IEOR 3600: Midterm Exam with Solutions. Spring 2008. 80 Points total.

1. (10 points) The number of cigars produced a day at a factory is known to be a random variable X with mean 500, and variance 225. Find an interval $[a, b]$ such that $P(X \in [a, b]) \geq .75$.

   SOLUTION: Chebychev's inequality (with $k = 2$) implies that $P(|X - \mu| \leq 2\sigma) \geq 0.75$, for any rv X. ("The probability that X falls within two standard deviations from its mean is at least .75".) $|X - \mu| \leq 2\sigma$ if and only if $-2\sigma \leq X - \mu \leq 2\sigma$; equivalently if and only if $X \in [a, b] = [\mu - 2\sigma, \mu + 2\sigma]$. Here $\mu = 500$ and $\sigma = \sqrt{225} = 15$; $a = 500 - 30 = 470$, $b = 500 + 30 = 530$; $[a, b] = [470, 530]$.

2. (15 points) X and Y have joint probability density

$$f(x, y) = \begin{cases} 6x, & 0 < x < 1,\ 0 < y < 1 - x; \\ 0, & \text{otherwise.} \end{cases} .$$

   (a) (5 points) Show that X and Y are *not* independent.
   SOLUTION:
   We shall derive both marginals, and show that their product is not the same as $f(x, y)$.

$$f_X(x) = \int_0^{1-x} 6x dy = 6(1 - x)x,\ x \in (0, 1).$$

$$f_Y(y) = \int_0^{1-y} 6x dx = 3(1 - y)^2,\ y \in (0, 1)$$

   $f_X(x)f_Y(y) = 6(1 - x)x3(1 - y)^2 \neq 6x = f(x, y)$.
   (Note that $\{(x, y) : 0 < x < 1,\ 0 < y < 1 - x\}$ is equivalently stated as $\{(x, y) : 0 < y < 1,\ 0 < x < 1 - y\}$, or $\{(x, y) : x > 0, y > 0,\ 0 < x + y < 1\}$)

   (b) (5 points) Compute $\sigma_{X,Y} = Cov(X, Y)$.
   SOLUTION: $E(X) = \int_0^1 f_X(x)dx = 0.50$, and $E(Y) = \int_0^1 f_Y(y)dy = 0.25$. $E(XY) = \int_0^1 \int_0^{1-y} f(x, y)dxdy = 0.1$.
   $Cov(X, Y) = E(XY) - E(X)E(Y) = -0.025$.

   (c) (5 points) Compute $P(X > 0.3 \mid Y = 0.5)$.
   SOLUTION: First we compute $f_{X|Y=.5}(x) = f(x, .5)/f_Y(.5) = 6x/3(1 - .5)^2 = 8x,\ 0 < x < .5$
   $P(X > 0.3 \mid Y = 0.5) = \int_{0.3}^{0.5} 8x dx = 0.64$.

3. (15 points) A stock is now $S_0 = \$50$ per share. Its price at any fixed time t (years) in the future is a (lognormal) random variable distributed as

$$S(t) = S_0 e^{\sigma\sqrt{t}Z + \mu t},$$

   where $Z \sim N(0, 1)$ is a standard unit normal, and $\mu = 0.20$ and $\sigma = 0.12$.

   (a) (5 points) Compute $E(S(2))$.
   SOLUTION: $E(S(t)) = S_0 e^{\overline{r}t}$, where $\overline{r} = \mu + \sigma^2/2 = 0.20 + 0.0072 = 0.2072$. $E(S(2)) = 50e^{(0.2072)2} = 50e^{0.4144} = 50(1.513) = 75.65$.

(b) (10 points) Find the probability that the stock's value at time $t = 2$ years from now will be at least \$75.

SOLUTION: $P(S(2) \geq 75) = P(e^{\sigma\sqrt{2}Z+2\mu} \geq 1.5) = P(\sigma\sqrt{2}Z + 2\mu \geq \ln(1.5)) = P(Z \geq (\ln(1.5) - 2\mu)/\sigma\sqrt{2}) = P(Z > 0.032) = P(Z < -0.032) = \Phi(-0.032) = 0.375$.

4. (25 points)

3% of new light bulbs are known to live less than 2 minutes, while 55% are known to live for over 6 months.

(a) (10 points) Out of a random sample of 100 new bulbs, estimate (give a numerical answer) the probability that exactly 3 live for less than 2 minutes.

SOLUTION: Poisson approximation to the binomial ($n = 100$ large, $p = 0.03$ small); $\lambda = np = 3$. $P(X = 3) \approx e^{-\lambda}\lambda^3/3! = e^{-3}(4.5) = 0.224$.

(b) (10 points) Out of a random sample of 100 new bulbs, estimate (give a numerical answer) the probability that more than 60 live for over 6 months.

SOLUTION: Normal approximation to the binomial ($n = 100$ large, $p = 0.55$ close to 0.5); $\mu = np = 55$, $\sigma = \sqrt{(100(0.55)(0.45))} = 4.97$. $P(X \geq 61) = P(X \geq 60.5) \approx P(Z \geq (60.5 - \mu)/\sigma)) = P(Z \geq (5.5/4.97)) = P(Z \geq 1.11) = P(Z \leq -1.11) = \Phi(-1.11) = 0.134$.

(c) (5 points) Suppose you keep buying new bulbs from a large random sample, sequentially, one at a time. Let $N =$ the number you buy until (for the first time) a bulb lives for less than 2 minutes. What is $E(N)$?

SOLUTION: N has a geometric distribution with "success" probability $p = 0.03$. $P(N = k) = (1 - p)^{k-1}p,\ k \geq 1$. $E(N) = 1/p = 33.3$.

5. (15 points) Three coins are in a box. Coin 1 is a fair coin, coin 2 lands heads (H) with probability 2/3 (hence tails (T) wp= 1/3), and coin 3 lands H with probability 1/3 (hence T wp= 2/3). You randomly select a coin from the box (but do not know which one you have chosen). You then flip the coin 4 times and get $HTHT$. What is the probability that you flipped the fair coin?

SOLUTION: Baye's Rule. $A_1 = \{\text{fair coin chosen}\}$, $A_2 = \{p = 2/3 \text{ coin chosen}\}$, $A_3 = \{p = 1/3 \text{ coin chosen}\}$; $B = \{HTHT\}$. We want $P(A_1 \mid B)$. (In general, $P(HTHT) = \binom{4}{2}p^2(1-p)^2$.)

$$= \frac{P(B \mid A_1)P(A_1)}{P(B \mid A_1)P(A_1) + P(B \mid A_2)P(A_2) + P(B \mid A_3)P(A_3)}.$$

$$= \frac{\binom{4}{2}(.5)^4(1/3)}{\binom{4}{2}(.5)^4(1/3) + \binom{4}{2}(2/3)^2(1/3)^2(1/3) + \binom{4}{2}(1/3)^2(2/3)^2(1/3)}$$

$$= \frac{(.5)^4(1/3)}{(.5)^4(1/3) + (2/3)^2(1/3)^2(1/3) + (1/3)^2(2/3)^2(1/3)}$$

$$= \frac{(.5)^4(1/3)}{(.5)^4(1/3) + (2/3)[(2/3)^2(1/3)^2]}$$

$$= \frac{(.5)^4}{(.5)^4 + (2/3)^3(1/3)}$$

$$= \frac{0.0625}{0.0625 + 8/81} = 0.388$$