IEOR 3600: HMWK 6 Solutions

1. Three players are playing a card game, denoted by P1,P2 and P3. Each time they play, they have probabilities of winning, independent of the past, of 1/3, 1/2, 1/6 respectively. If they play 10 times, then what is the probability that P1 wins 2 times, P2 wins 6 times, and P3 wins 2 times?

   SOLUTION: Multinomial distribution:

$$P(X_1 = 2, X_2 = 6, X_3 = 2) = \frac{10!}{2!6!2!}(1/3)^2(1/2)^6(1/6)^2 = .$$

2. X and Y have a joint continuous probability density function given by

$$f(x,y) = \begin{cases} 10xy^2, & 0 < x < y < 1; \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Confirm that f really is a probability density, that is, confirm that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1.$$

   SOLUTION:

$$\int_0^1\int_0^y 10xy^2dxdy = \int_0^1 5y^4dy = 1$$

   (b) Let $R = \{(x,y) : x \geq 0.5,\ y \geq 0.5\}$. Compute $P((X,Y) \in R) = P(X > 0.5,\ Y > 0.5) = \int\int_R f(x,y)dxdy$.
   SOLUTION: The joint density is non-zero only on the subset of R given by $\{(x,y) : 0.5 < x < y < 1\}$, and so

$$\int\int_R f(x,y)dxdy = \int_{0.5}^1\int_{0.5}^y 10xy^2dxdy =$$

   (c) Find the marginal density functions $f_X(x)$ and $f_Y(y)$ of X and Y. (Confirm that they really are probability density functions, e.g., that they integrate to 1.)
   SOLUTION: Both densities are on the interval $(0,1)$ (0 outside of that interval):

$$f_X(x) = \int_x^1 10xy^2dy =,\ x \in (0,1)$$

$$f_Y(y) = \int_0^y 10xy^2dx =,\ y \in (0,1)$$

   (d) Compute $E(X),\ E(Y),\ E(X^2),\ E(Y^2)$ and then give $Var(X),\ Var(Y)$.

$$\begin{aligned} E(X) &= \int_0^1 xf_X(x)dx = \\ E(Y) &= \int_0^1 yf_Y(y)dy = \\ E(X^2) &= \int_0^1 x^2f_X(x)dx = \\ E(Y^2) &= \int_0^1 y^2f_Y(y)dy = \\ Var(X) &= E(X^2) - E^2(X) = \\ Var(Y) &= E(Y^2) - E^2(Y) = . \end{aligned}$$

(e) Compute the covariance $\sigma_{X,Y} = Cov(X,Y) = E(XY) - E(X)E(Y)$, and the correlation coefficient

$$\rho = \frac{\sigma_{X,Y}}{\sigma_X \sigma_Y}.$$

SOLUTION: $\sigma_X = \sqrt{Var(X)},\ \sigma_Y = \sqrt{Var(Y)}$ are now known as are $E(X), E(Y)$ so we need only compute

$$E(XY) = \int_0^1 \int_0^y (xy) 10xy^2 dxdy = .$$

(f) Find the conditional probability density of Y given that $X = 0.25$, $f_{Y|0.25}(y)$. Use it to compute $P(Y > 0.5 \mid X = 0.25)$ and $E(Y \mid X = 0.25)$, and $Var(Y \mid X = 0.25)$.
SOLUTION: This is defined for $y \in (0.25, 1)$ as $f_{Y|0.25}(y) = f(.25, y)/f_X(.25)$, where we have $f_X(.25)$ from (c). Then

$$\begin{aligned}
P(Y > 0.5 \mid X = 0.25) &= \int_{0.5}^1 f_{Y|0.25}(y)dy = \\
E(Y \mid X = 0.25) &= \int_{0.25}^1 y f_{Y|0.25}(y)dy = \\
E(Y^2 \mid X = 0.25) &= \int_{0.25}^1 y^2 f_{Y|0.25}(y)dy = \\
E(Y \mid X = 0.25) &= E(Y^2 \mid X = 0.25) - E^2(Y \mid X = 0.25) = .
\end{aligned}$$

3. Suppose X and Y have joint density

$$f(x,y) = \begin{cases} x(1+3y^2)/4, & 0 < x < 2,\ 0 < y < 1; \\ 0, & \text{otherwise.} \end{cases}$$

Show that X and Y are independent (e.g., that $f(x,y) = f_X(x) f_Y(y),\ 0 < x < 2,\ 0 < y < 1$.)
SOLUTION: For $x \in (0,2)$,

$$f_X(x) = \int_0^1 f(x,y)dy = \frac{x}{2}.$$

For $y \in (0,1)$,

$$f_Y(y) = \int_0^2 f(x,y)dy = \frac{1+3y^2}{2}.$$

Indeed $f(x,y) = f_X(x) f_Y(y),\ 0 < x < 2,\ 0 < y < 1$.

4. Suppose that X and Y are such that conditional on $Y = y$, the distribution of X is exponential with rate y, that is, it has conditional density $f_{X|y}(x) = ye^{-yx},\ x > 0$. Further assume that Y has a continuous uniform distribution over the interval $(1,3)$. Find the joint density function $f(x,y)$.

SOLUTION: Using the general formula, $f_{X|y}(x) = f(x,y)/f_Y(y)$, we obtain $f(x,y) = f_{X|y}(x) f_Y(y) = ye^{-yx} f_Y(y)$. We are told that Y has a continuous uniform distribution over the interval $(1,3)$, thus $f_Y(y) = 0.5,\ y \in (1,3)$, ($= 0$ if $y \notin (1,3)$. Thus

$$f(x,y) = \begin{cases} 0.5ye^{-yx}, & x > 0,\ 1 < y < 3; \\ 0, & \text{otherwise.} \end{cases}$$

5. Given a rv X with density function $f_X(x)$and a rv Y given by the transformation $Y = h(X)$ for a smooth invertable function $h(x)$ (with inverse function $h^{-1}(y)$), here we utilize the general formula for deriving the density of Y:

$$f_Y(y) = f_X(h^{-1}(y))|J(y)|,$$

where $J(y) = (h^{-1})'(y)$, the derivative of the inverse function (called the *Jacobian*).

(a) Suppose that X has an exponential distribution, with density $f(x) = \lambda e^{-\lambda x},\ x > 0$. Find the density function of $Y = \sqrt{X}$ and the density function of $Y = 1/X$.
SOLUTION: For $h(x) = \sqrt{x}$, we have $h^{-1}(y) = y^2,\ J(y) = 2y$.

$$f_Y(y) = f_X(h^{-1}(y))|J(y)| = 2\lambda y e^{-\lambda y^2}.$$

For $h(x) = 1/x$, we have $h^{-1}(y) = 1/y,\ J(y) = -1/y^2$.

$$f_Y(y) = f_X(h^{-1}(y))|J(y)| = \lambda y^{-2} e^{-\frac{\lambda}{y}},\ y > 0.$$

(b) Suppose X has a uniform distribution over the interval $(0, 1)$. Find the density function of $Y = e^X$.
SOLUTION: For $h(x) = e^x$, we have $h^{-1}(y) = \ln(y),\ J(y) = 1/y$. Here $f_X(x) = 1,\ x \in (0, 1)$. Thus $f_X(h^{-1}(y)) = 1,\ \ln(y) \in (0, 1)$, or equivalently if $y \in (1, e)$.

$$f_Y(y) = \begin{cases} 1/y, & y \in (1, e); \\ 0, & \text{otherwise.} \end{cases}$$