{L^2(\Omega)} + \|\mathbf{H} - \widetilde{\mathbf{H}}\|_{L^2(\Omega)}. \tag{53}$$

The second bound of (48) then follows from (52) and (53). □

4.2 Reconstruction with fixed-point iteration

We now consider a fixed-point iteration algorithm for the reconstruction of the absorption coefficients. We again use the fact that if (σ_a, σ_b, u) solves the transport equation (1) to generate datum H, then we can replace the term $\sigma_a + \sigma_b\langle u \rangle$ in (1) with $H/\langle u \rangle$ to obtain a nonlinear transport equation for u:

$$\begin{aligned} \mathbf{v} \cdot \nabla u(\mathbf{x}, \mathbf{v}) + (\frac{H}{\langle u \rangle} + \sigma_s) u(\mathbf{x}, \mathbf{v}) &= \sigma_s(\mathbf{x}) K u(\mathbf{x}, \mathbf{v}), \quad \text{in } X \\ u(\mathbf{x}, \mathbf{v}) &= g(\mathbf{x}, \mathbf{v}), \qquad\qquad \text{on } \Gamma_- . \end{aligned} \tag{54}$$

For a given datum H, if we could solve this equation, we can reconstruct $\sigma_a + \sigma_b\langle u \rangle$. Note that we have some *a priori* bounds on $\langle u \rangle$ due to the *a priori* bounds we know on the coefficients. First, it is clear that the coefficient Σ_a to be reconstructed satisfies $\Sigma_a \geq \frac{H}{\overline{g}}$. Second, let us define

$$\eta(\mathbf{x}) := \frac{H(\mathbf{x})}{\overline{\sigma}_a + \overline{\sigma}_b \overline{g}}\,.$$

Then the transport solution u that generated this datum H satisfies: $\langle u \rangle \geq \eta$. Let $u_{\max}^H$ be the solution to the linear transport equation (5) with $\Sigma_a = \frac{H}{\overline{g}}$. We then conclude, before we perform any reconstruction, that the solution (54) that we are seeking has the property that

$$\eta \leq \langle u \rangle \leq \langle u_{\max}^H \rangle\,.$$

Starting with a given u_0, we define the following iteration for $k \geq 1$:

$$\begin{aligned} \mathbf{v}\cdot\nabla u_k(\mathbf{x},\mathbf{v}) + \Big(\frac{H}{\max(\langle u_{k-1}\rangle, \eta)} + \sigma_s\Big) u_k(\mathbf{x},\mathbf{v}) &= \sigma_s(\mathbf{x}) K u_k(\mathbf{x},\mathbf{v}), \quad \text{in } X \\ u_k(\mathbf{x},\mathbf{v}) &= g(\mathbf{x},\mathbf{v}), \quad \text{on } \Gamma_- \end{aligned} \tag{55}$$

where the function max is applied point-wise to its arguments.

Let $u_{\min}$ be the solutions to the linear transport equation (5) with absorption coefficient $\overline{\sigma}_a + \overline{\sigma}_b \overline{g}$. Here is an obvious observation on the iteration.

Lemma 4.4. *Let $\{u_k\}$ be a sequence generated by (55) from an initial point $u_0 \geq 0$. Then $u_{\min} \leq u_k \leq u_{\max}^H$, $\forall k \geq 1$.*

Proof. We first observe that this iteration will generate a sequence $\{u_k\}$ such that $0 \leq u_k \leq \overline{g}$, $\forall k \geq 1$. Therefore $\frac{H}{\overline{g}} \leq \frac{H}{\max(\langle u_{k-1}\rangle, \eta)} \leq \frac{H}{\eta} = \overline{\sigma}_a + \overline{\sigma}_b \overline{g}$. By monotonicity of the solution to the linear transport with respect to the absorption coefficient, we have $u_{\min} \leq u_k \leq u_{\max}^H$, $\forall k \geq 1$. □

We introduce the following space of functions with bounded angular average:

$$\mathcal{U} := \{u \in L^2(X) \mid \langle u_{\min} \rangle \leq \langle u \rangle \leq \langle u_{\max}^H \rangle \text{ a.e.}\} \tag{56}$$

This space is convex, bounded and closed under the L^2 topology. We make the following assumption:

$$(\mathcal{A}'') \qquad \eta \leq \langle u_{\min} \rangle.$$

We can then show the following result.

Corollary 4.5. *Let $\{\overline{u}_k\}$ and $\{\underline{u}_k\}$ be sequences generated from $\overline{u}_0 = u_{\max}^H$ and $\underline{u}_0 = u_{\min}$ respectively. Then $\overline{u}_k \to \overline{u}$ and $\underline{u}_k \to \underline{u}$ a.e. as $k \to \infty$ for some $\overline{u}, \underline{u} \in \mathcal{U}$.*

Proof. For any sequence $\{u_k\}$ generated by (55), let us define $\varphi_k := u_k - u_{k-1}$. Then φ_k solves

$$\begin{aligned} \mathbf{v}\cdot\nabla \varphi_k + \Big(\frac{H}{\max(\langle u_{k-1}\rangle, \eta)} + \sigma_s\Big)\varphi_k &= \sigma_s(\mathbf{x}) K \varphi_k + F, \quad \text{in } X \\ \varphi_k(\mathbf{x},\mathbf{v}) &= 0, \quad \text{on } \Gamma_- \end{aligned} \tag{57}$$

where

$$F := \frac{Hu_{k-1}}{\max(\langle u_{k-1}\rangle,\eta)\max(\langle u_{k-2}\rangle,\eta)}[\max(\langle u_{k-1}\rangle,\eta)-\max(\langle u_{k-2}\rangle,\eta)]\,.$$

When we start the iteration with $u_0 \in \mathcal{U}$, the iteration remains in $\mathcal{U}$. Therefore $\langle u_{k-1}\rangle$, $\langle u_{k-2}\rangle \geq u_{\min}$. With the assumption ($\mathcal{A}''$), we conclude that

$$\max(\langle u_{k-1}\rangle,\eta)-\max(\langle u_{k-2}\rangle,\eta) = \langle u_{k-1}\rangle - \langle u_{k-2}\rangle = \langle \varphi_{k-1}\rangle.$$

Therefore, in this case $F := \frac{Hu_{k-1}}{\max(\langle u_{k-1}\rangle,\eta)\max(\langle u_{k-2}\rangle,\eta)}\langle\varphi_{k-1}\rangle$. For the iteration (55) that starts with $u_{\max}$, we have that $\varphi_0 \leq 0$. Therefore $\{\varphi_k\}$ remains negative according to (57). This means that $\{\overline{u}_k\}$ is a decreasing sequence. The fact that it is also bounded from below by $u_{\min}$ then indicates that it converges to some $\overline{u} \in \mathcal{U}$. For the iteration (55) that starts with $u_{\min}$, we have that $\varphi_0 \geq 0$. Therefore $\{\varphi_k\}$ remains non-negative according to (57). This means that $\{\underline{u}_k\}$ is an increasing sequence. The fact that it is also bounded from above by $u_{\max}^H$ then indicates that it converges to some $\underline{u} \in \mathcal{U}$. □

We are ready to show that the iteration (55) converges to a unique fixed point in $\mathcal{U}$ that is the solution to the transport equation (54).

Theorem 4.6. *Assume that the solution to* (54) *is such that* $(\frac{H}{\langle u\rangle}, H, g) \in \Pi'_\alpha$ *for some* α. *Assume further that* $\Sigma_{a|\partial\Omega}$ *is known and* $\overline{\Sigma}_a\|\frac{gH}{\Sigma_{a|\partial\Omega}}\|_{L^\infty(\Gamma_-)} \leq \beta < 1$ *for some* β. *Then, under the assumption* ($\mathcal{A}$), *the iteration* (55) *converges to the unique solution of* (54) *in* $\mathcal{U}$.

Proof. Let $\{\underline{u}_k\}$ be the sequence generated from the starting point $\underline{u}_0 = u_{\min}$. By Corollary 4.5, $\underline{u}_k \to \underline{u}$. Moreover $\underline{u}$ solves

$$\begin{aligned} \mathbf{v}\cdot\nabla\underline{u}(\mathbf{x},\mathbf{v}) + (\frac{H}{\max(\underline{u},\eta)}+\sigma_s)\underline{u} &= \sigma_s(\mathbf{x})K\underline{u}(\mathbf{x},\mathbf{v}), && \text{in } X \\ \underline{u}(\mathbf{x},\mathbf{v}) &= g(\mathbf{x},\mathbf{v}), && \text{on } \Gamma_-\,. \end{aligned}$$

We then use the fact that $\langle\underline{u}\rangle \in \mathcal{U}$ and the assumption ($\mathcal{A}''$) to conclude that $\max(\underline{u},\eta) = \underline{u}$. Therefore, $\underline{u}$ is a solution to (54).

Let $\{u_k\}$ be a sequence generated from an arbitrary starting point in $\mathcal{U}$. Let $\phi_k := u_k - \underline{u}_k$. Then, using the same argument on F in Corollary 4.5, we check that ϕ_k solves

$$\begin{aligned} \mathbf{v}\cdot\nabla\phi_k + (\frac{H}{\max(\langle u_{k-1}\rangle,\eta)}+\sigma_s)\phi_k &= \sigma_s(\mathbf{x})K\phi_k + \widetilde{F}\langle\phi_{k-1}\rangle, && \text{in } X \\ \phi_k(\mathbf{x},\mathbf{v}) &= 0, && \text{on } \Gamma_- \end{aligned} \tag{58}$$

with $\widetilde{F} := \frac{H\,\underline{u}_k}{\max(\langle\underline{u}_{k-1}\rangle,\eta)\max(\langle u_{k-1}\rangle,\eta)}$. This gives the bound:

$$\begin{aligned} \|\phi_k\|_{L^2(X)} &\leq \|\widetilde{F}\langle\phi_{k-1}\rangle\|_{L^2(X)} \leq \|\widetilde{F}\|_{L^\infty(X)}\|\phi_{k-1}\|_{L^2(X)} \\ &\leq \|\frac{H}{\max(\langle u_{k-1}\rangle,\eta)}\|_{L^\infty(X)}\|\frac{\underline{u}_{k-1}}{\max(\langle\underline{u}_{k-1}\rangle,\eta)}\|_{L^\infty(X)}\|\phi_{k-1}\|_{L^2(X)}. \end{aligned} \tag{59}$$

We first observe that $\|\frac{H}{\max(\langle u_{k-1}\rangle,\eta)}\|_{L^\infty(X)} \le \overline{\Sigma}_a$. To bound the term $\|\frac{u_{k-1}}{\max(\langle \underline{u}_{k-1}\rangle,\eta)}\|_{L^\infty(X)} = \|\frac{u_{k-1}}{\langle \underline{u}_{k-1}\rangle}\|_{L^\infty(X)}$, we observe that $w = \frac{u_k}{\langle \underline{u}_k\rangle}$ solves the transport equation:

$$\begin{aligned} \mathbf{v}\cdot\nabla w + \Big(\frac{H}{\max(\langle \underline{u}_{k-1}\rangle,\eta)} + \mathbf{v}\cdot\nabla \ln\langle \underline{u}_k\rangle + \sigma_s\Big) w &= \sigma_s(\mathbf{x}) K w, \quad \text{in } X \\ w(\mathbf{x},\mathbf{v}) &= \frac{gH}{\Sigma_{a|\partial\Omega}}, \qquad \text{on } \Gamma_- \end{aligned}$$

where the boundary condition for w comes from the assumption that Σ_a (and therefore the density $\langle u_k\rangle_{|\partial\Omega} = \dfrac{H}{\Sigma_{a|\partial\Omega}}$) is known on the boundary of the domain. With the assumption that $(\frac{H}{\langle \underline{u}\rangle}, H, g) \in \Pi'_\alpha$, we have that $\frac{H}{\max(\langle \underline{u}_{k-1}\rangle,\eta)} + \mathbf{v}\cdot\nabla\ln\langle \underline{u}_k\rangle \ge \frac{\alpha}{2} > 0$ for sufficiently large k. Therefore, we conclude from the maximum principle that $\|w\|_{L^\infty(X)} \le \|\dfrac{gH}{\Sigma_{a|\partial\Omega}}\|_{L^\infty(\Gamma_-)}$. Therefore the bound in (59) can now be written as

$$\|\phi_k\|_{L^2(X)} \le \overline{\Sigma}_a \|\frac{gH}{\Sigma_{a|\partial\Omega}}\|_{L^\infty(\Gamma_-)} \|\phi_{k-1}\|_{L^2(X)}.$$

When $\overline{\Sigma}_a\|\dfrac{gH}{\Sigma_{a|\partial\Omega}}\|_{L^\infty(\Gamma_-)} \le \beta < 1$, this bound gives that $\|\phi_k\|_{L^2(X)} \to 0$. This means that u_k converges $\underline{u}$.

The above calculation shows that the iteration (55) sequence start with any initial point $u_0 \in \mathcal{U}$ converges to a solution of (54) whose average leaves in $\mathcal{U}$. This concludes the proof. □

The above result shows that in order to reconstruct the unknown absorption coefficients, we could use the fixed-point iteration (55) to find u. We then reconstruct the total absorption $\sigma_a + \sigma_b\langle u\rangle$ from $H/\langle u\rangle$. This procedure would allow us to reconstruct (σ_a, σ_b) from two different data sets H_1 and H_2. When we have a better *a priori* information on the coefficient to be reconstructed, we could modify η to further reduce the size of the space $\mathcal{U}$. This will in turn allow us to better reconstruct u.

When the media scatters isotropically, that is, when $\Theta(\mathbf{v},\mathbf{v}') \equiv 1$, we could make some of the assumptions we made in this section more explicit. The calculations are documented in the Appendix B.

5 Concluding remarks

In this work, we analyzed an inverse problem for a semilinear radiative transport equation, aiming at reconstructing two absorption coefficients of the transport equation from two internal data sets that are functionals of the transport solutions. We first established the well-posedness of the forward problem under small boundary sources. We then derived stability results on the inverse problem in the simplified settings where the scattering coefficient is known (either $\sigma_s \equiv 0$ or $\sigma_s > 0$). We also developed a reconstruction method based on a fixed-point iteration. Our results provide some mathematical understanding of quantitative photoacoustic imaging of two-photon absorption in the transport regime, complementing the results in [32] in the diffusive regime.

There are several interesting following up questions to the current work. For instance, it would be useful if we can remove some of the restrictive assumptions on the size of the gradient of the absorption coefficients to be reconstructed. Moreover, it would be of great interests to generalize the analysis we have to reconstruct simultaneously the absorption and the scattering coefficients triplet $(\sigma_a, \sigma_b, \sigma_s)$ from three sets of internal data. Note that in the case of linear transport equation, i.e. (1) without the semilinear term, the analysis in [6] shows that one can reconstruct (σ_a, σ_s) as well as partial information in the scattering phase function $\Theta(\mathbf{v}, \mathbf{v}')$ with data encoded in the full operator $\Lambda : g(\mathbf{x}, \mathbf{v}) \mapsto H(\mathbf{x})$. Whether or not one can reconstruct simultaneously σ_a and σ_s in the linear transport equation from a finite number of internal data is still a largely open question right now; see some progresses in [29, 20]. From application point of view, it is an interesting problem to see if one can reconstruct all the coefficients in the problem from the albedo data $\Lambda : u_{|\Gamma_-} \mapsto u_{\Gamma_+}$. This can probably be analyzed by combining the classical singular decomposition of Choulli and Stefanov [11, 12] with the linearization idea introduced by Isakov and collaborators [21, 22, 37].

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A Averaging lemma and Kellogg's theory

To improve the readability of the paper, we recall here two important results that we have used to prove the main results of the paper.

The first result is the averaging lemma in transport theory, developed in [19]. It characterizes the regularization effect of velocity averaging on the solution of transport equations. With the same notations as in the main text, the result can be stated as follows.

Theorem A.1 (Averaging Lemma)**.** *For $p \in (1, +\infty)$, let u be a function defined in X such that $u \in L^p(X)$, $\mathbf{v} \cdot \nabla u \in L^p(X)$, and $u|_{\Gamma_-} \in L^p(\Gamma_-)$. Then $\langle u \rangle$ belongs to the Sobolev space $W^{s,p}(\Omega)$ with $s = 1/2$ if $p = 2$ and $0 < s < \inf(p^{-1}, 1 - p^{-1})$ if $p \neq 2$. In addition, we have the inequality*

$$\|\langle u \rangle\|_{W^{s,p}(\Omega)} \leq \mathfrak{c} \left(\|u\|_{L^p(X)} + \|\mathbf{v} \cdot \nabla u\|_{L^p(X)} + \|u\|_{L^p(\Gamma_-)} \right) , \tag{60}$$

for some constant $\mathfrak{c} > 0$.

The second result we recall here is Kellogg's uniqueness theory for the Schauder Fixed-Point Theorem, developed in [23]. The theory provides a condition under which the Schauder fixed point is unique.

Theorem A.2 (Kellogg 1976 [23]). *Let $\mathcal{M}$ be a bounded convex open subset of a real Banach space, and $F : \overline{\mathcal{M}} \to \overline{\mathcal{M}}$ a compact continuous map which is continuously Fréchet differentiable on $\mathcal{M}$. If (i) for each $m \in \mathcal{M}$, 1 is not an eigenvalue of $F'(m)$, and (ii) for each $m \in \partial\mathcal{M}$, $m \neq F(m)$, then F has a unique fixed point in $\mathcal{M}$.*

B Inversion in isotropic media

We analyze here the inverse problem in Section 4 in the context of isotropic scattering. This is again done by analyzing a fixed-point iteration for solving (54). For simplicity, in the following we consider the boundary source $g(\mathbf{x}, \mathbf{v}) \equiv \overline{g}$ as a constant.

For any positive function $m(\mathbf{x}) > 0$, we define a map $\mathcal{C}$ through the relation:

$$\mathcal{C}(m) := \langle u \rangle,$$

where u solves the following linear transport equation with isotropic scattering:

$$\begin{aligned} \mathbf{v} \cdot \nabla u(\mathbf{x}, \mathbf{v}) + (H/m + \sigma_s) u(\mathbf{x}, \mathbf{v}) &= \sigma_s(\mathbf{x}) \int_{\mathbb{S}^{d-1}} u(\mathbf{x}, \mathbf{v}) d\mathbf{v}, \quad \text{in } X \\ u(\mathbf{x}, \mathbf{v}) &= \overline{g}, \quad \text{on } \Gamma_- . \end{aligned} \tag{61}$$

The map $\mathcal{C}$ is monotone increasing and is bounded from above by $\overline{g}$ in the space of positive functions, under the assumptions in $(\mathcal{A})$. We will show that $\mathcal{C}$ admits a unique fixed point in appropriate sense.

Let η be defined as in Section 4 and satisfy the assumption $(\mathcal{A}'')$, that is, $\mathcal{C}(\eta) \geq \eta$. We define the function space

$$\mathcal{M} = \{ m \in L^2(\Omega) \mid \eta \leq m \leq \overline{g} \quad a.e. \} . \tag{62}$$

Then $\mathcal{C}$ is monotone increasing and $\mathcal{C}(\mathcal{M}) \subset \mathcal{M}$. Moreover, $\mathcal{C}$ is compact and continuous on $\mathcal{M}$ in $L^2(\Omega)$ topology. The existence of solution on $\mathcal{C}$ then follows from the Schauder Fixed-Point Theorem.

To show the uniqueness of the fixed point of $\mathcal{C}$, we first observe that the equation (61) is equivalent to the following integral equation:

$$\langle u \rangle(\mathbf{x}) = \mathcal{J}_m \overline{g} + \mathcal{K}_m(\sigma_s \langle u \rangle), \tag{63}$$

where the integral operators $\mathcal{J}_m : L^p(\Gamma_-) \to L^p(\Omega)$ and $\mathcal{K}_m : L^p(\Omega) \to L^p(\Omega)$, $1 \leq p \leq \infty$, are defined as follows:

$$\begin{aligned} \mathcal{J}_m \overline{g} &= \overline{g} \int_{\mathbb{S}^{d-1}} E_m(\mathbf{x}, \tau_-(\mathbf{x}, \mathbf{v}), \mathbf{v}) d\mathbf{v} \, , \\ \mathcal{K}_m f &= \int_{\mathbb{S}^{d-1}} \int_0^{\tau_-(\mathbf{x},\mathbf{v})} E_m(\mathbf{x}, l, \mathbf{v}) f(\mathbf{x} - l\mathbf{v}) dl d\mathbf{v} \, , \end{aligned} \tag{64}$$

with the path integral operator E_m given as

$$E_m(\mathbf{x}, l, \mathbf{v}) = \exp\left[-\int_0^l (\frac{H}{m} + \sigma_s)(\mathbf{x} - s\mathbf{v})ds\right].$$

We first show the result on the fixed point starting from $\overline{g}$.

Lemma B.1. *Let $h := \lim_{n\to\infty} \mathcal{C}^n(\overline{g})$. For any $\psi \in L^\infty(\Omega)$ such that $\psi \geq 0$, the corresponding integral operators $\mathcal{J}_h$ and $\mathcal{K}_h$ satisfy*

$$J_h\overline{g} \geq \frac{h - \mu_h}{\overline{g} - \mu_h}\overline{g}, \quad \mathcal{K}_h\psi \leq \frac{\overline{g} - h}{\overline{g} - \mu_h} \sup\left[\frac{\psi}{\frac{H}{h} + \sigma_s}\right] \text{ with } \mu_h = \sup\left[\frac{\sigma_s h}{\frac{H}{h} + \sigma_s}\right]. \tag{65}$$

Proof. We first observe that:

$$\begin{aligned}\mathcal{K}_h\psi &= \int_{\mathbb{S}^{d-1}} \int_0^{\tau_-(\mathbf{x},\mathbf{v})} E_h(\mathbf{x}, l, \mathbf{v})\psi(\mathbf{x} - l\mathbf{v})dld\mathbf{v} \\ &\leq \int_{\mathbb{S}^{d-1}} (1 - E_h(\mathbf{x}, \tau_-(\mathbf{x}, \mathbf{v}), \mathbf{v})) \sup\left[\frac{\psi}{(\frac{H}{h} + \sigma_s)}\right] d\mathbf{v} \leq \left(1 - \frac{1}{\overline{g}}\mathcal{J}_h\overline{g}\right) \sup\left[\frac{\psi}{(\frac{H}{h} + \sigma_s)}\right].\end{aligned} \tag{66}$$

The function h solves the integral equation $h = \mathcal{J}_h\overline{g} + \mathcal{K}_h(\sigma_s h)$. Therefore,

$$h \leq \mathcal{J}_h\overline{g} + \left(1 - \frac{1}{\overline{g}}\mathcal{J}_h\overline{g}\right) \sup\left[\frac{\sigma_s h}{(\frac{H}{h} + \sigma_s)}\right], \tag{67}$$

which means

$$\left(1 - \frac{1}{\overline{g}} \sup\left[\frac{\sigma_s h}{(\frac{H}{h} + \sigma_s)}\right]\right) \mathcal{J}_h\overline{g} \geq h - \sup\left[\frac{\sigma_s h}{(\frac{H}{h} + \sigma_s)}\right]. \tag{68}$$

The proof is completed by bringing the above inequality into (66). □

Lemma B.2. *Assume that $\ell_\Omega = diam(\Omega) \leq 1$. Let $h := \lim_{n\to\infty} \mathcal{C}^n(\overline{g})$ and f be an arbitrary element of $\mathcal{M}$. We have that*

$$|\mathcal{C}(h) - \mathcal{C}(f)| \leq \gamma \left[\kappa\mu_f[1 - (1 - \ell_\Omega)\frac{h - \mu_h}{\overline{g} - \mu_h}] + \frac{\overline{g} - h}{\overline{g} - \mu_h}\overline{g}\right], \tag{69}$$

where $\gamma = \sup\left|\frac{h-f}{h\wedge f}\right|$, $\kappa = \sup \frac{\frac{H}{h\vee f}}{\frac{H}{h\vee f} + \sigma_s}$, $\mu_f = \sup \frac{\sigma_s f}{\frac{H}{h\vee f} + \sigma_s}$, and $\mu_h = \sup \frac{\sigma_s h}{\frac{H}{h} + \sigma_s}$ with the notations $h \wedge f := \min(h, f)$ and $h \vee f := \max(h, f)$.

Proof. We need to bound $|\mathcal{C}(h) - \mathcal{C}(f)|$. We first observe that

$$\begin{aligned}|\mathcal{C}(h) - \mathcal{C}(f)| &\leq |\mathcal{J}_h\overline{g} - \mathcal{J}_f\overline{g}| + |\mathcal{K}_h(\sigma_s(h - f))| + |(\mathcal{K}_h - \mathcal{K}_f)(\sigma_s f)| \\ &\equiv A_1 + A_2 + A_3\,.\end{aligned}$$

Using the fact that $|e^{-x}-e^{-y}|\le e^{-\min(x,y)}|x-y|$, $\forall x,y\in\mathbb{R}$, we have

$$|E_h(\mathbf{x},l,\mathbf{v})-E_f(\mathbf{x},l,\mathbf{v})|\le E_{h\vee f}(\mathbf{x},l,\mathbf{v})\left|\int_0^l \frac{H(h-f)}{hf}(\mathbf{x}-s\mathbf{v})ds\right|.$$

Note that $hf=(h\vee f)(h\wedge f)$, we obtain the estimates for A_1 and A_2:

$$\begin{aligned}
A_1 &= |\mathcal{J}_h\overline{g}-\mathcal{J}_f\overline{g}| \\
&\le \overline{g}\int_{\mathbb{S}^{d-1}} E_{h\vee f}(\mathbf{x},\tau_-(\mathbf{x},\mathbf{v}),\mathbf{v})\left|\int_0^{\tau_-(\mathbf{x},\mathbf{v})}\frac{H(h-f)}{hf}(\mathbf{x}-s\mathbf{v})ds\right|d\mathbf{v} \\
&\le \overline{g}\int_{\mathbb{S}^{d-1}}\left(\int_0^{\tau_-(\mathbf{x},\mathbf{v})} E_{h\vee f}(\mathbf{x},s,\mathbf{v})\left|\frac{H(h-f)}{hf}(\mathbf{x}-s\mathbf{v})\right|ds\right)d\mathbf{v} \\
&\le \overline{g}\sup\left|\frac{h-f}{h\wedge f}\right|\int_{\mathbb{S}^{d-1}}\left(\int_0^{\tau_-(\mathbf{x},\mathbf{v})} E_{h\vee f}(\mathbf{x},s,\mathbf{v})\left|\frac{H}{h\vee f}(\mathbf{x}-s\mathbf{v})\right|ds\right)d\mathbf{v}\,, \\
A_2 &= |\mathcal{K}_h(\sigma_s(h-f))| \\
&\le \mathcal{K}_h(\sigma_s(h\wedge f)\sup\left|\frac{h-f}{h\wedge f}\right|) \\
&\le \mathcal{K}_{h\vee f}(\sigma_s(h\wedge f)\sup\left|\frac{h-f}{h\wedge f}\right|) \\
&\le \sup\left|\frac{h-f}{h\wedge f}\right|\int_{\mathbb{S}^{d-1}}\left(\int_0^{\tau_-(\mathbf{x},\mathbf{v})} E_{h\vee f}(\mathbf{x},s,\mathbf{v})\,|\sigma_s(h\wedge f)(\mathbf{x}-s\mathbf{v})|\,ds\right)d\mathbf{v}\,,
\end{aligned}$$

The above estimates imply that

$$\begin{aligned}
A_1+A_2 &\le \sup\left|\frac{h-f}{h\wedge f}\right|\int_{\mathbb{S}^{d-1}}\left(\int_0^{\tau_-(\mathbf{x},\mathbf{v})} E_{h\vee f}(\mathbf{x},s,\mathbf{v})\left|\frac{\overline{g}H}{h\vee f}+\sigma_s(h\wedge f)\right|(\mathbf{x}-s\mathbf{v})ds\right)d\mathbf{v} \\
&\le \overline{g}\sup\left|\frac{h-f}{h\wedge f}\right|(1-E_{h\vee f}(\mathbf{x},\tau_-(\mathbf{x},\mathbf{v}),\mathbf{v}))\sup\frac{\frac{H}{h\vee f}+\sigma_s\frac{h\wedge f}{\overline{g}}}{\frac{H}{h\vee f}+\sigma_s} \\
&\le \overline{g}\sup\left|\frac{h-f}{h\wedge f}\right|(1-E_{h\vee f}(\mathbf{x},\tau_-(\mathbf{x},\mathbf{v}),\mathbf{v}))\ .
\end{aligned}$$

To estimate A_3, we observe that:

$$\begin{aligned}
A_3 &\le (\mathcal{K}_h - \mathcal{K}_f)(\sigma_s f) \\
&\le \int_{\mathbb{S}^{d-1}} \int_0^{\tau_-(\mathbf{x},\mathbf{v})} E_{h\vee f}(\mathbf{x}, l, \mathbf{v}) \left| \int_0^l \frac{H(h-f)}{hf}(\mathbf{x}-s\mathbf{v}) ds \right| \sigma_s f(\mathbf{x}-l\mathbf{v}) dl d\mathbf{v} \\
&\le \gamma \int_{\mathbb{S}^{d-1}} \int_0^{\tau_-(\mathbf{x},\mathbf{v})} E_{h\vee f}(\mathbf{x}, l, \mathbf{v}) \left(\int_0^l \frac{H}{h\vee f}(\mathbf{x}-s\mathbf{v}) ds \right) \sigma_s f(\mathbf{x}-l\mathbf{v}) dl d\mathbf{v} \\
&\le \gamma\kappa \int_{\mathbb{S}^{d-1}} \int_0^{\tau_-(\mathbf{x},\mathbf{v})} E_{h\vee f}(\mathbf{x}, l, \mathbf{v}) \left(\int_0^l (\frac{H}{h\vee f} + \sigma_s)(\mathbf{x}-s\mathbf{v}) ds \right) \sigma_s f(\mathbf{x}-l\mathbf{v}) dl d\mathbf{v} \\
&\le \gamma\kappa\mu_f \int_{\mathbb{S}^{d-1}} \int_0^{\tau_-(\mathbf{x},\mathbf{v})} E_{h\vee f}(\mathbf{x}, l, \mathbf{v}) \left(\int_0^l (\frac{H}{h\vee f} + \sigma_s)(\mathbf{x}-s\mathbf{v}) ds \right) (\frac{H}{h\vee f} + \sigma_s)(\mathbf{x}-l\mathbf{v}) dl d\mathbf{v} \\
&\le \gamma\kappa\mu_f \left[1 - (1-\ell_\Omega) \int_{\mathbb{S}^{d-1}} E_{h\vee f}(\mathbf{x}, \tau_-(\mathbf{x},\mathbf{v}), \mathbf{v}) d\mathbf{v} \right] .
\end{aligned}$$

We can then use the fact that $E_{h\vee f}(\mathbf{x}, l, \mathbf{v}) \ge E_h(\mathbf{x}, l, \mathbf{v})$ and Lemma B.1 to obtain that,

$$\begin{aligned}
&|\mathcal{C}(h) - \mathcal{C}(f)| \\
&\le \gamma \left[\overline{g} - \overline{g} \int_{\mathbb{S}^{d-1}} E_{h\vee f}(\mathbf{x}, \tau_-(\mathbf{x},\mathbf{v}), \mathbf{v}) d\mathbf{v} + \kappa\mu_f [1 - (1-\ell_\Omega) \int_{\mathbb{S}^{d-1}} E_{h\vee f}(\mathbf{x}, \tau_-(\mathbf{x},\mathbf{v}), \mathbf{v}) d\mathbf{v}] \right] \\
&\le \gamma \left[\overline{g} - \overline{g} \int_{\mathbb{S}^{d-1}} E_{h}(\mathbf{x}, \tau_-(\mathbf{x},\mathbf{v}), \mathbf{v}) d\mathbf{v} + \kappa\mu_f [1 - (1-\ell_\Omega) \int_{\mathbb{S}^{d-1}} E_{h}(\mathbf{x}, \tau_-(\mathbf{x},\mathbf{v}), \mathbf{v}) d\mathbf{v}] \right] \\
&\le \gamma \left[\kappa\mu_f [1 - (1-\ell_\Omega) \frac{h-\mu_h}{\overline{g}-\mu_h}] + \frac{\overline{g}-h}{\overline{g}-\mu_h} \overline{g} \right] .
\end{aligned}$$

This completes the proof. ∎

We are now ready to establish the uniqueness result. We define

$$\alpha := \sup \frac{H/\eta}{H/\eta + \sigma_s} \quad \text{and} \quad \beta := \sup \frac{\sigma_s}{H/\overline{g} + \sigma_s} .$$

Theorem B.3. *Assume that $\ell_\Omega := diam(\Omega) \le 1$. Let ψ be defined as*

$$\psi = \frac{1 + \alpha\beta - \alpha\ell_\Omega\beta^2}{2 - [1 - (1-\ell_\Omega)\alpha]\beta} \overline{g}. \tag{70}$$

When $\psi \le \eta$, the transport equation (54) admits a unique solution.

Proof. Let $h := \lim_{n\to\infty} \mathcal{C}^n(\overline{g})$ and $f := \lim_{n\to\infty} \mathcal{C}^n(\eta)$. It is clear that $\overline{g} \ge h \ge f \ge \eta$, $h \vee f = f$ and $h \wedge f = f$. Let γ, κ, μ_f and μ_h be given as in Lemma B.2. Then we have

$$\frac{h-f}{f} \le \gamma \left(\frac{\overline{g}}{f} \frac{\overline{g}-h}{\overline{g}-\mu_h} + \frac{1}{f} \kappa\mu_f [1 - (1-\ell_\Omega) \frac{h-\mu_h}{\overline{g}-\mu_h}] \right) . \tag{71}$$

To obtain the uniqueness, it is sufficient to have

$$\frac{\overline{g}}{f}\frac{\overline{g}-h}{\overline{g}-\mu_h}+\frac{1}{f}\kappa\mu_f[1-(1-\ell_\Omega)\frac{h-\mu_h}{\overline{g}-\mu_h}]\leq 1\,. \tag{72}$$

The equal sign case is safely included. This is because if the equal sign holds at some point $\mathbf{z}\in\Omega$, then we need, from the estimate in A_1, that

$$\int_0^{\tau_-(\mathbf{z},\mathbf{v})}\frac{H(h-f)}{hf}(\mathbf{z}-s\mathbf{v})ds=0\,, \tag{73}$$

for all $\mathbf{v}\in\mathbb{S}^{d-1}$. This means that $h-f\equiv 0$.

The inequality (72) is equivalent to:

$$t\kappa\ell_\Omega\mu_h+(1-t)\overline{g}\leq tf+(1-t)h\,, \tag{74}$$

with $t=\frac{(\overline{g}-\mu_h)}{2\overline{g}-\mu_h+\kappa\mu_f(1-\ell_\Omega)}\in(0,1)$. Using the definitions of the parameters, we have that $\mu_f\leq\mu_h\leq\beta\overline{g}$. This allows us to conclude that the left-hand-side of (74) is bounded by

$$t\kappa\ell_\Omega\mu_h+(1-t)\overline{g}\leq\frac{1+\kappa\beta-\kappa\ell_\Omega\beta^2}{2-[1-(1-\ell_\Omega)\kappa]\beta}\overline{g}\leq\psi\,, \tag{75}$$

since $\alpha=\sup\frac{H/\eta}{H/\eta+\sigma_s}\geq\kappa$. Now we can use the assumptions $\psi\leq\eta$ and the facts that $\eta\leq f\leq h$ to deduce (74). □

The next theorem gives the stability of the solution u with respect to changes in H.

Theorem B.4. *Let $\widetilde{u}$ and $\widehat{u}$ be the unique solutions to (54) with internal data $\widetilde{H}$ and $\widehat{H}$ respectively but the same source function $g\equiv\overline{g}$. Assume that $\ell_\Omega:=diam(\Omega)\leq 1$ and there exists a constant $0\leq r<1$, such that*

$$\psi:=\frac{1+\alpha\beta-\alpha\ell_\Omega\beta^2}{1+r(1-\beta)+(1-\ell_\Omega)\alpha\beta}\overline{g}\leq\langle u\rangle,\langle\hat{u}\rangle.$$

Then there exists a constant $c>0$ that

$$\|\langle\widetilde{u}\rangle-\langle\widehat{u}\rangle\|_{L^\infty(\Omega)}\leq c\|\widetilde{H}-\widehat{H}\|_{L^\infty(\Omega)}. \tag{76}$$

Proof. For a given function $m(\mathbf{x})>0$, let $\widetilde{w}$ and $\widehat{w}$ be respectively the solution to the transport equations

$$\begin{aligned}\mathbf{v}\cdot\nabla\widetilde{w}(\mathbf{x},\mathbf{v})+(\widetilde{H}/m+\sigma_s)\widetilde{w}(\mathbf{x},\mathbf{v}) &= \sigma_s(\mathbf{x})\int_{\mathbb{S}^{d-1}}\widetilde{w}(\mathbf{x},\mathbf{v})d\mathbf{v}, && \text{in } X\\ \widetilde{w}(\mathbf{x},\mathbf{v}) &= \overline{g} && \text{on } \Gamma_-\end{aligned} \tag{77}$$

and

$$\begin{aligned}\mathbf{v}\cdot\nabla\widehat{w}(\mathbf{x},\mathbf{v})+(\widehat{H}/m+\sigma_s)\widehat{w}(\mathbf{x},\mathbf{v}) &= \sigma_s(\mathbf{x})\int_{\mathbb{S}^{d-1}}\widehat{w}(\mathbf{x},\mathbf{v})d\mathbf{v}, && \text{in } X\\ \widehat{w}(\mathbf{x},\mathbf{v}) &= \overline{g} && \text{on } \Gamma_-\,.\end{aligned} \tag{78}$$

We define the maps $\widetilde{\mathcal{C}}$ and $\widehat{\mathcal{C}}$ via the relations:

$$\widetilde{\mathcal{C}}(m) := \langle \widetilde{w} \rangle \quad \text{and} \quad \widehat{\mathcal{C}}(m) := \langle \widehat{w} \rangle.$$

Then $\langle \widetilde{u} \rangle = \lim_{n\to\infty} \widetilde{\mathcal{C}}^n(\overline{g})$ and $\langle \widehat{u} \rangle = \lim_{n\to\infty} \widehat{\mathcal{C}}^n(\overline{g})$. For convenience, we denote $h := \langle \widetilde{u} \rangle$ and $f := \langle \widehat{u} \rangle$. It is then straightforward to check that

$$\left|\frac{h-f}{h\wedge f}\right| = \left|\frac{\widetilde{\mathcal{C}}(h)-\widehat{\mathcal{C}}(f)}{h\wedge f}\right| \le \left|\frac{\widetilde{\mathcal{C}}(h)-\widetilde{\mathcal{C}}(f)}{h\wedge f}\right| + \left|\frac{\widetilde{\mathcal{C}}(f)-\widehat{\mathcal{C}}(f)}{h\wedge f}\right|. \tag{79}$$

We first bound the first part on the right hand side. Following Lemma B.2, we define $\gamma = \sup\left|\frac{h-f}{h\wedge f}\right|$, $\kappa = \sup \frac{\widetilde{H}_j/(h\vee f)}{\widetilde{H}_j/(h\vee f)+\sigma_s}$, $\mu_h = \sup \frac{\sigma_s h}{\widetilde{H}_j/h+\sigma_s}$, and $\mu_f = \sup \frac{\sigma_s h}{\widetilde{H}_j/(h\vee f)+\sigma_s}$. Then $0 \le \mu_h, \mu_f \le \beta \overline{g}_j$ and $\alpha \ge \kappa$. We obtain

$$\begin{aligned}\left|\frac{\widetilde{\mathcal{C}}(h)-\widetilde{\mathcal{C}}(f)}{h\wedge f}\right| &\le \frac{\gamma}{h\wedge f}\left[\kappa\mu_f[1-(1-\ell_\Omega)\frac{h-\mu_h}{\overline{g}_j-\mu_h}] + \frac{\overline{g}_j-h}{\overline{g}_j-\mu_h}\overline{g}_j\right] \\ &\le \frac{\gamma}{h\wedge f}\left[\alpha\beta[\overline{g}_j-(1-\ell_\Omega)\frac{h-\beta\overline{g}_j}{(1-\beta)}] + \frac{\overline{g}_j-h}{1-\beta}\right].\end{aligned}$$

When $h, f \ge \psi$, it is easy to check that

$$\frac{1}{h\wedge f}\left[\alpha\beta[\overline{g}_j-(1-\ell_\Omega)\frac{h-\beta\overline{g}_j}{(1-\beta)}] + \frac{\overline{g}_j-h}{1-\beta}\right] \le r < 1.$$

Hence

$$\left|\frac{\widetilde{\mathcal{C}}(h)-\widetilde{\mathcal{C}}(f)}{h\wedge f}\right| \le r\sup\left|\frac{h-f}{h\wedge f}\right|. \tag{80}$$

To bound the second term on the right hand side of (79), we take $m = f$ in both (77) and (78) and take the difference $\phi = \widetilde{w} - \widehat{w}$. Simple algebra shows that ϕ satisfies the following equation:

$$\begin{aligned}\mathbf{v}\cdot\nabla\phi(\mathbf{x},\mathbf{v}) + (\widetilde{H}/f+\sigma_s)\phi(\mathbf{x},\mathbf{v}) &= \sigma_s(\mathbf{x})\int_{\mathbb{S}^{d-1}}\phi(\mathbf{x},\mathbf{v})d\mathbf{v} - \frac{\widetilde{H}-\widehat{H}}{f}\widehat{w}, && \text{in } X \\ \phi(\mathbf{x},\mathbf{v}) &= 0, && \text{on } \Gamma_-.\end{aligned}$$

By standard results from transport theory [15], there is a constant $c > 0$ that

$$\|\phi\|_{L^\infty(\Omega)} \le c\|\widetilde{H}-\widehat{H}\|_{L^\infty(\Omega)}.$$

Therefore, we have

$$\left|\frac{\widetilde{\mathcal{C}}(f)-\widehat{\mathcal{C}}(f)}{h\wedge f}\right| \le \frac{c}{\psi}\|\widetilde{H}-\widehat{H}\|_{L^\infty(\Omega)}.$$

Combining this with (80), we have the following stability bound:

$$\|h-f\|_{L^\infty(\Omega)} \le \frac{c\overline{g}}{(1-r)\psi}\|\widetilde{H}_j-\widehat{H}_j\|_{L^\infty(\Omega)} \le \frac{2c}{1-r}\|\widetilde{H}_j-\widehat{H}_j\|_{L^\infty(\Omega)}.$$

The proof is complete. □

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