IEOR 3600: Solutions to HMWK 1

1. (a) $\int_0^x f(u)du = 1 - e^{-\lambda x}$ and
   $\int_0^\infty f(x)dx = \lim_{x\to\infty} 1 - e^{-\lambda x} = 1$.
   (b) $\int_0^\infty xf(x)dx$. Integration by parts ($U = x$,
   $dV = f(x)dx$) easily yields the answer as $1/\lambda$.
   (c) $\int_0^\infty x^2 f(x)dx$. Integration by parts twice yields
   the answer as $2/\lambda^2$.
   (d) $\int_0^\infty \overline{F}(x)dx = \int_0^\infty e^{-\lambda x} = 1/\lambda$, same answer as (b).

2. (a) $\int_0^3 f(x)dx = 1/2$ and $\int_0^\infty f(x)dx = 1$.
   (b) $\int_0^\infty xf(x)dx = 3$.
   (c) $\int_0^\infty x^2 f(x)dx = 28/3$.
   (d) Straightforward: Since $f(x) = 0$ when $x \le 2$, we see that $F(x) = 0$ for $x \le 2$. Similarly, $f(x) = 0$ for $x \ge 4$ and since $\int_0^\infty f(x)dx = \int_0^4 f(x)dx = 1$, we have $F(x) = (1/2)\int_2^x du = (x-2)/2$ if $2 < x < 4$, and $F(x) = \int_0^4 f(x)dx = 1$, if $x \ge 4$. Computing $1 - F(x)$ for each of the three cases yields the $1 - F(x)$ formula.
   (e) Using the derived formula for $\overline{F}(x) = 1 - F(x)$, it is easily verified that $\int_0^\infty \overline{F}(x)dx = 3$, same as (b).

3. Let

$$f(x,y) = \begin{cases} 2e^{-x}e^{-2y}, & \text{if } x \ge 0,\ y \ge 0; \\ 0, & \text{otherwise.} \end{cases}$$

   Compute
   $\int_0^1 \int_0^4 f(x,y)dxdy = (1 - e^{-4})(1 - e^{-2})$ and $\int_0^\infty \int_0^\infty f(x,y)dxdy = 1$.

4. (a) Immediate since $(1 + x + x^2 + \cdots + x^n)\cdot(1 - x) = 1 - x^{n+1}$.
   (b) Follows since $\lim_{n\to\infty} x^{n+1} = 0$ when $|x| < 1$ and

$$\sum_{k=0}^{\infty} x^k = \lim_{n\to\infty} \sum_{k=0}^{n} x^k = \lim_{n\to\infty} \frac{1 - x^{n+1}}{1 - x}.$$

   (c) Differentiating $g(x) = \sum_{k=0}^{\infty} x^k$ term by term within (this is allowed here) to get $g'(x)$ yields the equality

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}.$$

   Multiplying both sides by x (and noting that $kx^k = 0$ for $k = 0$) yields the result.

5. Note that

$$P(\mathcal{S}) = \sum_{k=0}^{\infty} P(T^k H) = \sum_{k=1}^{\infty} (\frac{1}{2})^k.$$

Use (4b) above to conclude that $P(\mathcal{S}) = 1$, and (4c) to conclude that

$$\sum_{k=1}^{\infty} kP(T^{k-1}H) = 2.$$

This last answer "2" is the average number of times the coin needs to be flipped until landing H for the first time.

$$\sum_{k=1}^{\infty} (\frac{1}{2})^k = (1/2) \sum_{k=0}^{\infty} (\frac{1}{2})^k = (1/2)(1/(1-(1/2)) = 1.$$

$$\sum_{k=1}^{\infty} kP(T^{k-1}H) = \sum_{k=1}^{\infty} k(\frac{1}{2})^k = (1/2)/(1/2)^2 = 2.$$

Suppose that the coin, instead of being fair, lands H with "success" probability $p = 1/4$, and T with probability $q = 3/4$. Noting that then $P(T^k H) = q^k p$, re-do the above computation to obtain the average number of times the coin needs to be flipped until landing H for the first time. Finally, re-do for a general success probability $0 < p < 1$.

Noting that $1 - q = p$:

$$P(\mathcal{S}) = \sum_{k=0}^{\infty} P(T^k H) = p \sum_{k=0}^{\infty} q^k = p/p = 1.$$

$$\sum_{k=1}^{\infty} kP(T^{k-1}H) = p \sum_{k=1}^{\infty} kq^{k-1} = (p/q) \sum_{k=1}^{\infty} kq^k = 1/p$$

.