# IEOR 3600: HMWK 4 Solutions

1. (See text solutions)

2. A district $D$ within a state has a population of size $n = 1000$. Suppose that each voter will, independently, vote for candidate $A$ (versus $B$) with probability $p = 0.51$. Use the normal approximation to the binomial distribution to compute the probability that $A$ wins the popular vote of the district (e.g., receives more than 500 votes).

   Repeat for the case when $n = 10,000$.

   **SOLUTION:** $X$ is binomial $(n, p)$ with $\mu = np$, $\sigma = \sqrt{np(1-p)}$, where $p = 0.51$. $P(X \geq k) \approx P(Z \geq z)$, where

   $$z = \frac{k - 0.5 - \mu}{\sigma}.$$

   When $n = 1000$ we want to compute $P(X \geq 501)$, when $n = 10,000$ we want to compute $P(X \geq 5001)$.

   For $n = 1000$, $\mu = 510$, $\sigma = \sqrt{1000(0.51)(0.49)} = 15.81$ yielding

   $$z = \frac{500.5 - 510}{15.81} = -0.60.$$

   $P(Z \geq -0.60) = P(Z \leq 0.60) = \Phi(0.60) = 0.755$. (We are using symmetry about 0: $P(Z \geq -x) = P(Z \leq x)$.)

   For $n = 10000$, $\mu = 5100$, $\sigma = \sqrt{10000(0.51)(0.49)} = 50.0$ yielding

   $$z = \frac{5000.5 - 5100}{50} = -2.0.$$

   $P(Z \geq -2) = P(Z \leq 2) = \Phi(2) = 0.982$.

   Thus we see that as the population increases, the probability that $A$ wins increases. This intuitively makes sense because of the strong law of large numbers which asserts that as $n \to \infty$, the proportion of voters voting for $A$, $X/n$, converges exactly to .51, meaning that $X \approx n(0.51)$ as $n$ tends to $\infty$, or $P($ $X$ is "close" to $n(0.51)) \approx 1$ as $n$ gets large. Since $n(0.51)$ is much larger than $(n/2) + 1$ when $n$ is large, the result follows since $P(X \geq (n/2) + 1) \geq P($ $X$ is "close" to $n(0.51))$.

3. Suppose that $U$ has a uniform distribution over the continuous interval $(0, 1)$. For $x \in (0, 0.5)$ compute the conditional cumulative distribution function (cdf)
   $G(x) = P(U \leq x \mid U \leq 0.5)$. Show that $G(x)$ is the cdf of the uniform distribution over the smaller interval $(0, 0.5)$. Thus: Conditional on the event $U \leq 0.5$, the distribution of $U$ is uniform $(0, 0.5)$.

   **SOLUTION:** $G(x) = P(U \leq x, U \leq 0.5)/P(U \leq 0.50) = P(U \leq x)/P(U \leq 0.50) = x/(0.50)$, precisely the cdf of the $unif(0, 0.50)$ distribution.

   More generally prove that: If $X$ is uniform $(a, b)$, and $a < c < d < b$, then conditional on the event $X \in (c, d)$, the distribution of $X$ is uniform $(c, d)$: Compute, for $x \in (c, d)$, the conditional cumulative distribution function (cdf)
   $G(x) = P(X \leq x \mid X \in (c, d))$ and show that it is the cdf of the uniform distribution over the smaller interval $(c, d)$.

   **SOLUTION:** Similar to the above $Unif(0, 1)$ example, we have
   $G(x) = P(X \leq x)/P(X \in (c, d)) = [(x-c)/(b-a)]/[(d-c)/(b-a)] = (x-c)/(d-c)$, $x \in (c, d)$ and this is precisely the cdf of the $unif(c, d)$ distribution.

4. Suppose that $X$ has an exponential distribution at rate $\lambda = 2$; the probability density function is $f(x) = 2e^{-2x},\ x > 0$ and the cdf is $F(X \leq x) = 1 - e^{-2x},\ x > 0$. Find the cdf and density of the rv $Y = X^2$. Compute $E(Y)$ and $Var(Y)$ (Hint: $E(Y^2) = E(X^4)$.)

   **SOLUTION:**

   CDF: $G(x) = P(Y \leq x) = P(X^2 \leq x) = P(X \leq \sqrt{x}) = 1 - e^{-2\sqrt{x}},\ x > 0$. Density: $g(x) = G'(x) = \frac{e^{-2\sqrt{x}}}{\sqrt{x}},\ x > 0$.

   (Integration by parts yields:)
   $E(Y) = E(X^2) = \int_0^\infty x^2 \lambda e^{-\lambda x} dx = 2/\lambda^2 = 1/2$, since here $\lambda = 2$.
   $E(Y^2) = E(X^4) = \int_0^\infty x^4 \lambda e^{-\lambda x} dx = (4!)/\lambda^4 = 3/2$.
   $Var(Y) = E(Y^2) - E^2(Y) = 3/2 - (1/2)^2 = 5/4$.