IEOR 3600: HMWK 5 Solutions

1. A stock is now $S_0 = \$100$ per share. Its price at any fixed time t (years) in the future is a (lognormal) random variable distributed as

$$S(t) = S_0 e^{\sigma\sqrt{t}Z+\mu t},$$

where $Z \sim N(0,1)$ is a standard unit normal, and $\mu = 0.25$ and $\sigma = 0.15$.

(And recall that the expected value of the price at time t is given by $E(S(t)) = S_0 e^{rt}$, where $r = \mu + (\sigma^2/2)$.)

   (a) Compute $E(S(1))$, $E(S(2))$, and $E(S(3))$.
   $E(S(t)) = S_0 e^{rt}$, where $r = \mu + (\sigma^2/2) = 0.26125$.

$$\begin{aligned} E(S(1)) &= 100e^{0.26125} \\ &= 100(1.2986) \\ &= 129.86. \\ E(S(2)) &= 100(1.2986)^2 \\ &= 100(1.686) \\ &= 168.6 \\ E(S(3)) &= 100(1.2986)^3 \\ &= 100(2.190) \\ &= 219.0 \end{aligned}$$

   (b) Find the probability that the stock's value at time $t = 1$ year from now will be at least 50% higher; that is, $P(S(1) \geq (1.5)S_0)$.
   Taking logs converts the problem to computing $P(Z \geq z)$, where $z = (\ln(1.5) - \mu)/\sigma = 1.036$. $P(Z \geq z) = P(Z \leq -z)$ by symmetry about 0; thus we want $\Phi(-1.036) = 0.15$. (Interpolating from the Normal Table on Page 712 of Text).

   (c) Find the probability that the stock's value at time $t = 2$ is at least double what it was at time $t = 0$, that is, $P(S(2) \geq 2S_0)$.
   Taking logs converts the problem to computing $P(Z \geq z)$, where $z = (\ln(2) - 2\mu)/(\sqrt{2}\sigma) = 0.91$. $P(Z \geq z) = P(Z \leq -z)$ by symmetry about 0; thus we want $\Phi(-0.91) = 0.18$.

   (d) Find the value of t so that (for the first time) $P(S(t) \geq 2S_0) \geq 0.75$, the probability of doubling is at least 75%. For this value (denote it by $t*$), compute $E(S(t*))$.
   Taking logs converts the problem to that of finding the value of $t > 0$ such that

$$P(Z \geq \frac{\ln(2) - \mu t}{\sqrt{t}\sigma}) = 0.75.$$

   From the Normal Table, we observe that $P(Z \geq -0.675) = 0.75$; thus we need to find the value of $t > 0$ such that

$$\frac{\ln(2) - \mu t}{\sqrt{t}\sigma} = -0.675.$$

   This reduces to solving the following equation (in t): $\mu t - \sigma(0.675)\sqrt{t} - \ln(2) = 0$, which (via setting $x = \sqrt{t}$) reduces to the quadratic $\mu x^2 - \sigma(0.675)x - \ln(2) = 0$. This becomes $0.25x^2 - 0.101x - 0.693 = 0$, with positive root $x = 1.87$, and hence solution $t^* = (1.87)^2 = 3.51$. $E(S(t*)) = 100e^{0.26125(3.51)} = 250.2$.

2. From the text: Page 163, Ch.5: 1,2,3,8.