IEOR 3600: HMWK 1

1 Review: Not to be turned in for credit

Start Reading Chapter 2 of the textbook.

1. Given a constant $\lambda > 0$, let

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

   Compute the following:

   (a) $\int_0^x f(u)du$ and $\int_0^\infty f(x)dx$

   (b) $\int_0^\infty x f(x)dx$

   (c) $\int_0^\infty x^2 f(x)dx$

   (d) Let $F(x) = \int_0^x f(u)du$. Graph the function $\overline{F}(x) \stackrel{\text{def}}{=} 1 - F(x)$. Show that $\int_0^\infty x f(x)dx = \int_0^\infty \overline{F}(x)dx$.

2. Let

$$f(x) = \begin{cases} \frac{1}{2}, & \text{if } 2 < x < 4; \\ 0, & \text{otherwise.} \end{cases}$$

   Graph $f(x)$ and then compute the following:

   (a) $\int_0^3 f(x)dx$ and $\int_0^\infty f(x)dx$

   (b) $\int_0^\infty x f(x)dx$

   (c) $\int_0^\infty x^2 f(x)dx$

   (d) Let $F(x) = \int_0^x f(u)du$. Show that

$$F(x) = \begin{cases} 0, & \text{if } x \leq 2; \\ (x-2)/2, & \text{if } 2 < x < 4; \\ 1, & \text{if } x \geq 4, \end{cases}$$

   and that

$$1 - F(x) = \begin{cases} 1, & \text{if } x \leq 2; \\ (4-x)/2, & \text{if } 2 < x < 4; \\ 0, & \text{if } x \geq 4, \end{cases}$$

   Graph both $F(x)$ and $1 - F(x)$.

   (e) Show that $\int_0^\infty x f(x)dx = \int_0^\infty \overline{F}(x)dx$.

3. Let

$$f(x, y) = \begin{cases} 2e^{-x}e^{-2y}, & \text{if } x \geq 0,\ y \geq 0; \\ 0, & \text{otherwise.} \end{cases}$$

   Compute

   $\int_0^1 \int_0^4 f(x, y)dxdy$ and $\int_0^\infty \int_0^\infty f(x, y)dxdy$.

4. (a) Show that for any x,

$$(1+x+x^2+\cdots+x^n) = \frac{1-x^{n+1}}{1-x}.$$

(b) Use (a) to show that if $|x|<1$, then

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}. \tag{1}$$

(c) Let $g(x) = \frac{1}{1-x}$, $0<x<1$. Noting that $g'(x) = \frac{1}{(1-x)^2}$, argue from (b) that if $0<x<1$, then

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}.$$

5. Consider tossing a fair coin until it first lands Heads (H). The sample space (set of all possible outcomes of this experiment) is given by the infinite set

$$\mathcal{S} = \{H, TH, TTH, TTTH, \ldots\}.$$

For example, $TTTTTH$ is the outcome "the coin landed tails the first 5 flips then landed heads on the 6th flip", and

$$P(TTTTTH) = \frac{1}{2}\times\frac{1}{2}\cdots\times\frac{1}{2} = (\frac{1}{2})^6.$$

For simplicity of notation, let $T^kH =$ the outcome "the coin landed tails the first k flips then landed heads on the (k+1)th flip", $k\geq 1$, and $T^0H = H$. Then $P(T^kH) = (1/2)^{k+1}$. Note that

$$P(\mathcal{S}) = \sum_{k=0}^{\infty} P(T^kH) = \sum_{k=1}^{\infty} (\frac{1}{2})^k.$$

Use (4b) above to conclude that $P(\mathcal{S}) = 1$, and (4c) to conclude that

$$\sum_{k=1}^{\infty} kP(T^{k-1}H) = 2.$$

This last answer "2" is the average number of times the coin needs to be flipped until landing H for the first time.

Suppose that the coin, instead of being fair, lands H with "success" probability $p = 1/4$, and T with probability $q = 3/4$. Noting that then $P(T^kH) = q^kp$, re-do the above computation to obtain the average number of times the coin needs to be flipped until landing H for the first time. Finally, re-do for a general success probability $0<p<1$.