IEOR 3600: HMWK 3 Solutions

1. *Polling:* Suppose during a Presidential election (only two candidates for simplicity), it has been determined (by extensive sampling) that $p = 55\%$ of all voters will vote for candidate A and $q = 1 - p = 45\%$ for candidate B. If you randomly select 5 voters, what is the probability that exactly 3 will vote for candidate A?

   SOLUTION:

   $\binom{5}{3}(.55)^3(.45)^2 = 10(.55)^3(.45)^2 = 0.34.$

2. (*Continuation:*) Suppose that apriori we do not know the proportion p of voters who will vote for candidate A, but that when we randomly selected five voters, exactly 3 said they would vote for A. Find the value of p that maximizes the probability of this event (e.g., of getting exactly 3 out of the 5 saying they would vote for A.)

   SOLUTION:

   For the function
   $$f(p) = 10p^3(1-p)^2,$$
   we need to solve $f'(p) = 0$. To this end: solve $3p2(1-p)^2 - 2p^3(1-p) = 0$ or $3(1-p) - 2p = 0$;
   $p = 3/5$. (The other 2 solutions, $p = 0$, $p = 1$, are minimums.)

3. You come down with a headache. Your physician hands you two identical looking pills, telling you to take one when you get home. Unbeknownst to you: Pill 1 is a real drug and is known to stop headaches 75% of cases, whereas pill 2 is merely a placebo and is known to stop headaches in only 10% of cases. You took one and your headache did not go away. What is the probability that you took the placebo?

   SOLUTION:

   Bayes' Rule: $A_1 = \{\text{pill1 is chosen}\}$, $A_2 = A_1' = \{\text{ pill2 is chosen}\}$, $B = \{\text{ headache does not go away}\}$.

   $$\begin{aligned} P(A_2 \mid B) &= \frac{P(B|A_2)P(A_2)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2)} \\ &= \frac{(0.90)(0.50)}{(0.25)(0.50) + (0.90)(0.50)} \\ &= 0.7826 \end{aligned}$$

4. Three coins are in a box. Two of them are fair, and one of them has $p = 3/4$ of landing H. A coin is randomly selected from the box, to be flipped.

   (a) The coin is flipped two times, both times landing H. What is the probability that the coin was fair?

   SOLUTION:

   Bayes' Rule: $A_1 = \{\text{coin is fair}\}$, $A_2 = A_1' = \{\text{ coin is a } p = 3/4 \text{ coin}\}$, $B = \{\text{ coin lands H twice}\}$.

   $$\begin{aligned} P(A_1 \mid B) &= \frac{P(B|A_1)P(A_1)}{P(B|A_1)P(A_1) + P(B|A_2)P(A_2)} \\ &= \frac{(1/2)^2(2/3)}{(1/2)^2(2/3) + (3/4)^2(1/3)} \\ &= 0.8421 \end{aligned}$$

(b) Suppose the coin is flipped n times, and all n times landed H. Prove that the probability that the coin is fair converges to 0 as $n \to \infty$.
SOLUTION:
Let $B_n = \{$ coin lands H n times$\}$.

$$\begin{aligned}
P(A_2 \mid B_n) &= \frac{P(B_n|A_2)P(A_2)}{P(B_n|A_1)P(A_1) + P(B_n|A_2)P(A_2)} \\
&= \frac{(1/2)^n(2/3)}{(1/2)^n(2/3) + (3/4)^n(1/3)} \\
&= \frac{2}{2 + (3/2)^n} \to 0
\end{aligned}$$

5. A company makes boxes of various sizes as follows: The length (L) is a rv with probability mass function $P(L = 2) = 0.3,\ P(L = 4) = 0.7$. The width (W) is a rv with probability mass function $P(W = 3) = 0.3,\ P(W = 4) = 0.4,\ P(W = 5) = 0.3$. The height (H) is a rv with probability mass function $P(H = 2) = 1/3,\ P(H = 3) = 1/3,\ P(H = 4) = 1/3$. Assume that these random variables are independent.

   (a) Find $E(W)$ and $Var(W)$.
   SOLUTION: $E(W) = 4,\ Var(W) = E(W^2) - E^2(W) = 16.6 - 16 = 0.6$ (For example, $E(W^2) = 3^2(0.3) + 4^2(0.4) + 5^2(0.3) = 16.6$)

   (b) What is the expected value of the volume of a box? What is the variance of the volume of the box?
   SOLUTION: $V = LWH$ is the volume. By independence,
   $E(V) = E(LWH) = E(L)E(W)E(H) = 40.8$, and $E(V^2) = E(L^2W^2H^2) = E(L^2)E(W^2)E(H^2) = 1976.064$,
   $Var(V) = E(V^2) - [E(V)]^2 = 311.424$.

6. Suppose that N is a rv with a geometric distribution; $P(N = k) = (1-p)^{k-1}p,\ k \geq 1$, where $0 < p < 1$ is the probability of success. Show that $P(N > n+k \mid N > n) = P(N > k)$ for any $n, k \geq 1$. Thus given $N > n$, the remainder $N - n$ itself has the same geometric (p) distribution, independent of the value of n. This is known as the discrete *memoryless property* of the geometric distribution. For example, if N denotes the lifetime (in days) of a light bulb, then this property would imply that: Given that the bulb has already lived beyond 365 days, the probability that it will live at least another 365 days is the same as for a new bulb (here we are using $k = n = 365$). (Clearly a typical bulb would not have such a lifetime distribution, why?.)

   SOLUTION: Note that $P(N > k) = (1-p)^k,\ k \geq 0$ since it is the probability of at least k failures before the first success; e.g., "the first k coin tosses landed H".
   $P(N > n+k \mid N > n) = P(N > n+k\ ,\ N > n)/P(N > n) =$
   $P(N > n+k)/P(N > n)$
   $= (1-p)^{n+k}/(1-p)^n = (1-p)^k = P(N > k)$.