1. Consider a renewal point process \( \{ t_n \} \) with iid interarrival times \( \{ X_n \} \). We know that for the forward recurrence time process \( A = \{ A(t) : t \geq 0 \} \), wp1,

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t A(s)ds = \frac{E(X^2)}{2E(X)}.
\]

Notice that \( A \) is a regenerative process with regeneration times simply the \( t_n \) and regenerative cycle lengths \( \{ X_n \} \), and thus this result is just “the long-run average equals the expected value over a cycle divided by the expected cycle length.” Here, over a cycle we have \( R = \int_0^{X_1} A(s)ds = \int_0^{X_1}(X_1 - s) = X_1^2/2 \).

Suppose more generally that a point process \( \{ t_n : n \geq 1 \} \) is defined as follows: Let \( \{ Y_n \} \) be an iid sequence of non-negative rvs, and independently let \( \{ Z_n \} \) be another iid sequence of non-negative rvs. Define \( t_1 = Y_1, t_2 = Y_1 + Z_1, t_3 = Y_1 + Z_1 + Y_2, t_4 = Y_1 + Z_1 + Y_2 + Z_2 \) and so on.

(a) Note that now, unlike the renewal point process case, it is not true that the \( t_n \) are regeneration times (why?). Argue, however, that \( A \) is still a regenerative process and give the regeneration times and cycle lengths.

(b) Find an expression for

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t A(s)ds.
\]

2. Consider standard Brownian motion (BM) \( \{ B(t) : t \geq 0 \} \). \( B(0) = 0. \) Fix \( x > 0 \) and consider \( T_x = \) the hitting time to state \( x \). Now define \( \tau = \min \{ t > T_x : B(t) = 0 \} \). \( \tau \) is thus the first time that the BM returns back to 0 after hitting \( x \). Argue that BM is thus regenerative with regeneration time \( \tau \). Do the cycle lengths have finite first moment?

3. Recall a semi-Markov process in continuous time \( \{ X(t) : t \geq 0 \} \): It has a discrete state space, and transitions from state to state occur according to a discrete-time Markov chain with a transition matrix \( P = (P_{ij}) \). But when entering state \( i \), the chain remains there for an amount of time \( H_i \) (holding time in state \( i \)) that has a general distribution \( F_i \), independent of the past. (Such a process is not Markovian unless all the \( F_i \) are exponentials.) Suppose that the state space \( S \) is finite, and that each \( F_i \) has finite first moment \( 0 < E(H_i) < \infty \). Also suppose that the chain is positive recurrent; there is a unique probability solution \( \pi \) to \( \pi = \pi P \).

(a) Argue that for each initial state \( X(0) = i \), \( \{ X(t) : t \geq 0 \} \) is a (positive recurrent) regenerative process.

(b) Fix a state \( i \). Let \( j \neq i \). Consider the discrete-time chain with transition matrix \( P \). Argue that \( \pi_j/\pi_i \) is the expected number of visits to state \( j \) between visits to state \( i \). In other words if we start the chain off in state \( i \), and then wait until it re-enters state \( i \) again, and count how many times it hit \( j \) along the way, then \( \pi_j/\pi_i \) is the expected number of such times.

(c) Let \( P_i = \lim_{t \to \infty} \frac{1}{t} \int_0^t P(X(s) = i)ds, \ i \in S, \) denote the limiting distribution. Argue that in fact

\[
P_i = \frac{E(H_i)}{\sum_{j \in S} \pi_j E(H_j)}.
\]