1 Regenerative Processes

Continuous-time case

Given a stochastic process \( X = \{ X(t) : t \geq 0 \} \), suppose that there exists a (proper) random time \( \tau = \tau_1 \) such that \( \{ X(\tau + t) : t \geq 0 \} \) has the same distribution as \( X \) and is independent of the past \( C_1 = \{ \{ X(t) : 0 \leq t < \tau \}, \tau \} \). Then we say that \( X \) regenerates at time \( \tau \), meaning that it has “started over again” probabilistically, as if it was time \( t = 0 \) again, and its future is independent of its past. In particular, \( X(\tau) \) has the same distribution as \( X(0) \), and is independent of the regeneration time \( \tau \). \( C_1 \) is called the first cycle and \( X_1 = \tau_1 \) is the first cycle length. But if such a \( \tau_1 \) exists, then there must be a second such time \( \tau_2 > \tau_1 \) yielding an identically distributed second cycle \( C_2 = \{ \{ X(\tau_1 + t) : 0 \leq t < \tau_2 - \tau_1 \}, \tau_2 - \tau_1 \} \). Continuing in this fashion we conclude that there is a renewal process of such times \( \{ \tau_k : k \geq 1 \} \) with iid cycle lengths \( X_k = \tau_k - \tau_{k-1}, k \geq 1 \), and such that the entire cycles \( C_k = \{ \{ X(\tau_{k-1} + t) : 0 \leq t < X_k \}, X_k \} \) are iid.

Note how in particular, \( X(\tau_k) \) is independent of the time \( \tau_k \): Upon regeneration the value of the process is independent of what time it is.

The above defines what is called a regenerative process. It is said to be positive recurrent if the renewal process is so, that is, if \( 0 < E(X_1) < \infty \). Otherwise it is said to be null recurrent if \( E(X_1) = \infty \).

As an example: Any recurrent continuous-time Markov chain is regenerative. Fix any state \( i \) and let \( X(0) = i \). Then let \( \tau = \tau_{i,i} \) denote the first time that the chain returns back to state \( i \) (after first leaving it). By the (strong) Markov property, the chain indeed starts all over again in state \( i \) at time \( \tau \) independent of its past. Since \( i \) is recurrent, we know that the chain must in fact return over and over again, and we can let \( \tau_k \) denote the \( k^{th} \) such time. We also know that positive recurrence of the chain, by definition, means that \( E(\tau_{i,i}) < \infty \), which is thus equivalent to being a positive recurrent regenerative process.

Because \( \{ \tau_k \} \) is a renewal process and the cycles \( \{ C_k \} \) are iid, we can apply the renewal reward theorem in a variety of ways to a regenerative process so as to compute various time average quantities of interest. The general result can be stated in words as

the time average is equal to the expected value over a cycle divided by the expected cycle length.

For example, we can obtain the time-average of the process itself as

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t X(s)ds = \frac{E(R)}{E(X)}, \text{ wp}1,
\]

where \( X = X_1 \) and \( R = R_1 = \int_0^X X(s)ds \). The proof is derived by letting \( N(t) \) denote the number of renewals (regenerations) by time \( t \), and observing that

\[
\int_0^t X(s)ds \approx \sum_{j=1}^{N(t)} R_j,
\]

where

\[
R_j = \int_{\tau_{j-1}}^{\tau_j} X(s)ds.
\]
So we are simply interpreting $X(t)$ as the rate at which we earn a reward at time $t$ and applying the renewal reward theorem. The $R_j$ are iid since they are constructed from iid cycles; $R_j$ is constructed from $C_j$.

Functions of regenerative processes are regenerative with the same regeneration times

If $X$ is regenerative with regeneration times $\tau_k$, then for any function $f = f(x)$, the process $Y(t) = f(X(t))$ defines a regenerative process with the very same regeneration times. This allows us to obtain

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t f(X(s)) ds = \frac{E(R)}{E(X)}, \text{ wp1,}$$

where $X = X_1$ and $R = R_1 = \int_0^X f(X(s)) ds$.

We next precisely state and prove the main result.

**Theorem 1.1** If $X$ is a positive recurrent regenerative process, and $f = f(x)$ is a (measurable) function such that $E[\int_0^{X_1} |f(X(s))| ds] < \infty$, then

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t f(X(s)) ds = \frac{E(R)}{E(X)}, \text{ wp1}, \quad (1)$$

and

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t E[f(X(s))] ds = \frac{E(R)}{E(X)}, \quad (2)$$

where $R = R_1 = \int_0^X f(X(s)) ds$, and $X = X_1$.

**Proof**: Let $N(t) = \max\{j : \tau_j \leq t\}$ denote the number of renewals (regenerations) by time $t$. First note that if $f \geq 0$, then since $t_{N(t)} \leq t < t_{N(t)} + 1$, by defining $R_j = \int_{\tau_{j-1}}^{\tau_j} f(X(s)) ds$, (1) can be derived by using the following sandwiching bounds (upper and lower bounds),

$$\frac{1}{t} \sum_{j=1}^{N(t)} R_j \leq \frac{1}{t} \int_0^t f(X(s)) ds \leq \frac{1}{t} \sum_{j=1}^{N(t)+1} R_j. \quad (3)$$

Both these bounds converge to $E(R)/E(X)$ yielding the result: For the lower bound,

$$\frac{1}{t} \sum_{j=1}^{N(t)} R_j = \frac{N(t)}{t} \frac{1}{N(t)} \sum_{j=1}^{N(t)} R_j;$$

$N(t)/t \to 1/E(X)$ by the elementary renewal theorem, while $\frac{1}{N(t)} \sum_{j=1}^{N(t)} R_j \to E(R)$ by the strong law of large numbers since the $R_j$ are iid and have finite first moment by the theorem’s hypothesis.

Similarly, for the upper bound, noting that $(N(t) + 1)/t \to 1/E(X)$ (since $1/t \to 0$), and rewriting the upper bound as

$$\frac{N(t) + 1}{t} \frac{1}{N(t) + 1} \sum_{j=1}^{N(t)+1} R_j,$$
we obtain the same limit for this upper bound, thus completing the proof of (1) in this non-negative case. Note in passing from (3), that because the upper bound converges to the same (finite) limit as the lower bound, it must hold that

\[
\frac{R_{N(t)} + 1}{t} \rightarrow 0, \text{wp1},
\]

(4)

because \( R_{N(t)} + 1/t \) is precisely the difference between the upper and lower bounds.

If \( f \) is not non-negative, then first observe that since \( |f| \) is non-negative, we can apply the above proof to \( \int_{0}^{t} f(X(s))ds \) to conclude from (4) that \( R_{N(t)}^{*} \rightarrow 0, \text{wp1} \), where \( R_{j}^{*} = \int_{0}^{\tau_{j}} f(X(s))ds, \ j \geq 1 \). The proof of (1) is then completed by writing

\[
\frac{1}{t} \int_{0}^{t} f(X(s))ds = \frac{1}{t} \sum_{j=1}^{N(t)} R_{j} + \frac{1}{t} \int_{t_{N(t)}}^{t} f(X(s))ds,
\]

noting that the first piece on the rhs converges to the desired answer, and the second piece (the error) tends to 0 by (4),

\[
\left| \frac{1}{t} \int_{t_{N(t)}}^{t} f(X(s))ds \right| \leq \frac{1}{t} \int_{t_{N(t)}}^{t} |f(X(s))|ds \leq \frac{1}{t} \int_{t_{N(t)}}^{t} f(X(s))ds = R_{N(t)}^{*} \rightarrow 0.
\]

(\( E(|R_{j}|) < \infty \) since \( |R_{j}| \leq R_{j}^{*} \) and \( E(R_{j}^{*}) < \infty \) by the theorem’s hypothesis.)

To prove (2) we must show that \( \{R(t)/t : t \geq 1\} \) is uniformly integrable (UI), where \( R(t) \overset{\text{def}}{=} \int_{0}^{t} f(X(s))ds \). Since \( |R(t)| \leq \int_{0}^{t} f(X(s))ds \), it follows that

\[
|\frac{R(t)}{t} - Y(t)| \leq \frac{1}{t} \sum_{j=1}^{N(t) + 1} R_{j}^{*}.
\]

(5)

We know that \( Y(t) \rightarrow E(R)/E(X) < \infty \), wp1. Moreover, \( N(t) + 1 \) is a stopping time with respect to \( \{X_{j}, R_{j}^{*}\} \), so from Wald’s equation \( E(Y(t)) = E(N(t))/t + 1/t E(R^{*}) \rightarrow E(R)/E(X) \) (via the expected value version of the elementary renewal theorem); thus \( \{Y(t) : t \geq 1\} \) is uniformly integrable (UI)\(^1\) and since \( |R(t)|/t \leq Y(t) \), \( \{R(t)/t : t \geq 1\} \) is UI as well.

As an immediate application of Theorem 1.1 take \( f(x) = I\{x \leq b\} \) with \( b \) constant. Such a function is non-negative and bounded, so the hypothesis of Theorem 1.1 is immediately verified.

**Corollary 1.1** A positive recurrent regenerative process has a limiting stationary distribution: Letting \( X^{*} \) denote a rv with this distribution, we have

\[
P(X^{*} \leq b) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} I\{X(s) \leq b\}ds = \frac{E(R)}{E(X)}, \text{ wp1, } b \in \mathcal{R}
\]

where \( R = \int_{0}^{X_{1}} I\{X(s) \leq b\}ds \), and \( X = X_{1} \). Also,

\[
P(X^{*} \leq b) = \lim_{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} P(X(s) \leq b)ds = \frac{E(R)}{E(X)}, \ b \in \mathcal{R}.
\]

\(^1\) A collection \( \{X(t)\} \) of non-negative rvs for which \( X(t) \rightarrow X \) wp1 with \( E(X) < \infty \) is UI if and only if \( E(X(t)) \rightarrow E(X) \).
Examples

1. **Forward recurrence time** $A(t)$, backwards recurrence time $B(t)$ and spread $S(t)$ for a positive recurrent renewal process $\{t_n\}$ with iid interarrival times $X_n$. The regeneration times are these same renewal times. Theorem 1.1 yields for example

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t A(s) ds = \frac{E(X^2)}{2E(X)}, \text{ wp1},$$

and

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t E[A(s)] ds = \frac{E(X^2)}{2E(X)},$$

via $R = R_1 = \int_0^{X_1} A(s) ds = \int_0^{X_1} (X_1 - s) ds = X_1^2/2$.

Let $F(x) = P(X_1 \leq x)$ and let $F_e(x) = \int_0^x \frac{P(X_1 > y)dy}{E(X_1)}$, the equilibrium distribution of $F$. Corollary 1.1 yields for example

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t P(A(s) \leq x) ds = \frac{E(\min\{x, X_1\})}{E(X)} = \int_0^x \frac{P(X > y)dy}{E(X)} = F_e(x),$$

via $R = R_1 = \int_0^{X_1} I\{A(s) \leq x\} ds = \int_0^{X_1} I\{X_1 - s \leq x\} ds = \min\{x, X_1\}$.

2. **GI/GI/1 queue** $0 < \rho < 1$: $L(t)$ denotes the number of customers in the system at time $t$. Start the system off at time $t = 0$ with one customer just starting service, so that a busy period $B_1$ just begins. Let $\tau_k$ denote the $k^{th}$ time that an arrival finds the system empty (hence starts the $k^{th}$ busy period). $X_1 = B_1 + I_1$, where $I_1$ is the idle period following $B_1$. At time $\tau_k$, busy period $B_k$ begins, and is followed by idle period $I_k$; $X_k = B_k + I_k$, $k \geq 1$ are iid. $\{L(t) : t \geq 0\}$ regenerates (with value 0) at each time $\tau_k$. The $\{B_k\}$ are iid, as are the $\{I_k\}$, but for each $k$ the rvs $B_k$ and $I_k$ are typically dependent. It can be shown that $E(X) < \infty$ since $\rho < 1$.

Other related process, such as workload $\{V(t) : t \geq 0\}$ are also regenerative using these same regeneration times. One can, in principle, apply Theorem 1.1 to $\{L(t) : t \geq 0\}$ and $\{V(t) : t \geq 0\}$.

In the case when the arrival process is Poisson, the $M/G/1$ queue, there are alternative regeneration points that can be used by taking advantage of the memoryless property of the exponential distribution. For example, we could start the system empty, $L(0) = 0$, and then let $\tau_k$ denote the $k^{th}$ time at which the $k^{th}$ busy period ends. For the $M/M/1$ queue, we could even start the system off with one customer, $L(0) = 1$, and then let $\tau_k$ denote the $k^{th}$ time at which an arrival finds exactly one customer in the system.

### 1.1 Delayed regenerative processes

Given a regenerative process, $X = \{X(t) : t \geq 0\}$, it is natural to want to begin observing it starting at some future time $t = T$ (say) (random or deterministic) as opposed to at time $t = 0$; we thus would consider the process $X_T = \{X(T + t) : t \geq 0\}$. But if we do that, then (being in the middle of some cycle at time $T$) $X_T(0) = X(T)$ will not have the desired regeneration distribution as $X(0)$ did; the first cycle will be a remaining cycle, and thus not be the same (in distribution) as a typical future iid one. More generally, we can take any regenerative process, and simply replace the first cycle by something different in distribution from (but independent of) the others, and yet it will keep regenerating once the first cycle ends.
A regenerative process in which the first cycle has been replaced by one with a different distribution from (but is still independent of) the iid cycles following it is called a delayed regenerative process with delayed initial cycle $C_0$ and delayed initial cycle length $X_0$. Thus, there is an initial random time $0 \leq \tau_0 < \tau_1$, $X_0 = \tau_0$, and the initial cycle is given by $C_0 = \{\{X(t) : 0 \leq t < \tau_0\}\}X_0$. $X_1 = \tau_1 - \tau_0$, and the next cycle is $C_1 = \{\{X(\tau_0 + t) : 0 \leq t < X_1\}\}X_1$, and so on. $C_0$ is independent of the other iid cycles $\{C_k : k \geq 1\}$, but is allowed to have a different distribution. The renewal process $\{\tau_n : n \geq 0\}$ is a delayed renewal process with initial delay $\tau_0$. (We assume that $P(\tau_0 > 0) > 0$.)

When there is no delay cycle, a regenerative process is called non-delayed. Note that once the delay is finished, a delayed regenerative process becomes non-delayed, thus $\{X(\tau_0 + t) : t \geq 0\}$ is a regular non-delayed regenerative process, called a non-delayed version. A delayed regenerative process is called positive recurrent if its non-delayed version is so, that is, if $0 < E(X_1) < \infty$. It is not required that $\tau_0 = X_0$ have finite first moment, but we of course assume that it is a proper rv, that is, that $P(X_0 < \infty) = 1$, so that $C_0$ will end wp1.

As a common example, given a positive recurrent CTMC and using visits to state 1 as regeneration times, what if it started in state 2 instead? Clearly, when $X(0) = 2$, once the chain enters state 1 (which it will by recurrence), we are back to the non-delayed case; the delay is short lived. In this case $X_0 = \tau_{2,1} = \inf\{t > 0 : X(t) = 1 | X(0) = 2\}$, the hitting time to state 1 given $X(0) = 2$, and $C_0$ captures the evolution of the chain until it hits state 1.

Since a delay cycle $C_0$ eventually ends, limiting results such as Theorem 1.1 should and do remain valid with some minor regularity conditions placed on $C_0$ ensuring that its effect is asymptotically negligible as $t \to \infty$. We deal with this next. The reward and cycle used in such results must now be the first one, $R = R_1 = \int_{\tau_0}^{\tau_0+X_1} f(X(s))ds$, and $X = X_1$, not of course the initial one, $R_0 = \int_0^{\tau_0} f(X(s))ds$, $X_0$.

Corollary 1.1 remains valid without any further conditions:

**Proposition 1.1** A delayed positive recurrent regenerative process (with no conditions placed on the delay cycle $C_0$) has the same limiting stationary distribution as its non-delayed counterpart: Letting $X^*$ denote a rv with this distribution, we have

$$P(X^* \leq b) = \lim_{t \to \infty} \frac{1}{t} \int_0^t I\{X(s) \leq b\}ds = \frac{E(R)}{E(X)}, \text{ wp1, } b \in \mathbb{R}$$

where $R = R_1 = \int_{\tau_0}^{\tau_0+X_1} I\{X(s) \leq b\}ds$, and $X = X_1$. Also,

$$P(X^* \leq b) = \lim_{t \to \infty} \frac{1}{t} \int_0^t P(X(s) \leq b)ds = \frac{E(R)}{E(X)}, \text{ b } \in \mathbb{R}.$$

**Proof**: For $t > \tau_0$, we have wp1,

$$\frac{1}{t} \int_0^t I\{X(s) \leq b\}ds = \frac{R_0}{t} + \frac{1}{t} \int_{\tau_0}^t I\{X(s) \leq b\}ds,$$

where $R_0 = \int_0^{\tau_0} I\{X(s) \leq b\}ds \leq \tau_0$; thus $\frac{R_0}{t} \to 0$, wp1. The second piece is a non-delayed version, and can be re-written as

$$\frac{t - \tau_0}{t} \frac{1}{t - \tau_0} \int_{\tau_0}^t I\{X(s) \leq b\}ds,$$
hence converges wp1, as $t \to \infty$, to the desired $\frac{E(R)}{E(X)}$ by the first part of Corollary 1.1. Taking expected value of $\frac{1}{t} \int_0^t I\{X(s) \leq b\} ds$, and applying the bounded convergence theorem then yields the second result.

In general, some further conditions on $C_0$ are needed to obtain Theorem 1.1: The only further condition needed for (1) is

$$\int_0^{X_0} |f(X(s))| ds < \infty, \text{ wp1},$$

and for (2) a sufficient condition is

$$E\left[\int_0^{X_0} |f(X(s))| ds\right] < \infty.$$  

**Theorem 1.2** If $X$ is a delayed positive recurrent regenerative process, and $f = f(x)$ is a (measurable) function such that $E\left[\int_0^{\tau_0+X_1} |f(X(s))| ds\right] < \infty$, and also (9) holds, then

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t f(X(s)) ds = \frac{E(R)}{E(X)}, \text{ wp1},$$

and if additionally (10) holds, then also

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t E[f(X(s))] ds = \frac{E(R)}{E(X)},$$

where $R = R_1 = \int_0^{\tau_0+X_1} f(X(s)) ds$, and $X = X_1$.  

**Proof:** $R_0 = \int_0^{\tau_0} f(X(s)) ds$ and the proof of (11) is identical to the proof in Proposition 1.1 with condition (9) ensuring that $P(|R_0| < \infty) = 1$, so that $R_0/t \to 0$.

For (12) we argue UI as we did in proving (2), modifying the upper bound in (5) to a stochastic upper bound

$$|R(t)|/t \leq K(t) = \frac{R_0^*}{t} + Y(t),$$

where $R_0^* = \int_0^{\tau_0} |f(X(s))| ds$ is independent of $\{Y(t)\}$, and $Y(t)$ is constructed from a non-delayed version, and hence (as we proved) $E(Y(t)) \to \frac{E(R)}{E(X)}$. Since $E(R_0^*) < \infty$ by (10), it follows that $E(R_0^*/t) \to 0$ and hence $\{K(t)\}$ is UI; thus so is $\{R(t)/t\}$ as was to be shown.  

That (10) is not necessary for (12) can be seen by Proposition 1.1. That is because whenever $f$ is non-negative and bounded (such as $f(x) = I\{x \leq b\}$), (9) is automatically satisfied, but (10) might not be satisfied even though (12) still holds (by applying the bounded convergence theorem to (11)).

1.2 Stationary versions of a regenerative processes

Given a positive recurrent regenerative process $X = \{X(t) : t \geq 0\}$, we know that it always has a limiting stationary distribution from Corollary 1.1. But that is only a marginal distribution. It turns out that by creating a particular initial delay cycle $C_0^*$, we can make the regenerative process a stationary process, called a stationary version of the process, denoted by $X^*$ =
\( \{X^*(t) : t \geq 0\} \). This means that for every fixed \( T \geq 0 \), the entire shifted process \( \{X^*(T + t) : t \geq 0\} \) has the same distribution; the same as \( \{X^*(t) : t \geq 0\} \); e.g., all joint distributions such as the distribution of the vector \( (X^*(T + t_1), X^*(T + t_2), \ldots, X(T + t_k)) \) are the same for all \( T \). In particular, \( X^*(t) \) has the same distribution at \( X^*(0) \) for all \( t \geq 0 \), the distribution in Corollary 1.1. As an easy example, consider a positive recurrent CTMC, with stationary probabilities \( \{P_n\} \) (solution to the balance equations). Then, as we know, by starting off \( X(0) \) randomly distributed as \( \{P_n\} \), \( P(X(0) = n) = P_n \), the chain is stationary. Taking this stationary version, we choose \( \tau_0 = \inf\{t \geq 0 : X(t) = i\} \) and \( C_0^* \) is simply the evolution of the chain until state \( i \) is hit.

In general, though, how do we find \( C_0^* \)? First we must recall that the underlying renewal process has a stationary version by using delay \( \tau_0 \sim F_e \), where \( F(x) = P(X \leq x) \) is the cycle length distribution (for the iid cycle lengths \( \{X_n : n \geq 1\} \), and \( F_e(x) = \int_0^x P(X > u)du/E(X) \) is the equilibrium distribution. Letting \( \tau^* \) is a positive recurrent regenerative process, and \( f = f(x) \) is a (measurable) function such that \( E\left[ \int_0^{X_1} |f(X(s))| ds \right] < \infty \), then

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t f(X(s)) ds = E(f(X^*(0))), \text{ wp} 1. \tag{14}
\]
In particular, using \( f(x) = x \) yields
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t X(s)ds = E(X^*(0)), \text{ wp1.} \tag{15}
\]

As an example, consider a positive recurrent CTMC, and suppose we solved the balance equations and found the stationary probabilities \( \{P_n\} \). Then (15) says
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t X(s)ds = E(X^*(0)) = \sum_n nP_n. \tag{16}
\]

To prove (16), recall that \( \text{wp1, } P_n = \lim_{t \to \infty} \frac{1}{t} \int_0^t I\{X(s) = n\}ds. \)

Now let \( I_n(t) = \int_0^t I\{X(s) = n\}ds; \) thus \( I_n(t)/t \to P_n. \) Then
\[
\frac{1}{t} \int_0^t X(s)ds = \frac{1}{t} \sum_{n=0}^\infty nI_n(t) \]
\[
= \sum_{n=0}^\infty n(I_n(t)/t) \]
\[
\to \sum_{n=0}^\infty nP_n.
\]

The interchange of limit and infinite sum can be justified.

For the more general (15): Assume non-negativity. Then \( E(X^*(0)) = \int_0^\infty P(X^*(0) > b)db, \)
and we also know that \( P(X^*(0) > b) = \lim_{t \to \infty} \frac{1}{t} \int_0^t I\{X(s) > b\}ds, \) we thus obtain (changing the order of integration and limits; allowed by the bounded convergence theorem and Tonelli’s theorem):
\[
E(X^*(0)) = \int_0^\infty P(X^*(0) > b)db \]
\[
= \int_0^\infty \lim_{t \to \infty} \frac{1}{t} \int_0^t I\{X(s) > b\}dsdb \]
\[
= \lim_{t \to \infty} \frac{1}{t} \int_0^t \int_0^\infty I\{X(s) > b\}dbdt \]
\[
= \lim_{t \to \infty} \frac{1}{t} \int_0^t \int_0^\infty X(s)dbdt \]
\[
= \lim_{t \to \infty} \frac{1}{t} \int_0^t X(s)ds.
\]

The big idea in the background of all of this: \( X \) and \( X^* \) only differ up until time \( \tau_0 \) for \( X^* \), after which they are identical; thus their time-averages are identical (since the “up until \( \tau_0 \)” part is asymptotically insignificant). Thus, in fact,
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t f(X(s))ds = \lim_{t \to \infty} \frac{1}{t} \int_0^t f(X^*(s))ds \text{ wp1,} \tag{17}
\]

and we know we can obtain the same limits by taking expected values too. Thus, we can say that wp1,

$$
\lim_{t \to \infty} \frac{1}{t} \int_0^t f(X(s))ds = \lim_{t \to \infty} \frac{1}{t} \int_0^t E(f(X^*(s)))ds.
$$

(18)

But the rhs is stationary and so

$$
E(f(X^*(s))) = E(f(X^*(0)), s \geq 0, \text{ a constant}; \text{ thus}
\lim_{t \to \infty} \frac{1}{t} \int_0^t E(f(X^*(s)))ds = \lim_{t \to \infty} \frac{1}{t} \int_0^t E(f(X^*(0)))ds = E(f(X^*(0)));
$$

we have derived (14).

**Discrete-time regenerative processes**

For a stochastic sequence \( \{Y_n : n \geq 0\} \), the definition of regenerative is the same. In this case, \( C_1 = \{\{Y_n : 0 \leq n \leq \tau_1 - 1\}, \tau_1\} \), and the renewal process is a discrete time renewal process; \( N(n)/n \to 1/E(X) \) as \( n \to \infty \). Because of the discrete setting, the \( k^{th} \) cycle begins at time \( \tau_{k-1} \) and ends at time \( \tau_k - 1 \), and there are \( X_k \) values of the \( Y_n \) in the cycle. The analog of Theorem 1.1 is

**Theorem 1.4** If \( Y \) is a positive recurrent regenerative sequence, and \( f = f(x) \) is a (measurable) function such that \( E[\sum_{j=0}^{X_1-1} |Y_j|] < \infty \), then

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} f(Y_j) = \frac{E(R)}{E(X)}, \text{ wp1},
$$

(19)

and

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} E(f(Y_j)) = \frac{E(R)}{E(X)},
$$

(20)

where \( R = R_1 = \sum_{j=0}^{X_1-1} f(Y_j) \), and \( X = X_1 \).