IEOR 3600: HMWK 6

1. Three players are playing a card game, denoted by P1,P2 and P3. Each time they play, they have probabilities of winning, independent of the past, of $1/3,\ 1/2, 1/6$ respectively. If they play 10 times, then what is the probability that P1 wins 2 times, P2 wins 6 times, and P3 wins 2 times?

2. X and Y have a joint continuous probability density function given by

$$f(x,y) = \begin{cases} 10xy^2, & 0 < x < y < 1; \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Confirm that f really is a probability density, that is, confirm that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy = 1.$$

   (b) Let $R = \{(x,y) : x \geq 0.5,\ y \geq 0.5\}$. Compute $P((X,Y) \in R) = P(X > 0.5,\ Y > 0.5) = \int\int_R f(x,y)dxdy$.

   (c) Find the marginal density functions $f_X(x)$ and $f_Y(y)$ of X and Y. (Confirm that they really are probability density functions, e.g., that they integrate to 1.)

   (d) Compute $E(X),\ E(Y),\ E(X^2),\ E(Y^2)$ and then give $Var(X),\ Var(Y)$.

   (e) Compute the covariance $\sigma_{X,Y} = Cov(X,Y) = E(XY) - E(X)E(Y)$, and the correlation coefficient

$$\rho = \frac{\sigma_{X,Y}}{\sigma_X \sigma_Y}.$$

   (f) Find the conditional probability density of Y given that $X = 0.25$, $f_{Y|0.25}(y)$. Use it to compute $P(Y > 0.5 \mid X = 0.25)$ and $E(Y \mid X = 0.25)$, and $Var(Y \mid X = 0.25)$.

3. Suppose X and Y have joint density

$$f(x,y) = \begin{cases} x(1+3y^2)/4, & 0 < x < 2,\ 0 < y < 1; \\ 0, & \text{otherwise.} \end{cases}$$

   Show that X and Y are independent (e.g., that $f(x,y) = f_X(x)f_Y(y),\ 0 < x < 2,\ 0 < y < 1$.)

4. Suppose that X and Y are such that conditional on $Y = y$, the distribution of X is exponential with rate y, that is, it has conditional density $f_{X|y}(x) = ye^{-yx},\ x > 0$. Further assume that Y has a continuous uniform distribution over the interval $(1,3)$. Find the joint density function $f(x,y)$.

5. Given a rv X with density function $f_X(x)$and a rv Y given by the transformation $Y = h(X)$ for a smooth invertable function $h(x)$ (with inverse function $h^{-1}(y)$), here we utilize the general formula for deriving the density of Y:

$$f_Y(y) = f_X(h^{-1}(y))|J(y)|,$$

   where $J(y) = (h^{-1})'(y)$, the derivative of the inverse function (called the *Jacobian*).

   (a) Suppose that X has an exponential distribution, with density $f(x) = \lambda e^{-\lambda x},\ x > 0$. Find the density function of $Y = \sqrt{X}$ and the density function of $Y = 1/X$.

   (b) Suppose X has a uniform distribution over the interval $(0,1)$. Find the density function of $Y = e^X$.